CURVATURE AND THE BACKWARD SHIFT OPERATORS

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ABSTRACT. Let \( \varphi_\alpha \) be a Möbius transformation of the unit disk \( \mathbb{D} \), \( |\alpha| < 1 \). We characterize all the operators \( T \) in \( B_1(\mathbb{D}) \) which are unitarily equivalent to \( \varphi_\alpha(T) \) for all \( \alpha \) with \( |\alpha| < 1 \), using curvature techniques.

0. Introduction. The backward shift operator \( U^*_+ \) lies in the class \( B_1(\mathbb{D}) \), first introduced in Cowen and Douglas [1]. It is easy to compute the curvature \( K_{U^*_+}(\omega) \), which turns out to be \(- (1 - |\omega|^2)^{-2}\). For any operator \( T \) in \( B_1(\mathbb{D}) \) with \( \|T\| \leq 1 \), we have [3], \( K_T(\omega) \leq -(1 - |\omega|^2)^{-2} \). This inequality is best possible over all of \( \mathbb{D} \) since equality holds for \( T = U^*_+ \). Some time back R. G. Douglas asked if the inequality is best possible pointwise; that is, if \( T \in B_1(\mathbb{D}) \), \( \|T\| \leq 1 \) and \( K_T(\omega_0) = -(1 - |\omega_0|^2)^{-2} \) for some \( \omega_0 \) in \( \mathbb{D} \), does it follow that \( T \) is unitarily equivalent to \( U^*_+ \)?

In this note we obtain a characterization of those operators \( T \) in \( B_1(\mathbb{D}) \) that are unitarily equivalent to \( \varphi_\alpha(T) \) for all \( \alpha \), where \( \varphi_\alpha \) is a Möbius transformation of the disk, and answer the above problem in the negative.

1. The class \( B_1(\mathbb{D}) \) is defined as follows.

\[
B_1(\mathbb{D}) = \{ T \in \mathcal{L}(\mathcal{H}) : (i) \mathbb{D} \subset \sigma(T), \\
(ii) \bigwedge_{\omega \in \mathbb{D}} \ker(T - \omega) = \mathcal{H}, \\
(iii) \text{ran}(T - \omega) = \mathcal{H}, \\
(iv) \dim \ker(T - \omega) = 1 \text{ for all } \omega \in \mathbb{D} \}.
\]

For each operator \( T \) in \( B_1(\mathbb{D}) \), such that \( T(\gamma(\omega)) = \omega \gamma(\omega) \), it is possible to find a holomorphic family of eigenvectors \( \gamma(\omega) \) on \( \mathbb{D} \). Following Cowen and Douglas [1], we can define the curvature of an operator \( T \) in \( B_1(\mathbb{D}) \) to be

\[
K_T(\omega) = \frac{\partial^2}{\partial \omega \partial \bar{\omega}} \log \|\gamma(\omega)\|^{-2}.
\]

Let \( \varphi_\alpha(\omega) = (\alpha - \omega)/(1 - \overline{\alpha}\omega)^{-1} \) be a Möbius transformation of the unit disk, \( |\alpha| < 1 \). Whenever \( \|T\| \leq 1 \), the operator \( \varphi_\alpha(T) \) is well defined and a simple application of chain rule yields

\[
\|\varphi'_\alpha(\omega)\|^2 K_{\varphi_\alpha(T)}(\varphi_\alpha(\omega)) = K_T(\omega).
\]

In particular if \( T = U^*_+ \), we obtain

\[
K_{\varphi_\alpha(U^*_+)}(\varphi_\alpha(\omega)) = |\varphi'_\alpha(\omega)|^{-2} K_{U^*_+}(\omega) = -|\varphi'_\alpha(\omega)|^{-2} (1 - |\omega|^2)^{-2} = -(1 - |\varphi_\alpha(\omega)|^2)^{-2} = K_{U^*_+}(\varphi_\alpha(\omega)).
\]
The Cowen-Douglas theorem, which states that two operators in $B_1(D)$ are unitarily equivalent if and only if their curvatures are equal, implies $\varphi_\alpha(U^+_\alpha)$ is unitarily equivalent to $U^+_\alpha$ for all $\alpha$. We can now ask ourselves, which other operators in $B_1(D)$ share this property.

**Proposition.** If $T$ is in $B_1(D)$ and $\|T\| \leq 1$ then $\varphi_\alpha(T)$ is unitarily equivalent to $T$ for all $\alpha$ if and only if

$$K_T(\omega) = -c(1 - |\omega|^2)^{-2},$$

for some constant $c \geq 1$.

**Proof.** If $K_T(\omega) = -c(1 - |\omega|^2)^{-2}$, a calculation similar to the one above shows that $\varphi_\alpha(T)$ must be unitarily equivalent to $T$ for all $\alpha$.

Conversely, if $T$ is unitarily equivalent to $\varphi_\alpha(T)$ for all $\alpha$ then we must have

$$K_{\varphi_\alpha(T)}(\varphi_\alpha(\omega)) = |\varphi'_\alpha(\omega)|^2 K_T(\omega) = K_T(\varphi_\alpha(\omega)).$$

In particular, $|\varphi'_\alpha(0)|^{-2} K_T(0) = K_T(\alpha)$ so $K_T(\alpha) = (|\alpha|^2 - 1)^{-2} K_T(0)$ for all $\alpha$ in $D$. Let $c$ equal $K_T(0)$, then $K_T(\omega) \leq -(1 - |\omega|^2)^{-2}$ implies that $c \geq 1$.

Now, consider the weighted shift operator $T$ with weights $\omega_n = (c_n/c_{n+1})^{1/2}$, where $c_n$ is the $n$th coefficient in the generalized binomial expansion of $(1 - |\omega|^2)^{-c}$ for a fixed real number $c$. The adjoint of $T$ is in $B_1(D)$ (Seddighi [4]) and $\gamma(\omega) = (1 - |\omega|^2)^{-c}$ is a holomorphic family of eigenvectors for $T^*$. It is easy to compute

$$K_{T^*}(\omega) = -c(1 - |\omega|^2)^{-2}.$$

When $c$ is an even integer these operators can be identified with the adjoint of multiplication on the Hilbert space of square integrable holomorphic functions on $D$ with respect to the measure $d\mu = (i/2)(1 - |\omega|^2)^{2-2\eta} d\omega \wedge d\bar{\omega}$ (cf. Kra [2, pp. 89 and 95]). Thus, we are able to identify all of the operators that are unitarily equivalent to all their Möbius transforms $\varphi_\alpha(T)$.

It follows from the Proposition that if $T \in B_1(D)$ and $\|T\| \leq 1$, then the following two statements are equivalent.

1. $K_T(\omega_0) = -(1 - |\omega_0|^2)^{-2}$ for some $\omega_0$ and $\varphi_\alpha(T)$ is unitarily equivalent to $T$ for all $\alpha$.

2. $T$ is unitarily equivalent to $U^+_\alpha$.

However, $K_T(\omega_0) = -(1 - |\omega_0|^2)^{-2}$ does not necessarily imply that $T$ is unitarily equivalent to $U^+_\alpha$ as we will show by means of an example.

Let $T$ be a weighted shift operator with weights $\omega_0, \omega_1, \omega_2, \ldots$. We can consider $T$ to be an ordinary shift on a weighted sequence space (Shields [5]) with weights $\beta(0), \beta(1), \ldots$. For $\omega \in D$,

$$\gamma(\omega) = \left( \frac{1}{\beta(0)}, \frac{\omega}{\beta(1)}, \frac{\omega^2}{\beta(2)}, \ldots \right)$$

is an eigenvector for $T^*$ and

$$\|\gamma(\omega)\|^2 = \sum_{n=0}^{\infty} \frac{|\omega|^{2n}}{|eta(n)|^2}.$$
Assuming $T^*$ is in $B_1(D)$ (Seddighi [4] determines when a weighted shift is in $B_1(D)$), we compute

$$
\kappa_{T^*}(\omega) = -\left[\left(\sum_{n=0}^{\infty} (n+1) \frac{|\omega|^{2n}}{|\beta(n+1)|^2}\right) \left(\sum_{n=0}^{\infty} \frac{|\omega|^{2n}}{|\beta(n)|^2}\right)
- |\omega|^2 \left(\sum_{n=0}^{\infty} (n+1) \frac{|\omega|^{2n}}{|\beta(n+1)|^2}\right)^2 \left[\sum_{n=0}^{\infty} \frac{|\omega|^{2n}}{|\beta(n)|^2}\right]^{-2}\right].$

Putting $\omega = 0$, we see that

$$
\kappa_{T^*}(0) = -|\beta(0)|^2 |\beta(1)|^{-2}.
$$

Now, let $T$ be the weighted shift with weights $1, \frac{1}{2}, 1, 1, 1, \ldots$. It is easy to verify that $T^* \in B_1(D)$ and $\|T^*\| = 1$. Since $\beta(0) = 1$ and $\beta(1) = 1$, it follows that $\kappa_{T^*}(0) = -1$. Obviously $T^*$ is not unitarily equivalent to $U^*_+$. In fact, we can compute $h_{T^*}(\omega)$ explicitly for the weighted shift of our example and show that $h_{T^*}(\varphi_{\alpha}(\omega)) \neq |\varphi'_{\alpha}(\omega)|h_{T^*}(\omega)$, therefore $T$ is not unitarily equivalent to $\varphi_{\alpha}(T)$ for any $\alpha$.

REFERENCES


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