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TOPOLOGICAL LANDAU-GINZBURG MODEL OF TWO-DIMENSIONAL STRING THEORY

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ABSTRACT

We study a topological Landau-Ginzburg model with superpotential $W(X) = X^{-1}$. This is argued to be equivalent to $c = 1$ string theory compactified at the self-dual radius. We compute the tree-level correlation function of N tachyons in this theory and show their agreement with matrix-model results. We also discuss the nature of contact terms, the perturbed superpotential and the flow of operators in the small phase space. The role of gravitational descendants in this theory is examined, and the tachyon two-point function in genus 1 is obtained using a conjectured modification of the gravitational recursion relations.

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1. Introduction

Non-critical string theory in the background of matter with central charge $c \leq 1$ has been studied intensively over the past few years. The case $c = 1$ is the most interesting as there is a propagating degree of freedom, the massless 'tachyon', in addition to the discrete states. This theory also has a natural physical interpretation of a critical string moving in a two dimensional target space. For a comprehensive review, and references to original papers, see Ref.[1].

Discretization of the world-sheet via random matrix models first demonstrated the exact solvability of these models. This was subsequently better understood in terms of a topological field theory description. It was shown[2] that perturbations of pure topological gravity can reproduce an infinite subclass of $c < 1$ non-critical string models. The remaining models in the $c < 1$ series are obtained by coupling specific topological matter systems to topological gravity[3]. The relevant matter theories have a discrete series of topological central charges $\hat{c} = k/(k+2)$ where k is a positive integer..

The problem of finding a topological description of the $c = 1$ string remained unsolved for a considerable amount of time. In particular, naive attempts to continue the topological series for $c < 1$ by taking the limit of large k do not seem to work. However, more recently it was understood[4] that the correct topological field theory is the twisted Kazama-Suzuki[5] coset model $\mathrm{SL}(2)_k/\mathrm{U}(1)$ with $k = 3$. Another way to view this coset theory is to think of it as an $\mathrm{SU}(2)_k/\mathrm{U}(1)$ coset, where the level k is continued to the value -3. In this theory the topological central charge is $\hat{c} = 3$, the 'critical' value for topological field theories[2].

This model has been argued to reproduce $c = 1$ string theory where the $c = 1$ scalar field is compactified at the self-dual radius. It is shown in Ref.[4] that the entire spectrum of physical states in the latter theory (at zero cosmological constant) can be reproduced through a double cohomology in the coset model[1]. More important, a Lagrangian description of the same coset model[6] (which was previously used for the $\mathrm{SU}(2)_k/\mathrm{U}(1)$ case with positive k), gives explicit results which can be continued to $k = -3$. This remarkably leads to the evaluation of the genus-g partition function and certain tree-level 4-point functions, which agree with the corresponding results in the $c = 1$ matrix model (at self-dual radius) with *nonzero* cosmological constant, after a natural identification of chiral primaries with tachyons. Although this formulation in principle determines all correlators in every genus,

[1] In this picture, the cosmological operator is a screening-charge-like perturbation of the CFT, just as in the Liouville formalism, while one works in the Hilbert space of the unperturbed theory.

it is not so easy to extract explicit expressions for correlators other than those mentioned above.

This motivates us to turn to the Landau-Ginzburg (LG) description of topological matter, which has proved to be most convenient for explicit calculation in the context of minimal matter backgrounds[7][8]. In this description the LG superpotential determines the properties of the theory. The SU(2)$_k$/U(1) coset theory for positive integer level k is described by one chiral superfield X with the superpotential $W(X) = \frac{1}{k+2}X^{k+2}$. The physical operators defined by the BRS cohomology of the topological string correspond to the chiral primaries, while the topological algebra given by the factorization (of the zero-form operators) is derived from the isomorphism to the chiral ring $\mathbf{C}[X]/W'(X)$. In Ref.[7], tree-level correlators of the topological field theory were calculated and shown to agree with the results obtained from matrix models. Moreover, a direct correspondence between the LG superpotential and the KdV differential operator[9] of matrix models was demonstrated.

Since the topological coset SU(2)$_k$/U(1) and the Landau-Ginzburg models with superpotential $W(X) = \frac{1}{k+2}X^{k+2}$ give equivalent descriptions of $c < 1$ matter coupled to gravity, it is reasonable to guess that the correspondence holds even when the level k is continued to -3. Thus we are led to consider the LG theory with superpotential $W(X) = -X^{-1}$, as a candidate for a topological description of compactified $c = 1$ string theory. (It has already been observed[10] that the tachyon 4-point functions of $c = 1$ string theory are obtainable from the LG approach).

This theory requires extra work to define it precisely, as the superpotential is not polynomial and many of the properties described above, including the existence of a nilpotent chiral primary ring, do not strictly hold. The difficulties involved here are analogous to those encountered in defining an SL(2)$_k$ conformal field theory, where (unlike the case of SU(2)$_k$) it is not possible to rigorously determine the integrable representations and other basic properties just by analysing the Lagrangian. Nevertheless, consistency requirements are a powerful constraint in the study of Landau-Ginzburg theories, and we will see below that these suffice to find reasonable spectra of physical operators and formulae for correlation functions. We will show that the correlators of $c = 1$ string theory follow in a very simple way from our considerations, and moreover the complementary pictures of Refs.[7] and [11] both have natural analogues in the present model.

In the present work we will examine this theory in some detail and extract several of its properties which not only confirm its identification with the $c = 1$ string (directly at nonzero cosmological constant), but also reveal some interesting and unexpected features. We will then identify the gravitational descendants, and make a hypothesis for the gravitational recursion relations in this theory. With this hypothesis we show that one can explicitly obtain correlators on the torus which agree with matrix-model results. We also study the flow of the model in coupling-constant space.

2. Landau-Ginzburg Tachyons and Selection Rules.

Correlators of local operators in conventional LG theory (with superpotential $\frac{1}{k+2}X^{k+2}$) are obtained from the knowledge of the three-point function (structure constant) of three primaries on the sphere, c_{ijl}, and the two-point function (metric) η_{ij}. To compute correlators involving integrals of two-form operators, one computes the perturbed ring corresponding to the perturbed superpotential. The perturbations are by the physical operators, the chiral primaries ϕ_i. Let t_i, $0 \leq i \leq k$, denote the coupling constants corresponding to the scaling fields $\phi_i(t)$ of the perturbed ring. The central object to calculate is the perturbed structure constant $c_{ijl}(t)$—differentiating this an appropriate number of times and setting $t_i = 0$ for all i, gives all the correlators of the original theory. In this picture, the superpotential W and the primaries evolve as a function of t, that is they *flow* in the small phase space, whose coordinates are the couplings t_i.

After coupling this theory to gravity there arise an infinite number of physical operators, the so called gravitational descendants of the matter primaries. Correlators involving descendants can, however, be reduced to those of the primaries alone, by the use of the recursion relations[2][3]. Recently, Losev has shown that the LG description is adequate for the topological theory even after its coupling to gravity[11] (see also [12]). The m-th descendant of the primary X^i, $0 \leq i \leq k$, in this description, can be written as $\sigma_m(X^i) \sim X^{i+m(k+2)}$. These are matter secondaries which do not decouple in topological gravity. In Ref.[13], a recursive prescription is given to calculate the tree-level N-point correlator by reducing it in terms of $(N-1)$-point correlators. (It is important that the N-th insertion is a primary.)

With this background, let us turn now to the study of the LG model with superpotential $W(X) = -X^{-1}$. The superpotential must have U(1) charge 1, which implies that the field X has U(1) charge -1 and the various powers X^i have charge $-i$. We start by assuming that the physical fields are all (positive and negative) powers of X. It is not yet clear whether these should all be treated as gravitational primaries, and the distinction between primaries and secondaries in this set will become clear below.

Let us now examine the selection rules based on U(1) charge. It is known for LG theories of topological central charge $\hat{c}$ coupled to gravity, that the correlators satisfy anomalous U(1) charge conservation laws[3][7][6]. Suppose we consider the genus-g N-point function of gravitational primaries, each carrying U(1) charge q_i, then we have the selection rule

$$\sum_{i=1}^{N}(q_i - 1) = (g-1)(3-\hat{c}) \tag{1}$$

Note that this selection rule is also true for gravitational secondaries, if we assign an integer effective U(1) charge m to the mth gravitational secondary σ_m[3]. This fact will be useful in what follows.

Precisely at $\hat{c} = 3$, which is the case of interest here, we find the universal conservation law $\sum_{i=1}^{N}(q_i - 1)$, valid in every genus. Now in the $c = 1$ string there is a unique conserved quantity in every genus, and that is tachyon momentum. Thus we are led to identify LG fields of U(1) charge $k+1$ to discrete tachyons of momentum $-k$ (we choose the last sign for later convenience, as clearly a choice of convention is required here. This is due to the $\mathbf{Z}_2$ symmetry of the $c = 1$ string. In principle there could be an arbitrary factor as well, but that must equal 1 since both the tachyons of the self-dual compactified $c = 1$ string and the monomials of the LG theory are labelled by integers). Thus we are led to the correspondence

$$T_k = X^{k-1} \tag{2}$$

Later we will return to questions of overall normalization.

The zero-momentum tachyon $T_0 = X^{-1}$ becomes the cosmological operator in this identification. Note that this coincides with the superpotential of the theory, which is one reason to anticipate that the theory we are constructing is already at unit cosmological constant. The cosmological operator satisfies U(1) charge conservation for any number of insertions and in any genus, as it should. Note also that tachyons of positive and negative momenta should be thought of as having opposite target-space chirality.

The three-point function determines the structure constants of this theory. We can simply insert our identification of Landau-Ginzburg tachyons into the standard formula, which gives

$$\begin{aligned} c_{k_1k_2k_3} &= \langle T_{k_1}T_{k_2}T_{k_3}\rangle \\ &= \mathrm{Res}\left(\frac{X^{k_1-1}X^{k_2-1}X^{k_3-1}}{X^{-2}}\right) \\ &= \delta_{k_1+k_2+k_3,0} \end{aligned} \tag{3}$$

Here, 'Res' means the residue obtained by contour-integrating the argument in X around the circle at infinity in the complex X-plane.

From the structure constants we obtain the metric:

$$\eta_{k_1 k_2} = c_{k_1 k_2 1} = \delta_{k_1+k_2+1,0} \tag{4}$$

Note that for the metric, we must take the third index of the structure constant to correspond to the puncture operator, which actually carries a momentum $k = 1$ (see Eq.(2)).

We see that it was really necessary to take both positive and negative powers of the LG superfield X, since they are dual to each other. In a certain sense, this suggests that all the monomials in X and X^{-1} should be thought of as gravitational primaries. However, this interpretation cannot be taken literally as it will lead to incorrect results in computing N-point functions, as we will see in the next section. It will turn out that in correlation functions, only the positive-momentum tachyons behave as primaries, while the negative-momentum tachyons behave as gravitational secondaries of the cosmological operator.

3. Contact Terms and Four-point Function

The four-point function is the first non-trivial correlator, as it involves an integration over the moduli space $\overline{\mathcal{M}}_{0,4}$. There are two standard ways to compute this in theories of topological matter coupled to topological gravity. One is to evaluate the structure constants in the presence of a perturbation[7]:

$$c_{k_1 k_2 k_3}(t_4) = \mathrm{Res}\left(\frac{T_{k_1}(t_4) T_{k_2}(t_4) T_{k_3}(t_4)}{(W + t_4 T_{k_4})'}\right) \tag{5}$$

where on the right hand side one uses not only the perturbed superpotential, but also the perturbed forms of the fields, to first order in t_4. Differentiating this structure constant in t_4 gives the genus-0 4-point function $\langle T_{k_1} T_{k_2} T_{k_3} T_{k_4}\rangle_W$. We will perform this calculation below, but first we turn to the other computational procedure.

In the method of Ref.[11], the four-point function is obtained by differentiating the structure constants, which are however calculated leaving the fields unperturbed. Then one adds in contact terms arising from the collision of the fourth field with the other three. The result is

$$\begin{aligned}\langle T_{k_1} T_{k_2} T_{k_3} T_{k_4}\rangle_W =& \frac{\partial}{\partial t_4} \langle T_{k_1} T_{k_2} T_{k_3}\rangle_{W + t_4 T_{k_4}}\Big|_{t_4=0} + \langle C_W(T_{k_4}, T_{k_1}) T_{k_2} T_{k_3}\rangle_W \\ &+ \langle T_{k_1} C_W(T_{k_4}, T_{k_2}) T_{k_3}\rangle_W + \langle T_{k_1} T_{k_2} C_W(T_{k_4}, T_{k_3})\rangle_W ,\end{aligned} \tag{6}$$

where, $C_W(T_{k_i}, T_{k_j})$ is the contact term between the fields T_{k_i} and T_{k_j}. The distinction between contributions from the bulk and boundaries of moduli space is explicit in (6). The contact term is the contribution from the boundary, where the positions of two fields collide.

The contact term for the LG theory in question is determined by self-consistency. For LG theories with $k > 0$, it was shown in [11] that the contact term between two fields ϕ_i, ϕ_j is given by $C_W(\phi_i, \phi_j) = \frac{d}{dX}\left(\frac{\phi_i(X)\phi_j(X)}{W'(X)}\right)_+$, where the subscript denotes the prescription of keeping only the positive powers of X in the resulting expression. This is the unique choice leading to a symmetric expression for the four-point function. It is clear that this contact term gets contributions only if the OPE of the two fields which collide gives rise to a gravitational secondary.

It turns out that the same contact term (with a change of sign of the powers) is appropriate in the $k = -3$ model as well. We find that with our conventions, we must take

$$C_W(T_{k_i}, T_{k_j}) = \frac{d}{dX}\left(\frac{T_{k_i}(X)T_{k_j}(X)}{W'(X)}\right)_- \tag{7}$$

where the subscript indicates that we keep only the negative powers of X. Symmetry of the four-point correlator follows from the identity

$$\text{Res}\left\{\frac{d}{dX}\left(\frac{P}{W'}\right)_- \frac{Q}{W'} - \frac{P}{W'}\frac{d}{dX}\left(\frac{Q}{W'}\right)_-\right\} = \frac{1}{2}\text{Res}\left\{\frac{P'Q - PQ'}{(W')^2}\right\} \tag{8}$$

for two polynomials P, Q in X and X^{-1}. Notice that the relevant identity for positive integer level k has the subscript $+$ corresponding to keeping positive powers of X in the LHS of (8).

The contact term (7) between tachyons T_{k_i} and T_{k_j} is thus

$$C_W(T_{k_i}, T_{k_j}) = (k_i + k_j)T_{k_i+k_j}\theta(-k_i - k_j), \tag{9}$$

where $\theta(x)$ is the step function. Using Eq.(6) and the contact term (9), it is easy to show that the four-point function is

$$\begin{aligned} \langle T_{k_1}T_{k_2}T_{k_3}T_{k_4}\rangle &= \delta\,(\sum_{i=1}^{4} k_i)\,(1 - \text{max}|k_i|) \\ &= \delta\,(\sum_{i=1}^{4} k_i)\,[-\tfrac{1}{2}|k_1 + k_2| - \tfrac{1}{2}|k_1 + k_3| - \tfrac{1}{2}|k_2 + k_3| + 1] \end{aligned} \tag{10}$$

This result agrees with the tree-level correlator of four tachyons calculated in matrix models[14], evaluated at $\mu = -1$. The same correlator was also computed in ref.[15] in the continuum method.

In fact, in Ref.[15], a compact expression for N-point function of tachyons on the sphere is given. The correlator in question is in the kinematic region $k_2, \cdots, k_N > 0$ and $k_1 = -(k_2 + \cdots + k_N) < 0$. The answer can be expressed as

$$\langle T_{k_1} \cdots T_{k_N} \rangle = \left(\frac{\partial}{\partial \mu} \right)^{N-3} \mu^{-k_1 - 1} \tag{11}$$

In the following section we will rederive this result from the LG theory.

We are now in a position to make more precise the identification of the LG tachyons T_k with those in the standard continuum approach. Since the result of Ref.[15] is for the tachyons of so-called plus dressing[16][17], it is natural to identify the T_k to the positively dressed tachyons. The integer k, in units of $\sqrt{2}$, does indeed give the momentum of the continuum tachyon. Notice that in the continuum language, the LG tachyons are already *renormalized*, that is they are related to the tachyon fields with their leg-factors absorbed[2].

Before closing this section, let us point out that our results are automatically those for cosmological constant $\mu = -1$. As mentioned above, this is presumably due to the fact that the superpotential is just the cosmological operator itself. This idea can be tested by replacing the superpotential $-X^{-1}$ by μX^{-1}, in which case we should expect to obtain correlators with the right μ-dependence. Remarkably, this is just what happens. This will be shown in the section on gravitational descendants below.

[2] It is of course the *renormalized* cosmological operator which is actually the correct cosmological operator $\varphi e^{\sqrt{2}\varphi}$ of the $c = 1$ string theory.

4. Multipoint Correlators of Tachyons

The N-point function in LG theory is related to the $(N-1)$-point function by the recursion formula of Ref.[11][13]

$$\begin{aligned} \langle T_{k_1} \cdots T_{k_N} \rangle_W = & \frac{\partial}{\partial t_N} \langle T_{k_1} \cdots T_{k_{N-1}} \rangle_{W + t_N T_{k_N}} \Big|_{t_N=0} \\ & + \sum_{i=1}^{N-1} \langle T_{k_1} \cdots C_W(T_{k_N}, T_{k_i}) \cdots T_{k_{N-1}} \rangle_W . \end{aligned} \tag{12}$$

Unlike the expression for the four-point function, however, (12) is true only if the N-th insertion is a primary. We will see below that one recovers the expected matrix-model results only if the inserted fields are *positive-momentum* tachyons, T_k with $k \geq 0$. Thus we are led to think of only $k \geq 0$ tachyons as gravitational primaries. Fortunately, we can conveniently identify the $k < 0$ tachyons as gravitational descendants (this would not have been possible to do with the $k \geq 0$ tachyons, as we will see). Eventually we will also check that $k < 0$ tachyons obey recursion relations for descendants.

Using Eqs.(6),(9) and (12), one can systematically calculate correlators of 5, 6,... tachyons (with the restriction that $k_5, k_6, \ldots$ are all positive). In the kinematical region, $k_2, \cdots k_N > 0$ and $k_1 = -\sum_{i=2}^{N} k_i$, we now show that the answer turns out to be

$$\langle T_{k_1} \cdots T_{k_N} \rangle = \prod_{i=1}^{N-3} (k_1 + i) \tag{13}$$

in perfect agreement with Eq.(11) evaluated at $\mu = -1$!

We prove (13) in the general case by induction. Notice first that the contribution from the contact terms in (12) simplifies for the chosen kinematics, since only the contact term between T_{k_N} and T_{k_1} is non-zero. Using Eqs.(9) and the induction hypothesis (13), this contribution is

$$\sum_{i=1}^{N-1} \langle T_{k_1} \cdots C_W(T_{k_N}, T_{k_i}) \cdots T_{k_{N-1}} \rangle_W = \prod_{i=1}^{N-3} (k_1 + k_N + i - 1). \tag{14}$$

Notice that for $k_N = 1$, this is already the correct answer. This is consistent as in that case, the contribution from the first term vanishes (since the potential gets perturbed by a constant, leading to no change in the structure constants).

To find the contribution due to the first term, we will first prove the identity

$$\frac{\partial}{\partial t_N} \cdots \frac{\partial}{\partial t_{m+1}} \langle T_{k_1} \cdots T_{k_m} \rangle_{\tilde{W}} \Big|_{t_a=0} = \prod_{i=1}^{N-3} (k_1 + i) - \sum_{a=m+1}^{N} \prod_{i=1}^{N-3} (k_1 + k_a + i - 1)$$
$$- \sum_{\substack{a<b \\ m+1}}^{N} \prod_{i=1}^{N-3} (k_1 + k_a + k_b + i - 2) + \cdots$$
$$+ (-)^{N-m+1} \prod_{i=1}^{N-3} (k_1 + \sum_{a=m+1}^{N} k_a + i - N + m), \tag{15}$$

where $\tilde{W} = W + \sum_{a=m+1}^{N} t_a T_{k_a}$. This identity is obviously true for $m = 3$: The RHS is a polynomial of degree $(N-3)$ in $k_4, \cdots, k_N$ and has zeroes at each of $k_i = 1$, $i = 4, \cdots, N$; hence it is proportional to $\prod_{i=4}^{N} (k_i - 1)$. The polynomial is uniquely determined by the coefficient of the term $k_4 \cdots k_N$, which is easily found to be $(-)^{N-3}(N-3)!$. On the other hand, this is exactly what we get by explicit differentiation of the LHS. We will now assume that (15) true for an arbitrary m and prove by induction.

The LHS of (15) can be written as

$$\frac{\partial}{\partial t_N} \cdots \frac{\partial}{\partial t_{m+1}} \left[\frac{\partial}{\partial t_m} \langle T_{k_1} \cdots T_{k_{m-1}} \rangle_{\tilde{W} + t_m T_{k_m}} \Big|_{t_m=0} + \langle C_{\tilde{W}}(T_{k_m}, T_{k_1}) T_{k_2} \cdots T_{k_{m-1}} \rangle_{\tilde{W}} \right]_{t_a=0} . \tag{16}$$

Now, we know the first term from the induction hypothesis (15), and therefore to prove the above identity, we need to show that,

$$\frac{\partial}{\partial t_N} \cdots \frac{\partial}{\partial t_{m+1}} \langle C_{\tilde{W}}(T_{k_m}, T_{k_1}) T_{k_2} \cdots T_{k_{m-1}} \rangle_{\tilde{W}} \Big|_{t_a=0} = \prod_{i=1}^{N-3} (k_1 + k_m + i - 1)$$
$$- \sum_{a=m+1}^{N} \prod_{i=1}^{N-3} (k_1 + k_m + k_a + i - 2) + \cdots$$
$$+ (-)^{N-m+2} \prod_{i=1}^{N-3} (k_1 + k_m + \sum_{a=m+1}^{N} k_a + i - N + m - 1). \tag{17}$$

To prove Eq.(17) above, note that we can once again use the induction hypothesis as the correlator involved in (17) is an $(m-1)$-point function. Expanding the contact term $C_{\tilde{W}}(T_{k_m}, T_{k_1})$ order by order in t and using the induction hypothesis, it is tedious but straightforward to show that (17) is true. This completes the proof of (15).

A special case of (15) is $m = N-1$. In this case the LHS is precisely the first term is (12). Substituting for this from (15), we get the desired result (13).

Finally, one can try to go through the same procedure as above but using a negative-momentum tachyon as the perturbing field. Already at the level of the 5-point function, this gives an answer which differs from the correct one, which shows that one cannot consistently take negative-momentum tachyons to be gravitational primaries in this theory. We will discuss in a subsequent section how they can in fact be thought of as gravitational descendants, but contructed entirely in terms of matter-sector variables, analogous to the picture advocated by Losev[11] for the positive-level LG theory.

Actually from Eq.(7) we already find support for this picture. The contact term in this equation arises only if the OPE of the colliding fields results in a negative-momentum tachyon. Identifying these tachyons as secondaries leads to the conclusion that the contact terms in our theory arise only when a secondary is produced, exactly as in the $k > 0$ theories (see the comment below Eq.(6)).

5. Flow in the Small Phase Space

So far, we have been working in the basis of fields of Ref.[11][13], where the fields as well as the superpotential, (except in intermediate stages of computation), are independent of the couplings t. There exists another picture, that of Ref.[7], where they acquire non-trivial t-dependence—that is, they are said to *flow*. The relation between the two is through the formal generating function of the t-dependent multipoint correlator $\langle T_{k_1} \cdots T_{k_N} \rangle_W(t) \equiv \langle T_{k_1} \cdots T_{k_N} e^{\sum_{k_i>0} t_i T_{k_i}} \rangle_W$. This is equal to the multipoint correlator $\langle T_{k_1}(t) \cdots T_{k_N}(t) \rangle_{W(t)}$ in the t-dependent picture.

The t-dependent fields and superpotential are solutions to the differential equations[11][13]

$$\begin{aligned} \frac{\partial}{\partial t_i} T_{k_j}(t) &= C_{W(t)}(T_{k_i}(t), T_{k_j}(t)), \\ \frac{\partial}{\partial t_i} W(t) &= T_{k_i}(t), \end{aligned} \tag{18}$$

where, the index i is restricted to tachyons with positive k_i (primaries) only. It is easy to explicitly integrate Eqs.(18). To this end notice that the contact term (9) between two tachyons with positive momenta vanishes. This leads to a great simplification. First of all, the primaries do not flow at all,

$$T_k(t) = T_k \quad \text{for } k > 0, \tag{19}$$

while the superpotential is only a linear function of the t_i

$$W(t) = -X^{-1} + \sum_{i=1}^{\infty} t_i X^{i-1}. \tag{20}$$

The flow for the secondaries, that is tachyons with negative momenta, can be determined order by order in t. After a little algebra, the solution ($T_k(t)$ for $k<0$) is found to be

$$\begin{aligned} T_k(t) =& X^{k-1} + \sum_{\substack{k_i>0\\ k+k_i<0}} t_i(k+k_i)X^{k+k_i-1} + \sum_{\substack{k_i\neq k_j>0\\ k+k_i+k_j<0}} t_it_j(k+1)(k+k_i+k_j)X^{k+k_i+k_j-1} \\ &+ \sum_{\substack{k_i\neq k_j\neq k_l>0\\ k+k_i+k_j+k_l<0}} t_it_jt_l(k+1)(k+2)(k+k_i+k_j+k_l)X^{k+k_i+k_j+k_l-1} + \cdots \end{aligned} \tag{21}$$

Each term on the RHS is totally symmetric in all the t's, which is just the statement that the flows in the different directions commute.

It is now easy to calculate the perturbed three-point function between tachyons with momenta $k_1<0$ and $k_2,k_3>0$:

$$\begin{aligned} c_{k_1k_2k_3}(t) =& \mathrm{Res}\left(\frac{T_{k_1}(t)T_{k_2}T_{k_3}}{W'(t)}\right) \\ =& \delta_{k_1+k_2+k_3,0} + \sum_i t_i(k_1+1)\delta_{k_1+k_2+k_3+k_i,0} \\ &+ \sum_{i,j} t_it_j(k_1+1)(k_1+2)\delta_{k_1+k_2+k_3+k_i+k_j,0} + \cdots \end{aligned} \tag{22}$$

Differentiating this $(N-3)$ times and setting all the t_i to 0, we get the N-point correlator (13), as expected.

For other choices of the kinematics, it is equally straightforward to check that the correlators computed in the two bases give the same answer. As an illustration, we have computed the 5-point function in the configuration $k_1,k_2<0$ and $k_3,k_4,k_5>0$ in both the approaches. There are several different kinematic regions within this configuration. One example is the region $|k_2|>|k_5|>|k_1|>(|k_3|,|k_4|)$, for which the 5-point function is

$$\langle T_{k_1}T_{k_2}T_{k_3}T_{k_4}T_{k_5}\rangle = (k_2+1)(k_1-k_5+2) \tag{23}$$

The fact that we get the correct correlators in the basis of Ref.[7] with the use of Eqs.(18) serves, in particular, as an additional justification for our division of the Landau-Ginzburg fields into primaries and secondaries according to whether they are positive or negative momentum (and chirality) tachyons.

There is one important difference between our LG theory and those with positive level k. In the latter, the two point function between primaries is independent of the time t. The set of primary fields defines the tangent space at the fixed point (conformal point) in the space of topological theories. The two-point function defines a metric in this space, and since it is independent of t, this metric is constant and hence flat. On the other hand for the LG theory with $k = -3$, the non-vanishing two-point function is between a primary and a secondary, and from (21), we see that it acquires a non-trivial dependence on t.

6. Gravitational Descendants

The main point of Ref.[11] is that for $k > 0$ LG matter coupled to topological gravity, the gravitational descendants can be expressed entirely in terms of matter degrees of freedom. Since in our case the negative-chirality tachyons cannot be thought of as primaries, it is reasonable to suppose that in fact they are gravitational secondaries, expressed in terms of the matter sector fields. From U(1) charge conservation, we see that T_k has the same charge as $\sigma_m(T_{k'})$ whenever $k = k' - m$, in other words, taking the mth gravitational secondary lowers the charge by m units. Thus, assuming that the only primaries are T_k with $k > 0$, it is possible to construct secondaries of these which have the right charge to be negative-chirality tachyons. In fact, this can be done in infinitely many ways. We will see, however, that it is most natural to consider the negative-chirality tachyons in a unique way as descendants of the cosmological operator. Note that since gravitational descendants effectively lower the momentum, it would not have been possible for us to treat negative-chirality tachyons as primaries and positive-chirality as secondaries, once our initial conventions are fixed. Thus our hypothesis is consistent with the structure of the theory in a nontrivial way.

According to [11], the m-th descendant of the primary $\phi(X)$ can be written as

$$\sigma_m(\phi) = W'(X) \int^X dX_m\, W'(X_m) \cdots \int^{X_3} dX_2\, W'(X_2) \int^{X_2} dX_1\, \phi(X_1). \qquad (24)$$

Our strategy is to construct descendants of the cosmological operator, using the above prescription, and then compute a 4-point function in two ways: once using purely matter variables, which we have already done above, and then again using the gravitational descent equations[2]. Agreement between the two provides strong evidence for our identification.

From Eq.(24) above, we have in our case

$$\sigma_m(T_0) = \sigma_m(X^{-1}) = X^{-2} \int^X dX_m\, X_m^{-2} \cdots \int^{X_3} dX_2\, X_2^{-2} \int^{X_2} dX_1\, X_1^{-1} \qquad (25)$$

We see immediately that the very first integral produces a logarithmic factor, and that this is present only in our LG theory (which corresponds to the $c = 1$ string) and not in the LG theories which are associated to $c < 1$ strings. This is evidently an echo, in the topological description, of the logarithmic scaling violations in the $c = 1$ string. We will simply imagine that the integral is 'regularised', by slightly shifting the powers of X, and at the end we will have a divergent Γ-function. This will ultimately cancel out from both sides of our calculation.

With this understanding, evaluation of the above gives

$$\sigma_m(X^{-1}) = \Gamma(1-m)X^{-m-1} = \Gamma(1-m)T_{-m} \tag{26}$$

Now we put this in a correlator with three other tachyons, all of positive chirality:

$$\begin{aligned}\langle\sigma_m(T_0)T_{k_2}T_{k_3}T_{k_4}\rangle &= \Gamma(1-m)\langle T_{-m}T_{k_2}T_{k_3}T_{k_4}\rangle \\ &= (1-m)\Gamma(1-m) = \Gamma(2-m)\end{aligned} \tag{27}$$

where we have used the tachyon 4-point function formula Eq.(10), in the appropriate kinematic region.

Next, we consider the gravitational recursion relation. A general derivation of the relations in the present model is beyond the scope of this work, but we will use the principle of continuation from the $k > 0$ case as a guide. In that case, on the sphere we have[2]

$$\begin{aligned}\langle\prod_{i=1}^{4}\sigma_{m_i}(\phi_{r_i})\rangle =& \sum_r\langle\sigma_{m_1-1}(\phi_{r_1})\sigma_{m_2}(\phi_{r_2})\phi_r\rangle\eta^{rs}\langle\phi_s\sigma_{m_3}(\phi_{r_3})\sigma_{m_4}(\phi_{r_4})\rangle \\ &+ \sum_r\langle\sigma_{m_1-1}(\phi_{r_1})\phi_r\rangle\eta^{rs}\langle\phi_s\sigma_{m_2}(\phi_{r_2})\sigma_{m_3}(\phi_{r_3})\sigma_{m_4}(\phi_{r_4})\rangle\end{aligned} \tag{28}$$

(we have removed factors of m_1 on the RHS since these are absorbed by an $m!$ discrepancy between the norms used in Refs.[11] and [2]). Now let us specialise to the case where $m_2, m_3, m_4 = 0$, so the last three fields are primaries. In that case, it follows (for $k > 0$) that the second term on the RHS above does not contribute, by U(1) charge conservation, and the fact that primaries and secondaries in $k > 0$ models cannot carry the same charge.

Thus, we first drop this term and then take over the recursion relation for our case. Then we have

$$\langle\sigma_m(T_0)T_{k_2}T_{k_3}T_{k_4}\rangle = \sum_k\langle\sigma_{m-1}(T_0)T_{k_2}T_k\rangle\eta^{kk'}\langle T_{k'}T_{k_3}T_{k_4}\rangle \tag{29}$$

On the RHS we find that the answer is $\Gamma(2-m)$ (from Eq.(26)) times two momentum-conserving 3-point functions, which are equal to 1. Thus the answer is again $\Gamma(2-m)$, in agreement with Eq.(27).

Let us now re-introduce the cosmological constant, to provide an additional confirmation of the scenario that we have developed. We repeat the calculations in this section, but starting with the scaled superpotential μX^{-1}. From Eq.(25) the result is that the negative-chirality tachyons are scaled by a power of μ (while the positive ones are of course unaffected). The result is:

$$\begin{aligned} &T_k \to T_k \qquad (k>0) \\ &T_k \to (-\mu)^{|k|} T_k \qquad (k<0) \end{aligned} \tag{30}$$

With these scalings, we can repeat our computation of the perturbed structure constants, Eq.(5). Suppose we take $k_4 > 0$, $k_1, k_2, k_3 < 0$ in that equation, and scale the superpotential and negative chirality tachyons as described above. The result, after differentiating in t_4, is proportional to $\mu^{-k_1-k_2-k_3-2}$. But this is *precisely* the cosmological constant dependence of the tachyon 4-point function! Indeed, it is an easy exercise to check that in general the above scaling gives rise to the following μ-dependence:

$$\langle T_{k_1} \cdots T_{k_N} \rangle \sim \mu^{2-N+\frac{1}{2}\sum_i |k_i|} \tag{31}$$

which is precisely what is expected from matrix models or Liouville theory. This confirms not only that the superpotential should be thought of as the cosmological operator, but also that the construction of negative-chirality tachyons as secondaries is consistent.

The above observations provide a very interesting way to relate the LG theory being discussed here with the Kazama-Suzuki model of Ref.[4]. Taking $\mu \to 0$ simply sets the superpotential to 0, leaving behind what we might call the "free" LG model. (This is of course a singular change, as the resulting matter theory has $\hat{c} = 1$). Now, the result is *precisely* the theory obtained in Sec.(3.4) of Ref.[4], by starting with the Kazama-Suzuki coset and describing the SL(2) currents by a dual Wakimoto representation. This provides a direct link between the KS and LG pictures of the topological theory.

7. More on Gravitational Descendants

It is clear from the preceding section that the recursion relations obeyed by the negative-momentum tachyons, treated as gravitational descendants, are *not* the ones that would be obtained by just taking over the results of Ref.[2],[3] for the case of unitary topological matter coupled to topological gravity. In that context, one has to sum over all degenerations of the Riemann surface and at the point of degeneration, one has to sum over a complete set of matter primaries flowing through the pinch. This has no obvious analogue in the present case, simply because the gravitational primaries (positive tachyons) are dual to gravitational descendants (negative tachyons) in our model. The main point of difference seems to arise from the fact that our matter system is "nonunitary" as a superconformal theory before twisting.

One may conjecture that the right gravitational recursion relations are simply those of Ref.[2] in which all the tachyons, both primary and secondary (in dual pairs) flow through the pinch. But this relation does not then reproduce the right tachyon N-point functions that we have already derived in a previous section. Instead, we use the result of the previous section to conjecture a modified gravitational recursion relation. We will see that our conjecture is powerful enough to give not only the N-point functions on the sphere, but also the correct 2-point function on the torus.

From Eq.(29) we see that out of two possible degenerations of the sphere with four punctures, the one which contributes is the one with the *minimum* number of fields in the "right-hand" branch (the branch on which the gravitational descendant itself does not lie). Thus we are led to postulate the following general relation for an N-point function with a single gravitational secondary:

$$\langle\sigma_m(T_0)T_{k_2}\cdots T_{k_N}\rangle_0 = \sum_k \langle\sigma_{m-1}(T_0)T_{k_2}\cdots T_{k_{N-2}}T_k\rangle_0 \eta^{kk'}\langle T_{k'}T_{k_{N-1}}T_{k_N}\rangle_0 \tag{32}$$

The subscript on the correlators refers to the genus (0 in this case), this has been displayed explicitly as we will shortly turn to the case of the torus.

To check that this relation agrees with the known tachyon correlators is a simple exercise. Using Eq.(13), the LHS of the above equation is

$$\begin{aligned}\langle\sigma_m(T_0)T_{k_2}\cdots T_{k_N}\rangle_0 &= \Gamma(1-m)\langle T_{-m}T_{k_2}\cdots T_{k_N}\rangle_0\\ &= \Gamma(1-m)\prod_{i=1}^{N-3}(-m+i)\\ &= \Gamma(N-2-m)\end{aligned} \tag{33}$$

while momentum conservation and Eq.(13) give for the RHS

$$
\begin{aligned}
\sum_k \langle \sigma_{m-1}(T_0) T_{k_2} \cdots T_{k_{N-2}} T_k \rangle_0 \eta^{kk'} \langle T_{k'} T_{k_{N-1}} T_{k_N} \rangle_0 & \\
&= \langle \sigma_{m-1}(T_0) T_{k_2} \cdots T_{k_{N-2}} T_{k_{N-1}+k_N-1} \rangle_0 \langle T_{-k_{N-1}-k_N} T_{k_{N-1}} T_{k_N} \rangle_0 \\
&= \Gamma(2-m) \langle T_{1-m} T_{k_2} \cdots T_{k_{N-2}} T_{k_{N-1}+k_N-1} \rangle_0 \\
&= \Gamma(2-m) \prod_{i=1}^{N-4} (1-m+i) \\
&= \Gamma(N-2-m)
\end{aligned} \tag{34}
$$

Thus we see that the postulated recursion relation gives the correct answer for N-point functions on the sphere as long as just one gravitational descendant is present.

The above hypothesis for the sphere recursion relation, can now be applied to the torus. Again we postulate that of the various ways in which the torus can degenerate into a sphere times a torus, the minimum number of fields (in this case, a single one) lie in the "right branch" (the torus). In addition, there is the usual term corresponding to pinching a nontrivial cycle of the torus. Again, we restrict ourselves to the case where only one of the inserted fields is a gravitational secondary (negative tachyon). Thus we have

$$
\begin{aligned}
\langle \sigma_m(T_0) T_{k_2} \cdots T_{k_N} \rangle_1 = & \frac{1}{24} \sum_k \langle \sigma_{m-1}(T_0) T_{k_2} \cdots T_{k_N} T_k T_{-k-1} \rangle_0 \\
& + \sum_k \langle \sigma_{m-1}(T_0) T_{k_2} \cdots T_{k_N} T_k \rangle_0 \langle T_{-k-1} \rangle_1
\end{aligned} \tag{35}
$$

Note that while the first term involves an infinite sum over all positive and negative momenta, the second term has just a single contribution from $k = -1$ by momentum conservation.

We will check this for the two-point functions. On the LHS, we have

$$
\begin{aligned}
\langle \sigma_m(T_0) T_{k_2} \rangle_1 &= \Gamma(1-m) \langle T_{-m} T_{k_2} \rangle_1 \\
&= \Gamma(1-m) \langle T_{-m} T_m \rangle_1
\end{aligned} \tag{36}
$$

where we have $k_2 = m$ by momentum conservation, and the delta-function has been suppressed. On the RHS we have

$$
\begin{aligned}
& \frac{1}{24} \sum_k \langle \sigma_{m-1}(T_0) T_m T_k T_{-k-1} \rangle_0 + \langle \sigma_{m-1}(T_0) T_m T_{-1} \rangle_0 \langle T_0 \rangle_1 \\
& \quad = \Gamma(2-m) \left(\frac{1}{24} \sum_k \langle T_{1-m} T_m T_k T_{-k-1} \rangle_0 + \langle T_0 \rangle_1 \right)
\end{aligned} \tag{37}
$$

It follows that our recursion relation in this case is equivalent to

$$\langle T_{-m} T_m \rangle_1 = (1-m) \left(\frac{1}{24} \sum_k \langle T_{1-m} T_m T_k T_{-k-1} \rangle_0 + \langle T_0 \rangle_1 \right) \tag{38}$$

The right-hand side is easily evaluated by dividing the sum over k into regions corresponding to the distinct kinematic configurations in the sphere four-point function. We have to isolate a divergent term of the form $\sum_{k=1}^{\infty} k = \beta$, and the result is

$$\langle T_{-m} T_m \rangle_1 = -\frac{1}{24}(1-m)\left(m^2 - m + \beta - 24\langle T_0 \rangle_1\right) \tag{39}$$

Although we do not know how to give a meaningful value to the infinite constant β in a way that is uniquely dictated by the physics of this model, it can be fixed purely by self-consistency. To do this, recall that the torus partition function is proportional to $\log\mu$. Setting $m = 0$ in the above equation and rewriting the correlators of cosmological operators as derivatives in μ of the partition function, we find that for consistency, we must have $\beta = 0$.

Finally we insert the value $\langle T_0 \rangle_1 = \frac{1}{12}$ in Eq.(39) above to get

$$\langle T_{-m} T_m \rangle_1 = -\frac{1}{24}(1-m)\left(m^2 - m - 2\right) \tag{40}$$

which is precisely the matrix-model result[18]!

Note that several correlators in the Landau-Ginzburg theory we have been studying tend to have zeroes when some momentum k is equal to 1. This can in each case be traced to the following fact: the tachyon with unit momentum is represented in the LG theory by the identity operator. When treated as a perturbation of the superpotential, this clearly causes no change in W', which accounts for the zeroes at momenta $k = 1$. Of course, this holds only for those correlators where the kinematics is such that the only dependence on the given momentum comes from perturbing the superpotential. Contact terms give rise to a different dependence.

It should be clear that the gravitational recursion relations that we have been using above are not quite the standard ones for topological gravity — however, it is important that they involve a *subset* of the terms which appear in the standard case, and not any additional terms. We have conjectured the truncation to this subset of terms and shown that this consistently leads to correct results, but it would be worth trying to find a proof of these recursion relations. In any case, the discussion here applies only to the case of one gravitational secondary and the remaining fields primary, while the general case remains open.

We believe that with better understanding, this topological formulation of $c = 1$ string theory could be powerful enough to reproduce all higher-genus correlators, thus rivalling the highly successful matrix models.

8. Discussion and Conclusions

We have provided ample evidence that LG theory with superpotential X^{-1} describes the $c = 1$ string where the matter field is compactified on a circle with the self-dual radius. The N-point functions of tachyons on the sphere and the two-point function on the torus have been computed above, but in principle it should not be difficult to go further. The main obstacle to that is a complete understanding of the nature of gravitational recursion relations. The conjecture which we have presented and tested should prove useful in such an investigation.

The various discrete states of $c = 1$ string theory have not yet been clearly understood in the topological approaches. In Ref.[4], although they are clearly identified in the CFT description (which is in terms of the Hilbert space at $\mu = 0$), they are not straightforward to identify in the Lagrangian description of the same coset model (which, however, is automatically at nonzero μ). The same is true in the LG approach.

The obvious candidates for the states of ghost number 1 ($Y^+_{s,n}$ in the notation of Ref.[17]) come from the following identification with gravitational descendants:

$$Y^+_{s,n} \sim \sigma_m(T_k) \qquad (s = \tfrac{1}{2}(m+k),\ n = \tfrac{1}{2}(m-k)) \tag{41}$$

with $k > 0$. This has many appealing features: for the special cases $k = 0$ or $m = 0$ it reduces to the identifications we have already demonstrated above. Matter momentum conservation on the LHS gives the same relation as U(1) charge conservation on the RHS. Further confirmation will require the computation of correlation functions, for which we again need a better understanding of the gravitational recursion relations. Also, there is no sign of winding modes, as the above correspondence accounts only for the left-right symmetric (momentum) modes.

Let us comment on the mysterious way in which the topological theory discussed here fails to show explicit $\mathbf{Z}_2$ invariance, which in $c = 1$ string theory is simply sending the $c = 1$

free scalar field to minus itself (this is variously interpreted as time-reversal or parity)[3]. In the LG theory discussed here, such a symmetry would have to take X^{k-1} to X^{-k-1} which is clearly not obtainable by any transformation on the X superfield. This seems to originate in the fact that there is a shift of 1 unit between the tachyon momentum and the topological U(1) charge. One suggestive observation is the following: a generic perturbation of the LG theory, which can be thought of as the string field $\Psi(X)$, is given by

$$\Psi(X) = \sum_k t_k X^{k-1} \tag{42}$$

If we interpret X as a complex variable, this is precisely the mode expansion of a spin-1 current in a conformal field theory. If $\Psi(X)$ were really treated as a spin-1 conformal field, then the inversion $X \to X^{-1}$ would effect the change $k \to -k$ precisely because a spin-1 field picks up the appropriate Jacobian when the transformation is carried out. Moreover, transforming the variable to the cylinder by $X = e^Z$ would eliminate the 'shift' by 1 unit, and we would have perfect $\mathbf{Z}_2$ symmetry on the cylinder. This suggests that we need to understand better the 'target-space' properties of our theory, and the right variables in which to describe it. It is noteworthy that a target-space conformal dimension of 1 for the string field corresponding to the tachyons of $c = 1$ string theory has already been suggested by Witten and Zwiebach[17]. In their work, this comes about by studying the transformation properties under the target-space Virasoro algebra which arises as a subalgebra of W_∞.

The fact that the LG theory discussed in this paper has $\hat{c} = 3$ strongly suggests that it is even more closely linked to Calabi-Yau (CY) sigma-models than the conventional LG models, where one needs to take many copies to get the correct central charge[19]. Because the weight of the basic superfield X is negative, it seems likely that the related manifold will actually be a non-compact analogue of a CY hypersurface, in a weighted projective space with some negative weights.

The genus-g partition function of this theory is expected to be the Euler characteristic of the moduli space of genus-g Riemann surfaces[20][6][4]. If this could be computed directly from the present LG formulation, it would give an independent derivation of the fact that the Euler characteristic of moduli space is proportional to the Bernoulli numbers. Even more interesting, the generating function for tachyon correlators in genus-g for the

[3] We are grateful to E. Witten for stressing this point in the context of the coset description of Ref.[4], which evidently has the same feature.

self-dual radius theory is known[21] in the form of a Kontsevich-like matrix integral. It would be wonderful to derive this elegant integral representation directly from topological arguments in Landau-Ginzburg theory.

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