

The Topological Matrix Model of $c = 1$ String

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ABSTRACT

We derive a Kontsevich-type matrix model for the $c = 1$ string directly from the W_∞ solution of the theory. The model that we obtain is different from previous proposals, which are proven to be incorrect. Our matrix model contains the Penner and Kontsevich cases, and we study its quantum effective action. The simplicity of our model leads to an encouraging interpretation in the context of background-independent non-critical string field theory.

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1. Introduction

For $c < 1$ non-critical string theories, there exists a remarkable description of the generating function in terms of one-matrix models of Kontsevich type[1][2][3]. These give rise to a graphical expansion which corresponds to the “fat-graph” cell-decomposition of the moduli spaces $\mathcal{M}_{g,n}$ of Riemann surfaces[4]. These models play an important role in understanding topological gravity and hence, one hopes, the fundamental formulation of string theory.

One way to infer the equivalence of the $c < 1$ string theory and the corresponding Kontsevich-type models is to show that the latter satisfy the Virasoro and W_n constraints[5] which encode the complete perturbative solution of the former[6].

The solution of $c = 1$ string theory is given by W_∞ identities[7]. For the compactified theory at the self-dual radius, the singlet sector perturbation series is neatly encoded in the W_∞ recursion relations obtained by Dijkgraaf, Moore and Plesser[8] directly from matrix quantum mechanics via a coherent-state representation of the tachyon scattering process[9]. In this paper we will show that this solution can be used to directly derive a matrix-model of Kontsevich type, for the $c = 1$ string at the self-dual radius.

A Kontsevich-type matrix model for $c = 1$ (which we refer to below as the DMP model) was actually presented in Ref.[8], where it was obtained again from the coherent-state description. The model that we derive below differs from theirs in several distinctive ways. We examine very carefully the DMP model and show that it is incorrect – their model does not in fact satisfy the W_∞ constraints. We demonstrate this first by general arguments, and then by explicit computation of some tachyon correlators. Next we identify an error in their paper, after correcting which we indeed recover our model.

An earlier proposal for a Kontsevich-type model to describe $c = 1$ string theory, due to Chekhov and Makeenko[10] turns out to be somewhat closer to the correct model that we present here, although it is not quite right either.

The matrix model that we obtain is astonishingly simple, and we analyse it in some detail. In Section 2 we derive the model from W_∞ . In Section 3 we compare it with previous proposals and show that the latter do not describe $c = 1$ string theory. In Section 4 we give an alternative derivation of our model following the technique of Refs.[11],[8]. In Section 5 we examine the relation of our model to the Penner matrix model describing the Euler characteristic of punctured surfaces, and to the original Kontsevich model describing intersection theory on moduli space. In Section 6 we derive the quantum effective action

and show that it generates $c = 1$ string amplitudes via tree graphs. In Section 7 we observe that our Kontsevich-type model can be thought of as a background-independent string field theory coupled to an external source. Expanding about various possible minima then leads to the $c = 1$ and $c < 1$ strings, the latter arising from condensation of particular negative-momentum tachyons.

2. The $c = 1$ Kontsevich Model from W_∞

The tachyon operators T_n of 2D string theory compactified on a circle of unit radius are labelled by the integer-valued momentum n . It is convenient to introduce an infinite number of variables t_n with $n = 1, 2, \dots$, in correspondence with the tachyons of positive momentum n , and analogous variables \bar{t}_n for the negative-momentum tachyons. The generating functional for the correlation functions of all tachyons operators will be a function $Z(t, \bar{t})$ of the t_n and \bar{t}_n .

The W_∞ solution[8] of this string theory is encoded in the following recursion relation:

$$\frac{1}{\mu^2} \frac{\partial Z_{W_\infty}}{\partial \bar{t}_n}(t, \bar{t}) = \frac{1}{(n+1)} \oint dz : e^{-i\mu\phi(z)} \left(\frac{\partial_z}{i\mu} \right)^{n+1} e^{i\mu\phi(z)} : Z_{W_\infty}, \quad (2.1)$$

where the bosonic field $\phi(z)$ is the following operator:

$$\partial\phi(z) = \frac{1}{z} + \sum_{n>0} n t_n z^{n-1} - \frac{1}{\mu^2} \sum_{n>0} \frac{\partial}{\partial t_n} z^{-n-1}. \quad (2.2)$$

We now show that one can in fact construct a matrix model with logarithmic potential starting directly from the above expression. First, change variables in the W_∞ relation using the Frobenius-Miwa-Kontsevich transformation:

$$i\mu t_n = -\frac{1}{n} \text{tr} A^{-n}, \quad (2.3)$$

where A is a fixed $N \times N$ Hermitian matrix. In the limit of large N , all the couplings become independent, but at finite N everything continues to be valid in a subspace of the parameter space where only the first N t_k are independent. One finds

$$\begin{aligned} \frac{1}{\mu^2} \frac{\partial Z_{W_\infty}}{\partial \bar{t}_n}(t, \bar{t}) &= \frac{1}{n+1} \oint dz : \frac{z^{-i\mu} e^{\frac{1}{i\mu} \sum_{k>0} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}}}{\det(1 - zA^{-1})} \\ &\times \left(\frac{\partial_z}{i\mu} \right)^{n+1} z^{i\mu} e^{-\frac{1}{i\mu} \sum_{k>0} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}} \det(1 - zA^{-1}) : Z_{W_\infty}(t, \bar{t}). \end{aligned} \quad (2.4)$$

Inserting the eigenvalues a_i of the matrix A , one can pick up the residues of the poles at $z = a_i$, to get

$$\begin{aligned} \frac{1}{\mu^2} \frac{\partial Z_{W_\infty}}{\partial \bar{t}_n}(t, \bar{t}) &= \frac{1}{n+1} \cdot \frac{1}{(i\mu)^{n+1}} \sum_i \frac{a_i^{-i\mu} e^{\frac{1}{i\mu} \sum_{k>0} \frac{a_i^{-k}}{k} \frac{\partial}{\partial t_k}}}{\prod_{j \neq i} (a_j - a_i)} \\ &\times \left[\frac{\partial^{n+1}}{\partial z^{n+1}} \left(z^{i\mu} \prod_j (a_j - z) e^{-\frac{1}{i\mu} \sum_{k>0} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}} \right) \right]_{z=a_i} : Z_{W_\infty}(t, \bar{t}). \end{aligned} \quad (2.5)$$

Now, of the $n+1$ z -derivatives, at least one must act on the second factor in the brackets, otherwise the term will vanish. So one derivative can be picked out in $n+1$ ways to do this, cancelling the $n+1$ factor in the denominator. Next, using the identity

$$e^{\frac{1}{i\mu} \sum_{k>0} \frac{a_i^{-k}}{k} \frac{\partial}{\partial t_k}} \left[\frac{\partial}{\partial z} e^{-\frac{1}{i\mu} \sum_{k>0} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}} \right]_{z=a_i} = \frac{1}{i\mu} \sum_{k>0} a_i^{-k-1} \frac{\partial}{\partial t_k} = \frac{\partial}{\partial a_i}, \quad (2.6)$$

one rewrites Eq. (2.5) as

$$\begin{aligned} \frac{1}{\mu^2} \frac{\partial Z_{W_\infty}}{\partial \bar{t}_n}(t, \bar{t}) &= -\frac{1}{(i\mu)^{n+1}} \sum_i \frac{a_i^{-i\mu}}{\prod_{j \neq i} (a_j - a_i)} \\ &\times \left(\frac{\partial}{\partial a_i} \right)^n \left(a_i^{i\mu} \prod_{j \neq i} (a_j - a_i) Z_{W_\infty}(t, \bar{t}) \right). \end{aligned} \quad (2.7)$$

Recalling the well-known result

$$\text{tr} \left(\frac{\partial}{\partial A} \right)^n = \frac{1}{\Delta(a)} \sum_i \left(\frac{\partial}{\partial a_i} \right)^n \Delta(a), \quad (2.8)$$

where $\Delta(a) = \prod_{j < k} (a_j - a_k)$ is the Vandermonde determinant, one can change back from eigenvalues to the full matrix A :

$$\frac{1}{i\mu} \frac{\partial Z_{W_\infty}}{\partial \bar{t}_n}(t, \bar{t}) = \frac{1}{(i\mu)^n} (\det A)^{-i\mu} \text{tr} \left(\frac{\partial}{\partial A} \right)^n (\det A)^{i\mu} Z_{W_\infty}(t, \bar{t}). \quad (2.9)$$

This is a remarkable expression for W_∞ in terms of a single fixed matrix!

This identity can be solved through random matrices in the following way. Introduce the matrix integral

$$Z_K(t, \bar{t}) = (\det A)^{-i\mu} \int dM e^{\text{tr} V(M, t, \bar{t})} \quad (2.10)$$

with some, as yet unknown, potential V . Then Eq. (2.9) implies that

$$\left[\frac{1}{i\mu} \frac{\partial}{\partial \bar{t}_n} - \frac{1}{(i\mu)^n} \text{tr} \left(\frac{\partial}{\partial A} \right)^n \right] \left((\det A)^{i\mu} Z_K(t, \bar{t}) \right) = 0. \quad (2.11)$$

This determines

$$V(M, t, \bar{t}) = i\mu MA + i\mu \sum_{k>0} \bar{t}_k M^k + f(M) \quad (2.12)$$

where $f(M)$ is independent of (t, \bar{t}) . The boundary condition is that $Z_K(t, 0)$ must be independent of t (this comes from momentum conservation in the string theory). Now

$$\begin{aligned} Z_K(t, 0) &= (\det A)^{-i\mu} \int dM e^{i\mu \text{tr} MA + \text{tr} f(M)} \\ &= (\det A)^{-i\mu - N} \int dM e^{i\mu \text{tr} M + \text{tr} f(MA^{-1})}, \end{aligned} \quad (2.13)$$

from which it follows that

$$f(M) = -(i\mu + N) \log M \quad (2.14)$$

and consequently also the matrix integral must be over positive-definite Hermitian matrices M .

It follows that, up to an overall multiplicative constant independent of (t, \bar{t}) , $Z_{W_\infty}(t, \bar{t})$ is equal to the Kontsevich-type integral

$$Z_K(t, \bar{t}) = (\det A)^{-i\mu} \int dM e^{i\mu \text{tr} MA - (i\mu + N) \text{tr} \log M + i\mu \sum_{k>0} \bar{t}_k \text{tr} M^k}. \quad (2.15)$$

This can equivalently be written

$$Z_K(t, \bar{t}) = \int dM e^{i\mu \text{tr} M - (i\mu + N) \text{tr} \log M + i\mu \sum_{k>0} \bar{t}_k \text{tr} (MA^{-1})^k}. \quad (2.16)$$

In this form, our model is very similar to the matrix models studied recently by Kazakov et al.[12], with the difference that the Gaussian potential is replaced by the gamma-function integrand. The ‘‘Euclidean’’ continuation $\nu = -i\mu$ gives a convergent integral as long as $\nu > N - 1$.

3. Previous Kontsevich-type models for $c = 1$

The ‘‘Kontsevich-Penner’’ model of Dijkgraaf et al. is given by the following matrix integral:

$$Z_{DMP}(t, \bar{t}) = (\det A)^{-N+i\mu} \int dM e^{i\mu \operatorname{tr}[MA^{-1} - \log M + \sum_{k>0} \bar{t}_k M^{-k}]}, \quad (3.1)$$

with the parameters t_n defined as in Eq. (2.3) above.

Comparing with Eq. (2.15), we see that there are three differences: (i) given the convention in Eq. (2.3) for the relation between t_n and A , the power of A appearing in the first term is negative in the DMP model but positive in the correct one; (ii) the coefficient of the log term is $-i\mu$ in the DMP model but $-i\mu - N$ in our model; (iii) the perturbations representing the incoming tachyons are negative powers of M in the DMP model but positive powers in our model.

These differences are in no way conventional or removable by any change of variables. One of the simplest ways to see this is that in our model there are *two* linear terms in M , one coupled to A and the other to \bar{t}_1 . This fact is responsible for the puncture equation, as we show below. The DMP model has only one linear term in M , and does not satisfy the puncture equation. It is quite straightforward, if a little tedious, to compute correlators for the DMP model using Schwinger-Dyson equations, and we will give some examples below. By contrast, the Schwinger-Dyson equations are virtually trivial for our model, perhaps the greatest surprise of the present analysis.

As is well-known, Kontsevich-type matrix integrals make sense even for 1×1 matrices, where they compute correlators in some 1-dimensional subspace of the (t, \bar{t}) parameter space. Thus, one can start by computing $Z^{-1} \partial Z / \partial \bar{t}_n$ at $\bar{t} = 0$ in the DMP model of Eq. (3.1) above, and for our Kontsevich-type model (Eq. (2.15)), with the matrix M replaced by a single variable m , and with the constant matrix A set equal to a number a . This should be compared with the W_∞ answer for the same object, evaluated at $-i\mu t_n = \frac{a^n}{n}$. The calculations are elementary, and one finds

$$\langle m^{-n} \rangle_{DMP} = \left(\frac{-i\mu}{a} \right)^n \frac{\Gamma(-i\mu - n + 1)}{\Gamma(-i\mu + 1)}, \quad (3.2)$$

while

$$\langle m^n \rangle_K = \frac{1}{(i\mu a)^n} \frac{\Gamma(i\mu + 1)}{\Gamma(i\mu - n + 1)} \quad (3.3)$$

and, from W_∞ ,

$$-i\mu \langle T_{-n} \rangle_{W_\infty} = \frac{1}{(i\mu a)^n} \frac{\Gamma(i\mu + 1)}{\Gamma(i\mu - n + 1)}. \quad (3.4)$$

Another useful way to compare the DMP model and ours comes from the puncture equation. In the W_∞ solution, the simplest case $n = 1$ leads to the recursion relation

$$\frac{\partial \mathcal{F}}{\partial \bar{t}_1} = t_1 - (k+1)t_{k+1} \frac{\partial \mathcal{F}}{\partial t_k}, \quad (3.5)$$

where

$$Z_{W_\infty}(t, \bar{t}) = e^{\mu^2 \mathcal{F}}. \quad (3.6)$$

Translated into the language of the Kontsevich-type model, this implies that, for infinitesimal ϵ , the partition function should satisfy

$$Z(t_k + \epsilon(k+1)t_{k+1}, \bar{t}_1 + \delta_{k,1}\epsilon) = e^{\mu^2 \epsilon t_1} Z(t, \bar{t}). \quad (3.7)$$

Now, the transformation on Z in the LHS is equivalent to transforming the couplings \bar{t}_k and the constant matrix A as follows:

$$\begin{aligned} \bar{t}_1 &\rightarrow \bar{t}_1 + \epsilon \\ A &\rightarrow A - \epsilon. \end{aligned} \quad (3.8)$$

In the DMP model, this change does not lead to any Ward identity for Z , as one can check, so Eq. (3.1) does not satisfy the puncture equation. However, in our model Eq. (2.15), the variations of the two linear terms in M compensate each other under the transformation Eq. (3.8), leaving only a change from the determinant factor outside, which precisely gives Eq. (3.7).

One more interesting point to observe is that both the DMP model and ours can be rewritten after making the transformation $M \rightarrow M^{-1}$, which is a legitimate change of variables since M is a positive matrix. One finds the alternative forms

$$\begin{aligned} Z_{DMP}(t, \bar{t}) &= (\det A)^{-N+i\mu} \int dM e^{i\mu \operatorname{tr}[M^{-1}A^{-1} + (1 - \frac{2N}{i\mu}) \log M + \sum_{k>0} \bar{t}_k M^k]} \\ Z_K(t, \bar{t}) &= (\det A)^{-i\mu} \int dM e^{i\mu \operatorname{tr} M^{-1}A - (-i\mu + N) \operatorname{tr} \log M + i\mu \sum_{k>0} \bar{t}_k \operatorname{tr} M^{-k}}. \end{aligned} \quad (3.9)$$

In the DMP model, this inversion introduces an explicit N into the potential, which was not there before. However, in our model which already contained an N factor, it reappears in the same form, due quite simply to the change of matrix measure under inversion.

Indeed, our model in the singular limit $\mu \rightarrow 0$ is manifestly symmetric under inversion, as it becomes the matrix analogue of $\int dx/x$.

Finally, we present the results of explicit computations of a class of simple correlation functions for the DMP model, and for our model, to all orders in $\frac{1}{\mu}$. The former are obtained using Schwinger-Dyson equations in the form

$$0 = \int dM \frac{\partial}{\partial M_{ij}} \left(e^{i\mu \text{tr}[MA^{-1} - \log M]} f_{ij}(M, A) \right), \quad (3.10)$$

where $f_{ij}(M, A)$ is an arbitrary matrix-valued function. This process is rather tedious, particularly for the last line of the table, since one has to write down the above equation for a large number of choices of the function f_{ij} and then successively eliminate terms to get the desired result. In contrast, the relevant Schwinger-Dyson equations for our model follow from Eq. (2.9)¹.

Correlator	$c = 1$ Kontsevich model	DMP model
$\langle T_{-1} \rangle$	t_1	t_1
$\langle T_{-2} \rangle$	$2t_2 + t_1^2$	$\frac{\mu^2}{1+\mu^2} (2t_2 + t_1^2)$
$\langle T_{-3} \rangle$	$3t_3 + 6t_1t_2 + t_1^3 + \frac{1}{(i\mu)^2} 3t_3$	$\frac{\mu^4}{(1+\mu^2)(4+\mu^2)} (3t_3 + 6t_1t_2 + 2t_1^3)$
$\langle T_{-4} \rangle$	$4t_4 + 12t_1t_3 + 8t_2^2 + 12t_1^2t_2 + t_1^4$ $+ \frac{1}{(i\mu)^2} (20t_4 + 4t_2^2 + 12t_1t_3)$	$\frac{\mu^6}{(1+\mu^2)(4+\mu^2)(9+\mu^2)} (4t_4 + 12t_1t_3$ $+ 8t_2^2 + 20t_1^2t_2 + 5t_1^4$ $+ \frac{1}{(i\mu)^2} (4t_4 - 12t_2^2 + 12t_1t_3))$

Table I: Explicit computations of amplitudes

It is easy to see from this that the DMP model is inequivalent to our model, and to W_∞ .

¹ In collaboration with V. Kazakov, we have also carried out these computations using the technique of character expansions[12], applied to the version (2.16) of our model and the analogous one for the DMP model, and we obtained the same results. In particular, in this formalism it is manifest that both models are symmetric under exchange of t and \bar{t} . Character expansion turns out to be the most efficient technique for the DMP model.

Let us briefly mention that Chekhov and Makeenko[10] had proposed a model which they conjectured to be in some sense equivalent to the $c = 1$ string, though they did not state precisely what the full equivalence should be. Their matrix potential was

$$V(M, A) = N \left(MA + \nu \log M - \frac{1}{2} M^2 \right) \quad (3.11)$$

with $t_k = \frac{1}{k} \text{tr} A^{-k} - \frac{N}{2} \delta_{k,2}$ and with ν being the cosmological constant. This has some similarities to Eq. (2.15), but the dependence on ν and N is not the correct one.

4. The derivation via semi-infinite forms

In Ref.[8] a ‘‘coherent state’’ representation for the generating functional $Z(t, \bar{t})$ was derived. This representation involves a Fock space associated to bosonic creation and annihilation operators α_{-n} and α_n , satisfying the canonical commutation relations $[\alpha_m, \alpha_n] = m \delta_{m+n,0}$ with $m, n = 1, 2, \dots$. The α_n are conveniently collected into the conformal current $\partial\varphi(z) \equiv \sum_n \alpha_n z^{-n-1}$, which is related to the fermionic fields

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1} \quad \bar{\psi}(z) = \sum_{n \in \mathbb{Z}} \bar{\psi}_{n+\frac{1}{2}} z^{-n-1}, \quad (4.1)$$

by the familiar 2-dimensional bosonization formulas: $\partial\varphi(z) = :\bar{\psi}(z)\psi(z):$. The fermionic oscillators in Eq. (4.1) obey canonical anticommutation relations: $\{\psi_r, \bar{\psi}_s\} = \delta_{r+s,0}$, with $r, s \in \mathbb{Z} + \frac{1}{2}$.

The coherent state formula of Dijkgraaf et al. for the partition function of 2D string theory is

$$Z(t, \bar{t}) = \langle t | S | \bar{t} \rangle, \quad (4.2)$$

where $\langle t |$ and $|\bar{t}\rangle$ are coherent states associated to the positive and negative tachyons:

$$\langle t | \equiv \langle 0 | e^{i\mu \sum_{n=1}^{\infty} \alpha_n t_n} \equiv \langle 0 | U(t) \quad |\bar{t}\rangle \equiv e^{i\mu \sum_{n=1}^{\infty} \alpha_{-n} \bar{t}_n} | 0 \rangle \equiv U(\bar{t}) | 0 \rangle. \quad (4.3)$$

The operator S acts linearly on the fermionic fields:

$$S \psi_{-n-\frac{1}{2}} S^{-1} = R_{p_n} \psi_{-n-\frac{1}{2}} \quad S \bar{\psi}_{-n-\frac{1}{2}} S^{-1} = R_{p_n}^* \bar{\psi}_{-n-\frac{1}{2}}, \quad (4.4)$$

where R_{p_n} are reflection coefficients depending on the fermionic momentum $p_n = n + \frac{1}{2}$ and satisfying the unitarity condition $R_{p_n} R_{-p_n}^* = 1$.

The matrix model formulation of the $c = 1$ string theory leads to explicit expressions for the reflection coefficients[9],[8]:

$$R_{p_n} = (-i\mu)^{-p_n} \frac{\Gamma(\frac{1}{2} - i\mu + p_n)}{\Gamma(\frac{1}{2} - i\mu)}. \quad (4.5)$$

The strategy to derive a Konsevitch model from the coherent states formula (4.2) is to represent the fermionic Fock space in terms of semi-infinite forms. Let us make the same choice as in Ref.[8] for the semi-infinite form representing the fermionic Fock vacuum:

$$|0\rangle = z^0 \wedge z^1 \wedge z^2 \dots \quad (4.6)$$

Now we must take representatives for $\psi_{n+\frac{1}{2}}$ and $\bar{\psi}_{n+\frac{1}{2}}$ which, for $n > 0$, *annihilate* the vacuum. This is necessary since the coherent state formula (4.2) and the action of the S operator (4.4) are defined assuming such a convention, which is also the standard one in conformal field theory. A representation consistent with this convention is

$$\psi_{n+\frac{1}{2}} = z^n, \quad \bar{\psi}_{-n-\frac{1}{2}} = \frac{\partial}{\partial z^n}. \quad (4.7)$$

It follows from Eq. (4.4) that

$$S : z^n \rightarrow R_{-p_n} z^n. \quad (4.8)$$

The error in the derivation of the Konsevitch-Penner model of Ref. [8] is precisely a choice of representatives for the fermionic operators which is inconsistent with that of conformal field theory. With their choice, the action of S was $z^n \rightarrow R_{p_n} z^n$ (Eq. (5.26) of their paper).

In the following we briefly trace back the steps of the derivation of the Konsevitch model for $c = 1$ which starts from the coherent state formula, and show that, once the correct choice (4.7) is made, one recovers our matrix model.

Recalling that $[\alpha_n, \psi_{m+\frac{1}{2}}] = \psi_{m+n+\frac{1}{2}}$, the action of the coherent state operator $U(\bar{t})$ on the fermionic oscillators

$$U(\bar{t}) : \psi_{n+\frac{1}{2}} \rightarrow U(\bar{t})\psi_{n+\frac{1}{2}}U(\bar{t})^{-1} \quad (4.9)$$

reads in the semi-infinite forms representation as follows

$$\begin{aligned} U(\bar{t}) : z^n &\rightarrow e^{i\mu \sum_{k>0} \bar{t}_k \alpha_{-k}} z^n e^{-i\mu \sum_{k>0} \bar{t}_k \alpha_{-k}} \\ &= e^{i\mu \sum_{k>0} \bar{t}_k z^{-k}} z^n = \sum_{k=0}^{\infty} P_k(i\mu \bar{t}) z^{n-k}, \end{aligned} \quad (4.10)$$

where the $P_k(i\mu\bar{t})$ are the Schur polynomials.

Therefore the combined action of S and $U(\bar{t})$ is

$$\begin{aligned} S \circ U(\bar{t}) : z^n &\rightarrow w^{(n)}(z; \bar{t}) = S \sum_{k=0}^{\infty} P_k(i\mu\bar{t}) z^{n-k} S^{-1} \\ &= \sum_{k=0}^{\infty} P_k(i\mu\bar{t}) R_{-p_{n-k}} z^{n-k}. \end{aligned} \quad (4.11)$$

Recalling the expression (4.5) for the reflection coefficients and rewriting the gamma-function in terms of its integral representation one obtains

$$\begin{aligned} w^{(n)}(z; \bar{t}) &= \frac{(-i\mu)^{\frac{1}{2}}}{\Gamma(\frac{1}{2} - i\mu)} \int_0^{\infty} dm e^{-m} m^{-i\mu-1} \sum_{k=0}^{\infty} P_k(i\mu\bar{t}) \left(\frac{-i\mu z}{m} \right)^{n-k} \\ &= c(\mu) z^{-i\mu} \int_0^{\infty} dm m^{-n} e^{i\mu z m} m^{-i\mu-1} e^{i\mu \sum_{k>0} \bar{t}_k m^k}, \end{aligned} \quad (4.12)$$

where

$$c(\mu) \equiv \frac{(-i\mu)^{-i\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2} - i\mu)}. \quad (4.13)$$

From this we finally derive the expression for the state $S|\bar{t}\rangle$ in terms of semi-infinite forms

$$\begin{aligned} S|\bar{t}\rangle &= S \circ U(\bar{t}) z^0 \wedge z^1 \wedge z^2 \wedge \dots \\ &= w^{(0)}(z; \bar{t}) \wedge w^{(1)}(z; \bar{t}) \wedge w^{(2)}(z; \bar{t}) \wedge \dots \end{aligned} \quad (4.14)$$

One also needs to make use of the parametrization (2.3) for the coherent state $\langle t|$. If a_i , with $i = 1, \dots, N$ are the eigenvalues of the Hermitian matrix A in Eq. (2.3), then

$$\langle t| = \langle 0| \prod_{i=1}^N e^{-\sum_{n>0} \frac{\alpha_n}{n} a_i^n} = \langle N| \frac{\prod_{i=1}^N \psi(a_i)}{\Delta(a)}, \quad (4.15)$$

where the state $|N\rangle$ reads as follows in the semi-infinite form representation:

$$|N\rangle = z^N \wedge z^{N+1} \wedge z^{N+2} \dots \quad (4.16)$$

Putting together the bra in Eq. (4.15) with the ket in Eq. (4.14) one gets the formula

expressing $Z(t, \bar{t})$ in terms of determinants:

$$\begin{aligned}
Z(t, \bar{t}) &= \langle t|S|\bar{t} \rangle = \frac{\det w^{(j-1)}(a_i)}{\Delta(a)} \\
&= c(\mu)^N \left(\prod_j a_j \right)^{-i\mu} \\
&\quad \times \int_0^\infty \prod_j \left(\frac{dm_j}{m_j} e^{i\mu m_j a_j - i\mu \log m_j + i\mu \sum_{k>0} \bar{t}_k m_j^k} \right) \frac{\Delta(m^{-1})}{\Delta(a)}.
\end{aligned} \tag{4.17}$$

Converting the Vandermonde depending on m_i^{-1} to the standard one, and using the Harish-Chandra formula, one finds (up to overall factors independent of (t, \bar{t})):

$$\begin{aligned}
Z(t, \bar{t}) &= \left(\prod_j a_j \right)^{-i\mu} \int_0^\infty \prod_j \left(\frac{dm_j}{m_j^N} e^{i\mu m_j a_j - i\mu \log m_j + i\mu \sum_{k>0} \bar{t}_k m_j^k} \right) \frac{\Delta(m)}{\Delta(a)} \\
&= (\det A)^{-i\mu} \int dM e^{i\mu \operatorname{tr} M A - (i\mu + N) \operatorname{tr} \log M + i\mu \sum_{k>0} \bar{t}_k \operatorname{tr} M^k},
\end{aligned} \tag{4.18}$$

which is precisely our model, Eq. (2.15).

5. Relation to Penner and Kontsevich models

Let us set the couplings $t_k = \bar{t}_k = 0$ in Eq. (2.15). Then we are left with a partition function

$$Z(\mu, N) = \int dM e^{i\mu \operatorname{tr} M - (i\mu + N) \operatorname{tr} \log M}. \tag{5.1}$$

Rescaling and shifting M , we find

$$Z(\mu, N) = e^{N(i\mu + N)} \left(1 + \frac{i\mu}{N} \right)^N \int dM e^{(i\mu + N) \sum_{k=2}^\infty \operatorname{tr} \frac{M^k}{k}}, \tag{5.2}$$

which is proportional to the Penner integral

$$Z_{\text{Penner}}(\mu, N) = \int dM e^{-Nt \sum_{k=2}^\infty \operatorname{tr} \frac{M^k}{k}} \tag{5.3}$$

where $t = -(1 + \frac{i\mu}{N})$.

This integral was devised by Penner to count the Euler characters $\chi_{g,n}$ of Riemann surfaces with genus g and n punctures. However, as Distler and Vafa[13] showed, the double scaling limit $N \rightarrow \infty$ and $t \rightarrow t_c = -1$ with $\nu = N(t - t_c)$ fixed, actually counts the

Euler characteristic of *unpunctured* surfaces. Clearly, in Eq. (5.3) above this is just the limit $N \rightarrow \infty$ with μ fixed, and we have $\nu = -i\mu$ as the relation between the cosmological constant of [13] and ours. So the limit that is in the spirit of Kontsevich integrals is quite the same as Distler and Vafa's double-scaling limit, contrary to the claim in Ref.[8].

Let us now examine some other limits of our model. To start with, it is convenient to Euclideanize the cosmological constant via $\nu = -i\mu$. Next, setting $\nu = N$ and $\bar{t}_k = \delta_{k,3}$ one has:

$$Z_K(\nu = N, t, \bar{t} = \delta_{k,3}) = (\det A)^N \int dM e^{-N \operatorname{tr} M A - N \operatorname{tr} M^3}. \quad (5.4)$$

The log term and the expansion parameter ν have both disappeared simultaneously, and we have obtained a matrix integral which is very similar to the original Kontsevich model describing intersection theory on the moduli space of Riemann surfaces. Indeed, the above integral would be the matrix Airy integral but for the fact that integration is performed over positive-definite matrices. However, the asymptotic expansion of this integral, based as it is at the saddle-point $M \sim \sqrt{A}$, does not see this difference. Indeed, Kontsevich shows[1] that the matrix Airy integral gives rise to a sum over 2^N Kontsevich matrix models. In contrast, the integral in Eq. (5.4) above satisfies an inhomogeneous version of the Airy equation because of the boundary of the integration region at 0. It has a unique saddle-point by virtue of the positive-definiteness of M , so that up to the usual factors, it leads to precisely one Kontsevich model.

Thus the original Kontsevich model of two-dimensional pure gravity can be thought of as a special case of our $c = 1$ Kontsevich-type model (but not of the DMP model), after some suitable scalings and normalizations. The same is true for the generalized Kontsevich models, which appear by setting $\bar{t}_k = \delta_{k,p+1}$ for some $p > 2$. Note that in this picture, the choice of a fixed \bar{t}_k and t_k will ultimately correspond to a choice of (p, q) specifying a definite $c < 1$ minimal model coupled to gravity. The symmetry of the (p, q) minimal models in p and q would then be due to the symmetry of the $c = 1$ theory in t_k and \bar{t}_k . We will comment further on the significance of these points below.

Since the Kontsevich and generalized Kontsevich models are special cases of our model, it should follow that the Virasoro and W_n identities satisfied by the former arise from the W_∞ of the latter. This does not imply a completely straightforward connection, however, since the passage to Kontsevich models requires several rescalings and normalization factors. Additionally, the couplings of the Kontsevich model are defined in terms of the matrix A not through Eq.(2.3), but rather through a twisted version of it: $t_k \sim \operatorname{tr} A^{-k-\frac{1}{2}}$, the shift by $\frac{1}{2}$ being responsible for the ‘‘twisted free bosons’’ investigated in Refs.[6],[14]. Similar fractional shifts occur for the generalized Kontsevich models.

6. Quantum Effective Action

We have seen in Section 2 that correlators of negative tachyons T_{-n} of $c = 1$ string at $\bar{t} = 0$ are equal to the averages of $\text{tr} \frac{M^n}{\nu}$ taken with the matrix measure:

$$\begin{aligned} Z_\Gamma(A) &= \int dM e^{-\nu \text{tr} MA + (\nu - N) \text{tr} \log M} \\ &= (\det A)^{-\nu} \int dM e^{-\nu \text{tr} M + (\nu - N) \text{tr} \log M}. \end{aligned} \quad (6.1)$$

The matrix integral above has the obvious property that adding an external source J for M in the classical action leaves the form of the integrand invariant:

$$\begin{aligned} Z_\Gamma(A; J) &= \int dM e^{-\nu \text{tr} MA + (\nu - N) \text{tr} \log M - \text{tr} JM} \\ &= \left(\det \left(A + \frac{J}{\nu} \right) \right)^{-\nu} \int dM e^{-\nu \text{tr} M + (\nu - N) \text{tr} \log M}. \end{aligned} \quad (6.2)$$

Let us define the free energy to be minus the log of this expression, dropping the additive constant coming from the integral. Thus:

$$F_A(J) = \nu \text{tr} \log \left(A + \frac{J}{\nu} \right). \quad (6.3)$$

This allows us to derive explicitly the quantum action associated to this matrix measure. Define the “quantum” field \hat{M} via the equation

$$\hat{M} = \frac{\partial F_A(J)}{\partial J} = \left(A + \frac{J}{\nu} \right)^{-1}. \quad (6.4)$$

The quantum action $\Gamma(\hat{M})$ for \hat{M} is defined through a Legendre transformation of $F_A(J)$

$$\Gamma(\hat{M}) = F_A(J) - \text{tr} \hat{M} J, \quad (6.5)$$

and can be easily evaluated to give

$$\Gamma(\hat{M}) = -\nu N + \nu \text{tr} \hat{M} A - \nu \text{tr} \log \hat{M}. \quad (6.6)$$

The form of the quantum action is identical to that of the “classical” action in Eq. (6.1), the only difference being two simple renormalization effects: the appearance of a constant zero-point energy and the renormalization of the coefficient of the logarithm, which becomes N -independent.

The renormalization of the log term is extremely important. Even if we had set $\nu = N$ in Eq. (6.1) and thereby eliminated the log term in the classical action, it would still be present in the quantum action — thus it is *dynamically generated*. It cannot be tuned away as long as the background has all $\bar{t}_k = 0$.

The quantum action leads to the equation of motion

$$0 = \frac{\partial \Gamma(\hat{M})}{\partial \hat{M}} = A - \hat{M}^{-1}, \quad (6.7)$$

i.e. $\langle \hat{M} \rangle = A^{-1}$. This means that the quantum field \hat{M} has to be shifted around its vacuum expectation value,

$$\hat{M} = A^{-1} + \hat{m}, \quad (6.8)$$

and the quantum action becomes

$$\begin{aligned} \Gamma(\hat{M}) &= \nu \operatorname{tr} \log A + \nu \operatorname{tr} A \hat{m} - \nu \operatorname{tr} \log(1 + A \hat{m}) \\ &= \nu \operatorname{tr} \log A + \nu \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \operatorname{tr}(A \hat{m})^k. \end{aligned} \quad (6.9)$$

This expression, which encodes the full perturbation series for tachyon scattering in $c = 1$ string theory, might be called the Penner quantum action. It corresponds to our Kontsevich-type model shifted around the classical solution appropriate to $c = 1$ string theory.

The 1PI vertices for the field \hat{m} can be read off from Eq. (6.9); for example the 2-point and 3-point vertices are

$$\begin{aligned} \Gamma_{i_1 j_1; i_2 j_2}^{(2)} &= \nu A_{i_2 j_1} A_{i_1 j_2} \\ \Gamma_{i_1 j_1; i_2 j_2; i_3 j_3}^{(3)} &= \frac{1}{2} \nu [A_{i_3 j_1} A_{i_1 j_2} A_{i_2 j_3} + A_{i_2 j_1} A_{i_3 j_2} A_{i_1 j_3}]. \end{aligned} \quad (6.10)$$

Tree diagrams built out of these 1PI n -point vertices $\Gamma_{i_1 j_1; \dots; i_n j_n}$ together with the exact propagator $G_{i_1 j_1; i_2 j_2}^{(2)} = \langle \hat{m}_{i_1 j_1} \hat{m}_{i_2 j_2} \rangle = \frac{1}{\nu} A_{i_1 j_2}^{-1} A_{i_2 j_1}^{-1}$ generate all correlators $\langle M_{i_1 j_1; \dots; i_n j_n} \rangle$ and therefore reproduce all negative-tachyon expectation values. For example, $\nu \langle T_{-2} \rangle$ is given by

$$\langle \operatorname{tr} \hat{M}^2 \rangle = \operatorname{tr} A^{-2} + \langle \operatorname{tr} \hat{m}^2 \rangle = \operatorname{tr} A^{-2} + \frac{1}{\nu} (\operatorname{tr} A^{-1})^2, \quad (6.11)$$

while for $\nu \langle T_{-3} \rangle$ one obtains,

$$\begin{aligned} \langle \operatorname{tr} \hat{M}^3 \rangle &= \operatorname{tr} A^{-3} + \langle \operatorname{tr} A \hat{m}^2 \rangle + \langle \operatorname{tr} \hat{m}^3 \rangle \\ &= \operatorname{tr} A^{-3} + \frac{1}{\nu} \operatorname{tr} A^{-2} \operatorname{tr} A^{-1} + \frac{1}{\nu^2} (\operatorname{tr} A^{-3} + (\operatorname{tr} A^{-1})^3), \end{aligned} \quad (6.12)$$

in agreement with the results shown in the table of Section 3.

To summarise, the polynomials in t given by the negative-tachyon correlators of $c = 1$ string admit a neat diagrammatical interpretation, as a sum over connected and disconnected tree diagrams of the quantum Penner action (6.9).

7. Background Independence

Suppose that one tried to build up the most trivial matrix model possible, with a single Hermitian *positive-definite* matrix M . One might imagine choosing the potential to be zero, and then introducing an external source A :

$$Z(A) = \int dM e^{-\nu \operatorname{tr} MA}. \quad (7.1)$$

From Eq. (6.6) it follows that the quantum effective action, up to additive constants, is

$$\Gamma(\hat{M}) = \nu \operatorname{tr} \hat{M} A - N \operatorname{tr} \log \hat{M}. \quad (7.2)$$

Since a logarithmic term has appeared from renormalization effects, it is natural to add a “bare” log term in the original action. Choosing the coefficient of this term so that the quantum action becomes N -independent, we find:

$$Z(A) = \int dM e^{-\nu \operatorname{tr} MA + (\nu - N) \operatorname{tr} \log M}. \quad (7.3)$$

This is precisely our Kontsevich-type model! Viewed as a string field theory, the quantum equation of motion tells us that $\hat{M} = A^{-1}$, and expanding around this gives rise to the quantum Penner action that we have already discussed. Therefore the $c = 1$ string in this framework is nothing but a positive-definite matrix with zero potential, coupled to an external source.

What about other non-critical string backgrounds? Let us add the term $\operatorname{tr} M^{k+3}$, for some fixed $k \geq 0$, to the above potential. This corresponds to turning on a source for the tachyon T_{-k-3} in $c = 1$ language. One can no longer explicitly compute the quantum action. However, the matrix integral now has a saddle-point which is very far from $M = A^{-1}$. Indeed, tuning away the log term, the saddle-point is at $M \sim A^{\frac{1}{k+2}}$. Expanding around this saddle-point leads to the generalized Kontsevich model of level k [1][2], which describes the $(k + 2, q)$ minimal-model string backgrounds. In this sense,

all the $c < 1$ string backgrounds can be thought of as the different vacua to which our Kontsevich-type model flows when there is “condensation of negative tachyons”.

This picture needs to be studied in more detail, in particular to understand what is the mechanism by which the log term gets tuned away. Note that switching on T_{-1} does not shift the saddle-point since it can be absorbed in the source A , while T_{-2} leads to a background in which the free energy is quadratic in A and hence trivial.

8. Conclusions

We have solved the W_∞ constraints of $c = 1$ string theory via a Kontsevich-type matrix model. The resulting model is beautiful and natural, and we believe it should tell us something fundamental about string theory. In particular, this could lead to a framework to formulate background-independence in string field theory.

On the way, we obtained an elegant matrix version of the W_∞ constraints, Eq. (2.9), which has the form of a generalized heat-kernel equation, Eq. (2.11). One may be tempted to speculate that this is related to the holomorphic anomaly equation of Ref.[15] which, according to Ref.[16] expresses quantum background-independence in certain solvable topological string theories.

Our results also shed new light on the sense in which $c = 1$ string theory is like a $k \rightarrow -3$ limit of the k -minimal topological models coupled to gravity[17].

It should be emphasized that our model was constructed starting from the tachyon S -matrix and, apparently, does not contain the other kinds of states that one might expect to see in two-dimensional string theory, including the discrete “tensor” states and the states of the “wrong dressing”. However, our matrix-model in principle contains many more operators than the ones we have considered, in particular traces of negative powers, $\text{tr}M^{-k}$ and also more complicated objects such as $\text{tr}(M^{k_1}A^{k_2}M^{k_3}\dots)$. It remains to be seen whether these provide the missing states of $c = 1$.

Because of the resemblance of this model to those studied recently by Kazakov et al.[12], we expect that the powerful technique of character expansions can be used to gain more understanding of our model and its possible generalizations.

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