

# Gravitational Couplings and $Z_2$ Orientifolds

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The interplay between gravitational couplings on branes and the occurrence of fractional flux in low dimensional orientifolds is examined. It is argued that gravitational couplings need to be assigned not only to D-branes but also to orientifold planes. The fractional charges of the orientifold  $d$ -planes can be understood in terms of flux quantization of the  $d - 3$  form potential and modified Bianchi identities. Detailed results are presented for the case of the type IIB orientifold on  $T^6/Z_2$ , which is dual to F-theory on a complex 4-fold with terminal singularities.

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## 1. Introduction

Recent developments have demonstrated the importance of a variety of extended dynamical objects, branes[1], in string theory. One context where (Dirichlet) branes arise naturally is in the construction of orientifolds[2,3,4], where one can sometimes think of them as twisted sector states. Another kind of object that occurs in orientifold constructions is called the orientifold plane, the locus of fixed points of some discrete group.

Planes are usually assumed to be non-dynamical, as indeed they are at weak coupling. But it has been shown in a few contexts[5,6] that at strong coupling they can behave like dynamical objects. Another context where the distinction between branes and planes is blurred is in F-theory compactifications at constant self-dual coupling[7,8,9]. Here, F-theory branes move around in groups and can even produce exceptional symmetries, yet these configurations are continuously connected within the constant-coupling moduli space to the perturbative configurations of branes and planes.

An essential distinction between branes and planes is that the former carry Yang-Mills gauge fields on their world-volume, and have moduli for their locations, while the latter do not. Here we want to focus on a complementary feature in which some amount of symmetry is maintained between the two types of objects. Besides Yang-Mills couplings, branes also carry gravitational couplings localized on their world-volume. As we will see, orientifold planes also carry such localized gravitational couplings, essentially because they are loci of singular curvature. This fact neatly fits in with the various observations referred to above regarding the strong-coupling behaviour of planes.

An interplay between gauge and gravitational couplings is a key feature in maintaining consistency and anomaly-freedom in string compactifications. Hence exploring the apparently distinct origins of the two in theories with branes and planes may tell us something quite fundamental about the underlying theory.

Another, apparently unrelated, issue in orientifolds is that while branes always carry integer charge with respect to some  $p$ -form gauge field, planes can carry fractional charge in a very precise way. We will see that the presence of gravitational couplings on planes and branes together conspires to explain this fact and render these fractional charges consistent with Dirac quantization.

The appearance of fractional charges on planes can be argued for many low dimensional compactifications of M-Theory and string theory[10]. The most straightforward way to see this is by examining the orientifold of the type II string on  $T^n/Z_2$  where  $n$  is even

for IIB and odd for IIA. There are always 16 D-branes in the vacuum and  $2^n$  orientifold planes. Symmetry and charge conservation dictate that each plane carries  $2^{4-n}$  units of charge. This is fractional as soon as  $n$  is greater than 4.

Analogous situations occur in M-theory. In the  $T^5/Z_2$  orientifold, the fact that twisted-sector states are outnumbered by the fixed planes was noted and studied in Ref.[11]. A careful analysis of this situation in Refs.[12,13] revealed that as in stringy orientifolds, the planes carry fractional charge, and explained how this is in fact consistent.

The origin of fractional charge in this case is as follows. The  $T^5/Z_2$  compactification has 32 fixed points. Anomaly cancellation requires 16 copies of  $N = 2$  tensor multiplets in six dimensions. Assuming each tensor multiplet comes from a space-filling 5-brane, this would give rise to two interesting phenomena: (i) the  $C^{(3)} \wedge I_8$  term from the  $D = 11$  supergravity ( $C^{(3)}$  being the three form of M-theory) cancels anomalies locally on the brane by anomaly inflow, and (ii) a magnetic charge +1 appears on each of the 5-branes. Therefore charge cancellation requires the planes to carry  $-\frac{1}{2}$   $C^{(3)}$ -field magnetic charge while the anomalies automatically cancel locally, on both branes and planes, by the inflow term in the lagrangian.

Some more interesting cases arise in low-dimensional compactifications of string theory and M-theory. For example, consider the type II theories on  $T^8/Z_2$  orbifolds and orientifolds. Type IIB theory on the  $T^8/Z_2$  orientifold has 256 orientifold planes and 16 D1-branes. Charge cancellation would require orientifold planes to carry  $-\frac{1}{16}$  units of  $\tilde{B}_{\mu\nu}$  charge.

Type IIA on  $T^8/Z_2$  orbifold has 256 twisted sectors, but they do not contribute any massless multiplets. The massless states in the twisted sector can only arise in the RR sector since the left or right moving fermions in the NS sector give vacuum energy greater than zero. But the RR ground state in this case does not survive GSO projection[14]. However, due to the existence of a  $B_{\mu\nu}$  tadpole in two dimensions[15] a consistent compactification requires  $\chi/24$  one-branes (fundamental type IIA strings) to condense in the vacuum, where  $\chi$  is the Euler characteristic of the compact manifold. For  $T^8/Z_2$  the orbifold Euler characteristic is 384, and thus tadpole cancellation requires 16 type IIA strings in the vacuum. Then charge cancellation would require the fixed points to carry  $-\frac{1}{16}$  units of  $B_{\mu\nu}$  charge.

This lifts to M-theory and F-theory on the same orbifold. In the M-theory case the planes have  $-\frac{1}{16}$  units of 3-form charge while in F-theory there are really only  $2^6$  rather

than  $2^8$  fixed points (we count only fixed points on the base) and these carry  $-\frac{1}{4}$  units of 4-form charge. Indeed, this is in the class of  $T^n/Z_2$  orientifolds referred to above, with  $n = 6$ . Note that the orbifold  $T^8/Z_2$  has terminal singularities, and hence requires irrelevant rather than marginal operators in order to be blown up. String and M propagation on it, however, appear to be smooth.

There is also a dual pair with chiral supersymmetry in  $1 + 1$  dimensions. Type IIB on the  $T^8/Z_2$  orbifold is chiral and has potential gravitational anomalies. The 256 twisted sectors carry a total gravitational anomaly of  $\frac{64}{3}$  (in the units of Pontryagin numbers) and this is cancelled by the total anomaly of 256 chiral bosons from the fixed points, which is  $-\frac{64}{3}$ [16]. The same anomaly cancellation occurs for M-Theory on  $T^9/Z_2$  (which is conjectured to be dual to IIB on  $T^8/Z_2$ ) with some crucial differences. The fixed points, 512 in number, now give chiral fermions and their anomalies cancel the anomalies coming from the untwisted sectors.

Another low-dimensional case is type IIA on the  $T^9/Z_2$  orientifold. This has the usual 16 D0-branes in the vacuum and 512 orientifold points. Charge cancellation would now require the orientifold points to carry  $-\frac{1}{32}$  units of  $A_\mu$  charge. We will make some new observations about these low-dimensional cases later on.

There is an interesting relationship between Bianchi identities and flux quantization that we will exploit in this paper. The Bianchi identity of a  $p$ -form potential (in type IIA or IIB strings or in M-theory) in  $d$  uncompactified dimensions is related to charge quantization of the field in  $d - 4$  uncompactified dimensions. As an example, the Bianchi identity of the  $C^{(3)}$  field of M theory on  $S^1/Z_2$  is related to the flux quantization of  $G^{(4)} = dC^{(3)}$  for M theory on  $T^5/Z_2$ [13]. After compactification on a circle this descends to an analogous relation for type IIA on  $S^1/Z_2$  and  $T^5/Z_2$ . Other examples include the flux quantization of  $G^{(5)} = dC^{(4)}$  for type IIB compactified on  $T^6/Z_2$  orientifold ( $C^{(4)}$  is the self-dual RR four-form potential of type IIB). This is related to the Bianchi identity of type IIB on the  $T^2/Z_2$  orientifold. Similarly flux quantization for IIA on  $T^7/Z_2$ , IIB on  $T^8/Z_2$  and IIA on  $T^9/Z_2$  is related to Bianchi identities on  $T^3/Z_2$ ,  $T^4/Z_2$  and  $T^5/Z_2$  respectively.

This paper is organized as follows. In Section 2 we argue that orientifold planes carry gravitational WZ terms, and discuss how Bianchi identities get modified in the presence of planes and branes. In Section 3 we show that the modified Bianchi identities indeed lead to consistent behaviour of branes when transported around planes even if the latter carry fractional charge, so that Dirac quantization is always satisfied. In Section 4 we discuss Wess-Zumino terms of  $R^4$  type that are supported on planes as well as branes. In Section

5 we focus on the special case of the  $T^6/Z_2$  orientifold, where 3-branes condense in the vacuum. Gravitational couplings in 3+1 dimensional supersymmetric gauge theories have received some attention recently[17,18,19]. We interpret our results in the context of N=4 compactifications. Finally in Section 6 we comment on low-dimensional cases.

## 2. Orientifolds and Fractional Charges

Orientifolds are constructed by gauging the world sheet parity transformation along with some target space discrete symmetry of type II string theory. This gauging gives non-vanishing disc tadpoles. That means orientifold planes are charged with respect to the field whose disc tadpole is non-vanishing. These charges can be cancelled by inserting an appropriate number of space-filling D-branes. In the simplest case, the  $Z_2$  orientifolds, we need 16 D-branes to cancel the charge carried by orientifold planes.

These orientifolds can also be understood via T-duality. Consider type IIB string theory in ten dimensions. Orientifolding this theory in ten dimensions gives the type I string. All other  $Z_2$  orientifolds in lower dimensions can then be understood by T-dualizing type I string theory after toroidal compactification. In ten dimensions, the orientifold 9-plane carries charge  $-16$  with respect to the 10-form potential. This is cancelled by condensing 16 D-9 branes in the vacuum, which gives the well known  $SO(32)$  gauge group of type I string theory. The orientifold 9-plane splits into two orientifold 8-planes after compactification and T-duality on a circle. These orientifold 8-planes carry charge  $-8$  each with respect to 9-form potential. Thus again we need 16 D-8 branes to cancel the charge on the orientifold planes.

A special vacuum is the one for which the D-8 branes cancel the orientifold charge locally. In this case the 16 D-8 branes are placed on the two orientifold planes in bunches of 8 each. As we compactify further and T-dualize along the compact directions, the number of orientifold planes keep doubling with each action of T-duality, whereas the total charge carried by all the orientifold planes remains equal to  $-16$  which is equally distributed among them. Thus for compactifications on higher-dimensional tori, orientifold planes carry fractional charge, and special vacua with local charge cancellation do not exist unless the compactification tori are squashed to merge orientifold planes. (It is intriguing that this occurs just at the value of uncompactified dimension (6) below which the m(atric) theory proposal[20] starts to become problematic[21].)

The first instance where fractional charges on the orientifold planes occur is the  $T^5/Z_2$  orientifold compactification of type IIA string theory. Using the relation of M-theory and type IIA string theory, the same conclusion can be reached for the  $T^5/Z_2$  orientifold of M-theory. In both cases, the orientifold planes carry half-integral magnetic charge with respect to the three form field  $C^{(3)}$ . This phenomenon was explained by Witten[13], who showed that the fluxes of the four form field strength  $G^{(4)} = dC^{(3)}$  are quantized in half-integral units. Whenever the  $G$  flux through a four-cycle  $M$  of an eleven-dimensional manifold  $Y$  is half integral, the first Pontryagin class  $p_1(Y)$  of  $Y$  restricted to  $M$  is an integer divisible by two but not by four. The flux which is integrally quantized belongs to  $G^{(4)} - (p_1(Y)/4)$ .

Since orientifolds produce vacuum charges which are canceled by branes, the space-time action for the orientifolds can be written as

$$I_{orient} = I_{bulk} + \sum_{i=1}^{16} I_{DBI}^{(i)}, \quad (2.1)$$

where the first term on the right hand side is the bulk space-time action of type IIA(IIB) string theory, subject to the orientifold projection, and the second term is the sum over Dirac-Born-Infeld actions for the 16 space-filling D-branes coupled to type IIA(IIB) potentials.

The first term, the bulk string theory action, is written in ten dimensions, whereas the space-filling D-brane actions fill the space transverse to  $T^n/Z_2$ . The sum is taken over 16 points on  $T^n/Z_2$  where these D-branes are localised, hence in fact the second term on the RHS of (2.1) is accompanied by  $n$  dimensional  $\delta$ -functions specifying locations of D-branes on  $T^n/Z_2$ . We will always consider curved non-compact space, i.e., both orientifold planes and the D-brane world volumes are curved.

As mentioned above, in curved space the D-brane world-volume has additional couplings. These are Wess-Zumino terms which are wedge products of the  $p$ -form field with powers of the curvature two-form  $R$ . In particular, on the worldvolume of a D-5-brane in curved space, there is an additional Wess-Zumino term coupling the RR two-form  $B$  to the first Pontryagin class  $p_1(R)$ [22]. One can see this by a simple anomaly inflow argument[23].

Now consider the orientifold of type IIB string theory in ten dimensions. In this case we have an orientifold 9-plane accompanied by 16 D-9-branes, leading to type I string

theory. The modified Bianchi identity for the three-form field strength due to the anomaly cancellation condition in the type I string is

$$dH = \frac{1}{2} [p_1(R) - p_1(F)]. \quad (2.2)$$

where  $p_1(R)$  is the first Pontryagin class of the spin manifold  $Y$  and is defined as  $\frac{1}{8\pi^2} \text{tr}(R \wedge R)$  and similarly  $p_1(F) = \frac{1}{8\pi^2} \text{tr}(F \wedge F)$ . For a spin manifold,  $p_1$  is divisible by 2, hence  $\lambda \equiv \frac{p_1}{2}$  defines an integer cohomology class. Let us try to understand this equation from the orientifold point of view. The D9-brane worldvolume action is given by

$$S_{D9} = S_{DBI} + S_{WZ}, \quad (2.3)$$

where  $S_{BI}$  is the usual Dirac-Born-Infeld action and  $S_{WZ}$  is the contribution of the Wess-Zumino terms. We will not write all these terms explicitly. The relevant Wess-Zumino terms[23] are<sup>1</sup>

$$\int *B \wedge \frac{1}{16\pi^2} \text{tr}(F \wedge F) \quad \text{and} \quad \frac{1}{24} \int *B \wedge \frac{1}{16\pi^2} \text{tr}(R \wedge R). \quad (2.4)$$

where  $*B$  is the Poincare dual of the RR two form  $B$ .

We also have the space-time action of type IIB string theory. The term relevant for our purpose is

$$S_{IIB} \sim \frac{1}{2} \int H^2 + \dots, \quad (2.5)$$

where,  $H$  is the field strength of the RR sector two-form field  $B$ . The equation of motion for  $*B$  would then be given by

$$dH = \frac{1}{16\pi^2} (-\text{tr}F \wedge F + \frac{2}{3} \text{tr}R \wedge R) \quad (2.6)$$

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<sup>1</sup> By the anomaly inflow argument, the WZ term that actually occurs on the brane world volume is proportional to

$$\int_{B_p} C \wedge \text{tr}_n \exp\left(\frac{F}{2\pi}\right) \sqrt{\hat{A}(R)}$$

where  $\hat{A}(R) = 1 - \frac{p_1}{24} + \frac{7p_1^2 - 4p_2}{5760} + \dots$  and  $p_i$  are the Pontryagin numbers. For our case it suffices to take  $C = *B$ .

where the RHS is the contribution coming only from 16 D-9-branes. Since orientifold planes are not dynamical they do not couple to  $F$  but since we are considering both D-branes as well as orientifold planes in curved space, planes can couple to  $R$ . We claim that orientifold planes contribute a further

$$\frac{1}{3} \frac{\text{tr}(R \wedge R)}{16\pi^2} = \frac{p_1}{6} \quad (2.7)$$

to the RHS of the above equation.

One way to see this is the following. In case of D-branes both the terms in (2.4) occur at the disc order with three insertions. For the orientifold plane also these terms should contribute at the same order, except that now the disc is replaced by  $RP^2$ . The first term in Eq.(2.4) requires two open-string insertions, whereas the second term has all three closed-string insertions. Since  $RP^2$  has no boundary and open string vertices are inserted on boundaries,  $RP^2$  does not contribute to the first term, which is equivalent to the statement that the orientifold planes have no open-string dynamics or they do not couple to Yang-Mills fields. Closed-string vertex insertions are in the bulk of the worldsheet and hence are allowed on  $RP^2$ . Thus the three point vertex on  $RP^2$ , i.e., the orientifold plane, contributes a term proportional to the second term in Eq.(2.4). Both the disc and  $RP^2$  are tree-level diagrams, and at tree level only D-branes and orientifold planes can contribute these terms. Since we already know the modified Bianchi identity as well as the contribution of D-branes, the term Eq.(2.7) has to come from the orientifold plane. We will see that this interpretation makes sense when we consider other cases, particularly the 8-dimensional example.

Though the number of orientifold planes multiplies on compactification followed by T-duality, the total contribution of orientifold planes towards the appropriate Bianchi identity remains the same. In other words, if  $C^{(n)}$  is an RR  $n$ -form, then the  $\frac{1}{3}C^{(n)} \wedge \frac{\text{tr}(R \wedge R)}{16\pi^2}$  term residing on the orientifold planes is equally distributed among all the orientifold planes and the total contribution of orientifold planes to the Bianchi identity for  $C^{(n)}$  is equal to Eq.(2.7). In the case of D-5-branes, Ref.[22] could not fix the sign of the Wess-Zumino term containing the Pontryagin class. Relating this term to the Bianchi identity, it is easy to see that there is a relative minus sign between the two terms occurring in (2.4). Incorporating this contribution of orientifold planes gives us the correct Bianchi identity

$$dH = \frac{1}{16\pi^2} (-\text{tr}F \wedge F + \text{tr}R \wedge R). \quad (2.8)$$

Another related way of seeing this is the following. In ten dimensions, the 3-form field strength in the type-I string has a kinetic term

$$S_I = \frac{1}{2} \int H \wedge *H \quad (2.9)$$

where

$$H = dB + \omega_{3L} - \omega_{3Y}. \quad (2.10)$$

Here,  $d\omega_{3L} \equiv \frac{1}{16\pi^2} \text{tr}R \wedge R$  and similarly for  $d\omega_{3Y}$  with R replaced by F. This leads to a cross term

$$\int *dB \wedge (\omega_{3L} - \omega_{3Y}) \quad (2.11)$$

Now let  $B^{(6)}$  be the dual of  $B$  defined by  $*dB = dB^{(6)}$ . Then the above coupling becomes the WZ term

$$\frac{1}{16\pi^2} \int B^{(6)} \wedge (\text{tr}(F \wedge F) - \text{tr}(R \wedge R)) \quad (2.12)$$

where  $B^{(6)}$  is a (dual) RR potential in type I.

From the coefficient of the term above, it is evident that the curvature terms have the right coefficient to arise from 24 9-branes. But in reality there are only 16 9-branes, which contribute 2/3 of the desired factor, and the remaining 1/3 is ascribed to the planes as in Eq.(2.7).

Consider now compactification of type I theory on a circle followed by T-duality. This gives us type I' theory which can also be obtained from type IIA theory on an  $S^1/Z_2$  orientifold. As mentioned earlier, T-duality doubles the number of orientifold planes. In this case we get two orientifold planes and (2.7) is distributed equally between them. The vacuum with local charge cancellation is the one where eight D8 branes are located on one orientifold plane and eight on the other. Let us focus only on one of the orientifold planes. The total orientifold action (modulo projection) in the vicinity of one orientifold plane is

$$S_{orient} = S_{IIA} + S_{D8}. \quad (2.13)$$

The relevant terms in this action are:

$$S_{orient} \sim \frac{1}{2} \int G^2 + \int *C^{(5)} \wedge \frac{\text{tr}(F \wedge F)}{16\pi^2} \delta(x^9) - 8 \int \frac{1}{24} *C^{(5)} \wedge \frac{\text{tr}(R \wedge R)}{16\pi^2} \delta(x^9), \quad (2.14)$$

where  $x^9$  is the compact circle coordinate, the orientifold plane that we are concentrating on is localised at  $x^9 = 0$ ,  $*C^{(5)}$  is the 5-form dual to the RR three-form potential  $C^{(3)}$  in

type IIA string theory. The field strength  $G^{(4)} = dC^{(3)} + \dots$  where extra terms have to be introduced to “solve” the Bianchi identity as we discuss below. The gauge field strength  $F$  takes values in the group  $SO(16)$  whereas the curvature ( $R$ ) terms all add up, so that the WZ term of a single D8-brane is multiplied by 8. Lastly the  $\delta$  function tells us that the branes are orthogonal to  $x^9$ .

The equation of motion for the field  $*C^{(5)}$  is given by

$$dG^{(4)} = \frac{1}{16\pi^2} \left( \frac{1}{3} \text{tr} R \wedge R - \text{tr} F \wedge F \right) \delta(x^9) \quad (2.15)$$

where the RHS is a contribution coming entirely from the branes. We have argued that the  $R \wedge R$  contribution from the orientifold planes is equally distributed among the planes. In the present case, a single orientifold plane contributes  $\frac{1}{6} \frac{\text{tr}(R \wedge R)}{16\pi^2} \delta(x^9) = \frac{p_1}{12} \delta(x^9)$ . Adding this contribution to the Bianchi identity of  $G$  we get

$$dG^{(4)} = \frac{1}{16\pi^2} \left( \frac{1}{2} \text{tr} R \wedge R - \text{tr} F \wedge F \right) \delta(x^9). \quad (2.16)$$

This is the analogue, for type IIA string theory on  $S^1/Z_2$ , of an equation derived in the second paper of Ref.[24] in the context of the strong coupling limit of this theory, namely M-theory on  $S^1/Z_2$ . That equation was used in Ref.[13] to show that  $G^{(4)}$  can have half integral fluxes in M-theory.

Despite the similarity in the final equation, there is an important difference between the derivation of our result and that in Ref.[24]. In the latter case, there are really no branes, just fixed planes, since there are no moduli to break  $E_8 \times E_8$ . For string theory orientifolds below 10 dimensions, perturbatively there are always branes and planes, and they can be separated from each other. Hence it is essential to understand the contribution of each one separately, to obtain the correct Bianchi identity in the special charge-cancelling configuration as we have just done. An essential role was played here by the gravitational coupling on the fixed planes.

Now we turn to the  $T^2$  compactification of type I string theory. This is equivalent, by T-duality, to the  $T^2/Z_2$  orientifold of type IIB strings. This model has been studied in great detail by Sen[5]. Here we have four orientifold planes which carry seven-brane charge  $-4$ . This charge can be canceled by putting four seven branes on the top of each orientifold plane. From what we have said earlier every orientifold plane in this case will contribute a factor  $\frac{1}{12} \frac{\text{tr}(R \wedge R)}{16\pi^2} \delta(x^8) \delta(x^9) = \frac{p_1}{24} \delta(x^8) \delta(x^9)$ . This is consistent with results of [5]: when seven branes are taken away from the orientifold plane, the plane splits into two seven

branes which are mutually non-local and also non-local with respect to the original seven branes. The curvature term that we expect from the orientifold plane is exactly twice as much as that contributed by a single D-brane. We therefore see that each orientifold plane in this case can split into two 7-branes which share the curvature terms. Once we take this into account the Bianchi identity becomes

$$dG^{(5)} = \frac{1}{16\pi^2} \left( \frac{1}{4} \text{tr} R \wedge R - \text{tr} F \wedge F \right) \delta(x^8) \delta(x^9), \quad (2.17)$$

where  $G^{(5)} = dC^{(4)} + \dots$  and  $C^{(4)}$  is the self-dual RR four-form in type IIB string theory. Although the 24 7-branes of F-theory that emerge from the nonperturbative analysis carry different  $(p, q)$  charges, hence are related to each other by  $SL(2, Z)$  S-duality, they all must carry the same worldvolume term contributing to the above Bianchi identity since  $C^{(4)}$  is  $SL(2, Z)$ -invariant. This is a nice confirmation that in this situation, planes really can turn into branes.

Compactifying further on K3, one finds a 4-dimensional theory that is dual to F-theory on  $K3 \times K3$ . The 24 7-branes wrapped over K3 become 24 anti-3-branes, which give rise to  $-24$  units of tadpole in the 4-form potential because of the WZ term. This is cancelled by condensing 24 fundamental 3-branes in the 4d vacuum, as predicted in [15]. So we apparently have 24 anti-3-branes and 24 3-branes, though the anti-branes (which are “embedded” in the 7-branes) arise from the WZ coupling and do not break extra supersymmetry.

As is clear from the above discussion, this trend continues as we compactify type I string theory down to lower dimensions. The Bianchi identities for seven, six and five dimensional compactifications are

$$\begin{aligned} d^*G^{(6)} &= \frac{1}{16\pi^2} \left( \frac{1}{8} \text{tr} R \wedge R - \text{tr} F \wedge F \right) \delta(x^7) \delta(x^8) \delta(x^9) \\ d^*G^{(7)} &= \frac{1}{16\pi^2} \left( \frac{1}{16} \text{tr} R \wedge R - \text{tr} F \wedge F \right) \delta(x^6) \delta(x^7) \delta(x^8) \delta(x^9) \\ d^*G^{(8)} &= \frac{1}{16\pi^2} \left( \frac{1}{32} \text{tr} R \wedge R - \text{tr} F \wedge F \right) \delta(x^5) \delta(x^6) \delta(x^7) \delta(x^8) \delta(x^9). \end{aligned} \quad (2.18)$$

We will have more to say about these Bianchi identities and their relation to fractional charges in the next section.

The 7-dimensional case also shows some interesting features. In this case there are 8 orientifold 6-planes, along with 16 6-branes. The planes carry  $-2$  units of magnetic charge with respect to the RR 1-form. The WZ term involving the RR 3-form  $C^{(3)}$  is shared

equally between the branes and planes in this case. This suggests that there may be a situation in which the orientifold planes can behave as (single) branes, and indeed there is. Compactify the 7-dimensional theory on  $K3$  to 3 spacetime dimensions. This is dual to M-theory on  $K3 \times K3$ , for which again it is known[15] that the vacuum contains 24 condensed 2-branes. These are precisely there to cancel the 2-branes sitting “inside” the 16 6-branes and the 8 6-planes.

It is intriguing that apparently static objects like orientifold planes actually contain so much dynamics.

### 3. M-Theory on $S^1/Z_2$ and type IIB on $T^2/Z_2$

In the previous section we saw how the Bianchi identity is modified in the neighbourhood of an orientifold plane. It follows from these modified Bianchi identities that the charges associated with RR  $p$ -form potentials can be fractional. Orientifold planes are charged with respect to these  $p$ -form RR potentials, and the charge fractionalization due to modified Bianchi identities is closely related to the fractional charges on the orientifold planes. In this section we will show how these fractional charges are consistent with the Dirac quantization condition in string theory and M-theory.

To illustrate this we will first consider M-theory compactified on  $S^1/Z_2$ [24]. The  $Z_2$  action here is an orientifold action which takes  $C \rightarrow -C$  and leaves other fields invariant. This theory has to satisfy ten dimensional anomaly cancellation conditions at the orientifold points of  $S^1/Z_2$ . The anomaly can be canceled by putting 8 space filling nine branes at each end of the world. This gives rise to the  $E_8 \times E_8$  heterotic string theory with each end of the world contributing one  $E_8$  gauge symmetry. In the M-theory picture, these branes have no moduli and therefore they are stuck at the two ends – indeed, they are more like static planes than dynamical branes, although one of our conclusions has been that there is not so much of a distinction between the two objects.

We have already used the well-known result that the three form field strength  $H$  in the heterotic string theory satisfies the modified Bianchi identity

$$dH = \frac{1}{16\pi^2} (\text{tr}R \wedge R - \text{tr}F \wedge F) \tag{3.1}$$

where  $F$  is the gauge field strength taking values in  $E_8 \times E_8$  gauge group. This equation takes quite a different form in M-theory. If we are close to one of the orientifold points on the circle, only one of the two  $E_8$  gauge symmetries is visible. At the same time only

half of the Pontryagin class of the curvature contributes. Thus from this point of view, the Bianchi identity is

$$dH = \frac{1}{16\pi^2} \left( \frac{1}{2} \text{tr} R \wedge R - \text{tr} F_1 \wedge F_1 \right), \quad (3.2)$$

where the subscript 1 stands for one of the  $E_8$  groups. When we lift this equation to M-theory, the three-form field strength  $H$  goes over to the four-form field strength  $G^{(4)}$  to give

$$dG^{(4)} = \frac{1}{16\pi^2} \left( \frac{1}{2} \text{tr} R \wedge R - \text{tr} F_1 \wedge F_1 \right) \delta(x^{10}), \quad (3.3)$$

where  $x^{10} = 0$  is the location of the orientifold plane. Witten[13] observed that since  $R \wedge R$  is quantized in integers, the magnetic charge of the four-form field strength  $G^{(4)}$  is quantized in half-integers.

The existence of such half-integral charge and its consistency with Dirac quantization can be established by studying the world-volume theory of a membrane[13]. To do this let us first wrap the world-volume of the membrane on a closed three cycle  $T$  of the eleven (or ten) dimensional manifold. What we want to find out is what happens to the membrane path integral when we take it around a circle. The WZ coupling of the membrane world volume theory to  $C^{(3)}$  is given by

$$\exp(i \int_T C^{(3)}) \quad (3.4)$$

which when transported along the circle gives

$$\exp(i \int_{T \times S^1} G^{(4)}). \quad (3.5)$$

There is another factor which contributes to the phase of the membrane path integral. This is related to the parity anomaly in the path integral over world-volume fermions, from which it follows that the interaction  $\int C^{(3)}$  on the brane world-volume is modified as

$$C^{(3)} \rightarrow C^{(3)} + \frac{1}{2} \text{tr}(\omega_{3,L} + \omega_{3,N}) \quad (3.6)$$

where  $\omega_{3,L}$  and  $\omega_{3,N}$  are the Chern-Simons 3-forms associated respectively to the tangent and normal bundles to the brane world-volume.

Thus the extra phase on transporting the membrane world-volume over a circle is proportional to the first Pontryagin class of the normal bundle to  $T \times S^1$  (since the tangent bundle is trivial):

$$\exp(2\pi i [\frac{1}{2} \int \frac{\text{tr} F \wedge F}{16\pi^2}]) = \exp(i\pi \frac{p_1(N)}{2}) \quad (3.7)$$

where  $d\omega_{3,N} = \frac{1}{16\pi^2} \text{tr}F \wedge F = \frac{p_1(N)}{2}$ .

Thus the total phase, which must be equal to 1, is given by[13]

$$(-1)^{\int_{T \times S^1} \frac{p_1(N)}{2}} \exp(i \int_{T \times S^1} G^{(4)}). \quad (3.8)$$

It follows that  $G^{(4)}/2\pi$  has a half-integral period precisely on those cycles on which the integral of  $p_1(N)/2$  is odd. Therefore, what is really observable is not the periods of  $G^{(4)}/2\pi$  which are half-integral but  $G^{(4)}/2\pi - p_1(N)/4$  which is always integral. In general, what appears in this condition is the full Pontryagin class  $p_1$  of the tangent bundle of the ambient spacetime. Thus we see that the Dirac quantization condition is obeyed by  $G^{(4)}/2\pi - p_1/4$  charges.

As mentioned in the previous section, orientifold planes in the  $T^5/Z_2$  compactification of M-theory or of type IIA string theory have half-integral magnetic charge with respect to  $G^{(4)}$ . If we can find a four cycle in  $T^5/Z_2$  where  $\frac{\text{tr}(R \wedge R)}{16\pi^2} = \frac{p_1}{2}$  integrates to an odd integer, this would give us half integral charge. (In this case it is the Pontryagin class of the tangent, rather than normal, bundle to the 4-manifold that contributes.) The four cycle which encloses an orientifold fixed point in  $T^5/Z_2$  has this desired property. Though this cycle,  $S^4/Z_2$ , is non-orientable, its Stiefel-Whitney class  $w_4$ , which is equivalent to  $R \wedge R \text{ mod } 2$  in the orientable case, is unity[13].

Consider now the  $T^6/Z_2$  compactification of type IIB string theory. The modified Bianchi identity (2.17) in type IIB compactified on  $T^2/Z_2$  orientifold tells us that self-dual five-form charges are quantized as  $n - 1/4$ , where  $n$  is an integer. The solution of the Bianchi identity Eq.(2.17) relevant for this case can be written

$$G^{(5)} = dC^{(4)} + \left( \frac{1}{4} \omega_{3,L} - \omega_{3,Y} \right) \delta(x^8) \delta(x^9) \quad (3.9)$$

or, alternatively, as

$$G^{(5)} = dC^{(4)} + \frac{1}{16\pi^2} \left( \frac{1}{4} \text{tr}R \wedge R - \text{tr}F \wedge F \right) \times \frac{1}{2} (\epsilon(x^8) \delta(x^9) - \epsilon(x^9) \delta(x^8)) \quad (3.10)$$

where  $\epsilon(x)$  is the step function.

In contrast to the case of the  $S^1/Z_2$  Bianchi, in this case neither of the solutions is free of a  $\delta$ -function. This is related to the fact that in the present case, spacetime does not just acquire an end-of-the-world boundary but rather ends on submanifolds of (real) codimension 2.

The two solutions above differ by the addition to  $G^{(5)}$  of the exact form

$$d\left(\left(\frac{1}{4}\omega_{3,L} - \omega_{3,Y}\right) \times \frac{1}{2}(\epsilon(x^8)\delta(x^9) - \epsilon(x^9)\delta(x^8))\right) \quad (3.11)$$

The  $T^6/Z_2$  orientifold compactification is the place where we expect that the charge carried by the orientifold planes is  $-1/4$ . To measure this charge, we consider a five cycle,  $S^5/Z_2$  which encloses the orientifold fixed point and integrate the self-dual five-form  $dC^{(4)}$  over this five cycle. As is clear from Eq.(2.17), in the absence of gauge fields there is another contribution coming from the term  $\frac{1}{4}\frac{\text{tr}(R\wedge R)}{16\pi^2}\delta(x^5)$ . Since we are only interested in fractional charges we can safely ignore the gauge field contribution which is always an integer. The charge of the orientifold plane, which is  $-1/4$ , is obtained by integrating the five-form  $dC^{(4)}$  over  $S^5/Z_2$ . What remains to be done, by analogy with similar manipulations in Ref.[13], is to integrate  $\frac{1}{16\pi^2}\text{tr}(R\wedge R)\delta(x^5)/4$  over  $S^5/Z_2$ .

At this point we give a more general argument which explains the existence of fractional charge when we integrate the quantity

$$T \equiv \frac{1}{2^m} \frac{\text{tr}(R\wedge R)}{16\pi^2} \delta(x_i)\delta(x_j)\dots\delta(x_p). \quad (3.12)$$

over the manifold  $S^{4+q}/Z_2$ . Here  $m, i, j, \dots, p$  are the relevant integers and  $q$  is the number of delta functions. For the  $T^6/Z_2$  case  $m = 2, i = 5$  and  $q = 1$ .

$S^{4+q}$  is defined by

$$S^{4+q} : x_1^2 + x_2^2 + \dots + x_{4+q}^2 + x_{5+q}^2 = 1. \quad (3.13)$$

And  $S^{4+q}/Z_2$  is defined modding out with the antipodal map  $x_i \rightarrow -x_i$  for all  $i$ . Now the integration is simple to perform. The  $\delta(x_i), \delta(x_j), \dots$  factors fix us at the locus  $x_i = 0, x_j = 0, \dots$ . This locus is a section of  $S^{4+q}/Z_2$  which is nothing but  $S^4/Z_2$  because there are  $q$  delta functions. It is well known that  $S^4/Z_2$  can be naturally embedded in  $S^{4+q}/Z_2$ . This embedding corresponds to setting  $q$  coordinates of  $S^{4+q}/Z_2$  to zero. Integrating out the delta function precisely implements this action<sup>2</sup>. Therefore now we only have to integrate  $\frac{1}{2^m} \frac{\text{tr}(R\wedge R)}{16\pi^2}$  over  $S^4/Z_2$ .

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<sup>2</sup> The simplest geometrical way to think of it is that the boundary of a ball in 3 dimensions is  $S^2$ , and its equator is  $S^1$ . A  $Z_2$  modding will make the equator  $S^1/Z_2$ . So if there is a quantity to be integrated over  $S^2/Z_2$  with a delta-function along say the  $z$  direction, it will reduce to an integral over  $S^1/Z_2$ . This also seems like  $S^2/Z_2$  being represented as a ‘‘fibration’’ over  $S^1/Z_2$  with a fibre  $S^1$ . Since the  $Z_2$  action has done nothing to the fibre the integral of delta function will be just 1.

The quantity  $\lambda \equiv \frac{1}{16\pi^2} \text{tr}(R \wedge R)$  is congruent modulo two to the Stiefel-Whitney class  $w_4$ . By a standard computation[25] one can show that

$$\int_{S^4/Z_2} w_4 = 1 \pmod{2}. \quad (3.14)$$

Together with the  $\frac{1}{2^m}$  factor, this would point to the existence of fractional fluxes for the corresponding fields. For the  $T^6/Z_2$  example considered we see that the period of the five-form  $G^{(5)}$  is fractional precisely on those five-cycles on which the integral of  $\frac{1}{4} \frac{\text{tr}(R \wedge R)}{16\pi^2} \delta(x^5)$  contributes the compensating fraction, so that the total charge is effectively integer and the Dirac quantization condition is then satisfied by the charges of the field  $G^{(5)} - \frac{1}{4} \frac{\text{tr}(R \wedge R)}{16\pi^2} \delta(x)$ .<sup>3</sup> The crucial thing that has entered into the discussion is that the object which we want to integrate has support only a  $S^4/Z_2$  submanifold of  $S^{4+q}/Z_2$ .

It is interesting to consider whether other physical effects are related to the occurrence of the fractional fluxes that we have been discussing. For example, fractional fluxes similar to those discussed above would appear in other orientifolds based on higher discrete groups than  $Z_2$  and correspondingly with lower supersymmetry. However, there do seem to be some important distinctions between the situation discussed in the context of  $S^1/Z_2$  orientifolds[24] and the more general cases considered here. In the former case, one can

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<sup>3</sup> In case of  $S^5/Z_2$ , there is another way to show how  $G^{(5)} - \frac{1}{4} \frac{\text{tr}(R \wedge R)}{16\pi^2} \delta(x)$  satisfies the Dirac quantisation condition. To see this we use the fact that  $S^5$  is a generalized Hopf fibration over  $CP^2$  with  $S^1$  fibre. The antipodal map which takes  $S^5$  to  $S^5/Z_2$  acts trivially on  $CP^2$  but it halves the volume of the fibre. Therefore,  $S^5/Z_2$  is also an  $S^1$  fibration over  $CP^2$ . The difference between these two bundles is that the Chern class of the latter is double of that of the former. To evaluate  $\frac{1}{4} \frac{\text{tr}(R \wedge R)}{16\pi^2} \delta(x^5)$ , one first integrates along the fibre to reduce the top form on  $S^5/Z_2$  to the top form on the base, i.e.,  $CP^2$ . Integration along the fibre can be done if the cohomology classes have compact support in the vertical direction[26]. Since the Chern class is doubled, integration along the fibre gives

$$\frac{1}{4} \frac{\text{tr}(R \wedge R)}{16\pi^2} \Big|_{CP^2} \int_{S^1} \delta(x^5) dx^5 = \frac{1}{2} \frac{\text{tr}(R \wedge R)}{16\pi^2} \Big|_{CP^2}.$$

The RHS of the equation can be written in terms of the Pontryagin class of  $CP^2$  as  $p_1(R)/4$ . Thus the integration along the fibre gives the top form on  $CP^2$  which is one quarter of its first Pontryagin class. Since the integral of  $p_1(R)$  over  $CP^2$  is equal to 3, integrating it on  $CP^2$  one obtains the contribution of the curvature terms, which is equal to 3/4.

write the modified Bianchi identity at an orientifold plane as a completely nonsingular boundary condition, while in the latter cases it needs to be expressed in a singular form. Presumably related to this is the fact that the former case has wider applicability: besides orientifold planes, fractional flux for a 4-form field strength can occur in M-theory compactifications on any complex 4-fold with a 4-cycle whose first Pontryagin class is a multiple of 2 but not 4. It remains to be seen whether charges of the type  $\frac{1}{2^n}$  for field strengths  $G^{(n+3)}$  can be realized in smooth compactifications, but from the present considerations this does not seem likely. In the same vein,  $p$ -branes for  $p \geq 3$  do not have anomalies with the right discrete ambiguity  $Z_{2^{p-1}}$  to play a role similar to the parity anomaly on the 2-brane. It remains to understand what exactly happens to the parity anomaly on the 2-brane when it is T-dualized to a higher brane. Comments in this direction appear in Refs.[27][28][29] but a careful analysis remains to be carried out.

In this section we have seen how the fractional charges on the orientifold planes, which could potentially be inconsistent with the Dirac quantization condition, actually conspire to give consistent results in the case of  $T^n/Z_2$  orientifolds with  $n \geq 5$ . In subsequent sections we examine other aspects of gravitational couplings and some details of low-dimensional orientifolds.

#### 4. $R^4$ Wess-Zumino terms on high-dimensional branes and planes

We have seen that  $R^2$  couplings of Wess-Zumino type appear on orientifold planes as well as branes. Here we will show that the same is true for  $R^4$  couplings, though for obvious reasons these can only appear on  $p$ -branes and  $p$ -planes for  $p \geq 7$ .

In 10 dimensions, the type I string has a term  $B \wedge X_8$  which plays an essential role in the Green-Schwarz anomaly cancellation mechanism. Here  $X_8$  is the 8-form

$$\begin{aligned} X_8 &= \frac{1}{(2\pi)^4} \left( \frac{1}{48} \text{tr} F^4 - \frac{1}{192} \text{tr} F^2 \text{tr} R^2 + \frac{1}{384} \text{tr} R^4 + \frac{1}{1536} (\text{tr} R^2)^2 \right), \\ &= \frac{1}{(2\pi)^4} \left( \frac{1}{48} \text{tr} F^4 - \frac{1}{192} \text{tr} F^2 \text{tr} R^2 \right) + \tilde{X}_8, \end{aligned} \quad (4.1)$$

where,

$$\tilde{X}_8 = \frac{1}{128} (p_1)^2 - \frac{1}{96} p_2. \quad (4.2)$$

The first Pontryagin class  $p_1$  was defined earlier, and the second Pontryagin class is given by

$$p_2 = \frac{1}{(2\pi)^4} \left( -\frac{1}{4} \text{tr} R^4 + \frac{1}{8} (\text{tr} R^2)^2 \right) \quad (4.3)$$

Let us first look at the terms which contain gauge fields. It is easy to see that these terms can be obtained by expanding the Wess-Zumino terms on the D-branes. In case of multiple coincident branes all we need to do is to define the trace in the fundamental representation of the appropriate gauge group generated by coincident branes. In the case at hand, we have 16 coincident D9-branes in the presence of an O9-plane, which leads to SO(32) gauge symmetry. The terms involving gauge fields obtained from expanding the Wess-Zumino term on the world volume correctly reproduce the  $F^4$  and  $F^2 R^2$  terms in  $X_8$ . Hence, as one would expect, there is no need to assign any gauge couplings to the orientifold plane.

Now we will address the analogous issue for the  $R^4$  terms. This time it will prove necessary to assign specific couplings to the O9-plane, as was the case for  $R^2$  terms. As before, we decompose this term into the contribution from the bulk, the branes and the orientifold plane, all of which are of course coincident in 10 dimensions. No term of the above form is present in type IIB in 10 dimensions. The contribution on a single 9-brane, which we denote  $B \wedge B_8$ , is extracted from the anomaly inflow formula, which leads to

$$B_8 = \frac{1}{320} \left( \frac{1}{8} (p_1)^2 - \frac{1}{9} p_2 \right) \quad (4.4)$$

This can be conveniently recast in terms of  $\tilde{X}_8$  and  $\text{tr} R^4$ , and we find

$$B_8 = \frac{1}{16} \left( \frac{4}{3} \tilde{X}_8 - \frac{1}{480} \frac{\text{tr} R^4}{(2\pi)^4} \right) \quad (4.5)$$

Since we require that the contribution from 16 branes plus that from the plane must provide the total  $R^4$  term, we have  $16B_8 + P_8 = \tilde{X}_8$ , where  $B \wedge P_8$  is the plane contribution.  $P_8$  is found to be<sup>4</sup>

$$P_8 = -\frac{1}{3} \tilde{X}_8 + \frac{1}{480} \frac{\text{tr} R^4}{(2\pi)^4}. \quad (4.6)$$

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<sup>4</sup> Note that the following equation differs from the expression that appeared in the original version and the published form of this paper. This is due to an error of a factor of 2 in the normalisation of Eq.(4.1) above. We were motivated to check and revise these expressions due to the appearance of some recent preprints[30,31] (see also [32]), in which our proposal for  $R^4$  couplings on orientifold planes was checked by explicit computation of string amplitudes. The computations confirmed that such couplings exist on O-planes as we had predicted, but showed an error in our precise coefficients. The following equation, Eq.(4.6), now agrees with the conclusions of these papers.

This can be re-expressed in terms of Pontryagin classes as

$$P_8 = \frac{1}{640}(p_1)^2 - \frac{7}{1440}p_2 \quad (4.7)$$

One might think that Eqs.(4.5) and (4.6) together contradict the fact that a 7-plane can split into a pair of 7-branes. However, the  $R^4$  terms on 7-branes and 7-planes are of the form  $\tilde{\phi} \wedge B_8$  and  $\tilde{\phi} \wedge P_8$  respectively, where  $\tilde{\phi}$  is the RR scalar. When 7-planes split into  $(p, q)$  7-branes, since  $\tilde{\phi}$  is not  $SL(2, \mathbb{Z})$  invariant one cannot say what the  $(p, q)$  branes should carry. In this respect the situation is similar to that for the 8-form charge carried by 7-planes and 7-branes, which according to Ref.[5] does not split additively because of the non-Abelian nature of the monodromy. This is in contrast to the  $R^2$  term, where the RR potential  $C^{(4)}$  that appears is  $SL(2, \mathbb{Z})$  invariant, and the term splits additively.

## 5. Gravitational couplings in 3 + 1 dimensional gauge theory

It has been observed that certain supersymmetric gauge theories in 3 + 1 dimensions have partition functions which are modular under  $SL(2, \mathbb{Z})$  with nontrivial weight. The resolution to this apparent failure of exact  $SL(2, \mathbb{Z})$  invariance, or modular anomaly, is that these theories have specific couplings to gravity which produce (cancelling) modular anomalies.

Here we will realize the relevant gauge theories on world-volumes of 3-branes, and will investigate the relationship between the gravitational couplings required for consistency of gauge theories and those generated by branes and planes on their world-volumes.

Consider  $N = 4$  super-Yang-Mills in 3+1d. This is the world-volume field theory of a 3-brane. It can be considered to be topologically twisted (namely, the physical and twisted theories are equivalent), when written on flat spacetime or (after Euclideanization) on hyper-Kähler 4-manifolds. As we have seen, the gravitational Wess-Zumino couplings on the 3-brane world-volume are known to be

$$\frac{1}{16\pi^2} \frac{1}{24} \text{tr} \left( C^{(0)} R \wedge R \right) \quad (5.1)$$

where  $C^{(0)}$  is the RR scalar of type IIB. However, this coupling is not  $SL(2, \mathbb{Z})$  invariant or even covariant.

To discover the correct extension of the above coupling, we need to realize a known vacuum of string theory in terms of condensed branes. The appropriate vacuum in this

case is the orientifold of type IIB on  $T^6/Z_2$ , which has already made an appearance above. This vacuum has  $N = 4$  spacetime supersymmetry, and its gauge sector is an  $N = 4$  super-Yang-Mills theory. In this way of describing the vacuum, there are 16 3-branes along with 64 orientifold planes. As far as world-volume gravitational couplings are concerned, we have argued that the planes carry 1/2 the fraction carried by the branes, so that to find the contribution on a single brane world-volume, we need to divide relevant terms in the spacetime action by 24.

In Ref.[17] it is observed that the spacetime  $R^2$  coupling in 4d  $N = 4$  compactifications is, at tree level<sup>5</sup>, proportional to

$$\text{tr} \left( C^{(0)} R \wedge R + e^{-\phi} R \wedge *R \right) \quad (5.2)$$

This is argued in Ref.[17] to be corrected by the replacement

$$\begin{aligned} C^{(0)} &\rightarrow \text{Re} \left( \frac{\log \eta(\tau)^{24}}{2\pi i} \right) \\ e^{-\phi} &\rightarrow \text{Im} \left( \frac{\log \eta(\tau)^{24}}{2\pi i} \right) \end{aligned} \quad (5.3)$$

where  $\tau \equiv C^{(0)} + ie^{-\phi}$ . In the limit of constant dilaton and axion, the action gets a contribution depending only on the topological invariants  $\chi$  (the Euler characteristic) and  $\sigma$  (the signature) of the spacetime 4-manifold:

$$-(\chi - \frac{3}{2}\sigma) \log \eta^{12} - (\chi + \frac{3}{2}\sigma) \log \bar{\eta}^{12} \quad (5.4)$$

It follows that each brane carries  $\frac{1}{24}$  of this term. To leading order and considering only the term involving  $\sigma$ , this precisely coincides with Eq.(5.1), given that  $\sigma = p_1/3$  where  $p_1$  is the correctly normalized first Pontryagin class. The full gravitational coupling on the brane, to second order in derivatives, is thus

$$-\frac{1}{4}(2\chi - 3\sigma) \log \eta - \frac{1}{4}(2\chi + 3\sigma) \log \bar{\eta} \quad (5.5)$$

One way to check that this is correct is to note that on hyper-Kähler manifolds, the modular anomaly from this must cancel that coming from the gauge partition function, for which we have the result[33]:

$$Z_{gauge} \left( -\frac{1}{\tau} \right) = \tau^{\frac{\chi}{2}} Z_{gauge}(\tau) \quad (5.6)$$

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<sup>5</sup> Our conventions differ slightly from those in Ref.[17], since we want to make the anomaly term purely holomorphic rather than anti-holomorphic.

To check cancellation of the modular anomaly, we examine the  $SL(2, Z)$  transformation law for Eqn.(5.5) after setting  $\sigma = -\frac{2}{3}\chi$  which is the case for hyper-Kähler manifolds. Thus we need to know how the term in the functional integral

$$Z_{grav} = \exp(-\chi \log \eta) = \eta^{-\chi} \quad (5.7)$$

transforms. Using  $\eta(-\frac{1}{\tau}) = \tau^{\frac{1}{2}}\eta(\tau)$  we find that the gravitational contribution to the modular anomaly is

$$Z_{gauge}(-\frac{1}{\tau}) = \tau^{-\frac{\chi}{2}} Z_{gauge}(\tau) \quad (5.8)$$

which exactly cancels the gauge contribution.

## 6. Some Issues Concerning Fractional Charge in Dimensions $d < 3$

Although not specifically related to gravitational couplings on branes and planes, there is a curious situation in which gravitational anomalies turn into fractional charges on planes upon compactification. We discuss this below and explain how chiral supersymmetry in this problem cures an apparent paradox.

Consider the orientifold of type IIA on  $T^9/Z_2$ . By T-duality, this vacuum, where all of space is compactified, is realized with 16 0-branes located at points in the internal torus (of course in such low dimensions the concept of moduli space is not strictly appropriate). The  $2^9 = 512$  orientifold points each carry a charge  $-\frac{1}{32}$  with respect to the RR 1-form. This vacuum may be considered a limit of M-theory on  $T^9/Z_2$  to 2 spacetime dimensions, further compactified on a circle, as the circle shrinks. However, in the M-theory case[11,14,16] one expects 512 chiral fermions to appear in the twisted sector, located symmetrically at the 512 fixed points.

Thus it would appear that on compactification of the M-theory orientifold on a further circle, 512 chiral fermions in 1 + 1 noncompact dimensions must suddenly turn into 16 D0-branes, while the gravitational anomaly carried by each of the  $2^9$  fixed planes (which are really fixed lines) turns into  $-\frac{1}{32}$  units of 1-form charge. The fermions in 1 + 1 dimensions were forced to sit at the fixed points to bring about local gravitational-anomaly cancellation, so it is hard to understand how they go over into 16 objects in 0 + 1 dimensions which apparently cannot bring about local 1-form charge cancellation.

The resolution to this lies in the supersymmetry of this problem. In 1 + 1 dimensions, the above orientifold of M-theory has (0, 16) *chiral* supersymmetry. The algebra is:

$$\{Q_-^i, Q_-^j\} = \delta_{ij} P_- \quad (6.1)$$

The chiral fermions which appear as twisted sectors are singlets of supersymmetry, which means they have + chirality in these conventions, and hence are annihilated by both sides of the algebra by virtue of the Dirac equation  $P_- \psi_+ = 0$ .

On compactification to  $0 + 1$  dimensions, we end up with a supersymmetric quantum mechanics that is also chiral, in the sense that now the supercharges  $Q^i$  satisfy

$$\{Q^i, Q^j\} = \delta_{ij}(P - Z) \tag{6.2}$$

where  $Z$  is a central charge. D 0-branes are BPS, which means they are annihilated by the RHS of this algebra. Thus in fact the 0-branes propagating in the fully compactified space are singlets of the residual supersymmetry, rather than multiplets with 32 states (16 bosonic, 16 fermionic) as they are in higher dimensions. As a result, in type IIA on  $T^9/Z_2$  the twisted sector of the orientifold can be thought of as being made up of 512 supersymmetry singlets, and local gravitational anomaly cancellation goes directly over into local charge cancellation.

## 7. Discussion

We have seen that curved orientifold planes carry gravitational couplings of WZ type. Quite plausibly they also carry other types of gravitational couplings such as  $R^4$  terms (not of Wess-Zumino type) or their dimensional reductions. This would be interesting to investigate, along with the possible appearance of similar terms on curved D-branes.

It has also been argued here that fractional fluxes can consistently be carried by orientifold planes. The general conclusion from this discussion would be that such phenomena as Dirac quantization can be modified by suitable (topological) gravitational couplings, as first noted in Ref.[13].

While all this adds some insight into the fascinating interplay between gauge and gravitational interactions in string and M-theory, a deeper understanding of this interplay would be desirable. Also, it would be interesting to understand gravitational couplings in  $N = 2$  supersymmetric gauge theories[19] from the point of view of 3-branes in suitable backgrounds.

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