ON A CLASS OF STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

By Alladi Ramakrishnan and P. M. Mathews
(*Department of Physics, University of Madras*)

Received April 27, 1953
(Communicated by Dr. G. N. Ramachandran, F.A.Sc.)

Summary

The solutions of the stochastic integro-differential equation involving, in addition to the random transitions of the stochastic variable, a deterministic change, are obtained under various approximations. It is shown that the solution in the case of a negative deterministic change is strikingly different from the solution when the change is positive. Examples from physics and astro-physics are cited to illustrate such stochastic processes.

The following type of process is well known in the theory of stochastic processes.

$\pi(E|E_0, t)\, dE$ is the probability that the continuous stochastic variable $E(t)$ assumes a value between E and $E + dE$ at t (t is a one-dimensional parameter with respect to which the process progresses), given that it had the value E_0 at $t = 0$. We assume the process is homogeneous and Markovian with respect to t. We are given that:

$R(E'|E)\, dE' dt$ is the probability that the stochastic variable jumps from E to an interval between E' and $E' + dE'$ in the parametric interval dt. $R(E'|E) = 0$ if $E' > E$.

The stochastic integro-differential equation for the above process can be directly written down using its Markovian character

$$\frac{\partial \pi(E|E_0, t)}{\partial t} = -\pi(E|E_0, t) \int_0^E R(E'|E)\, dE' + \int_E^\infty \pi(E'|E_0, t) \times R(E|E')\, dE'. \tag{1}$$

In some physical problems we may have to deal with stochastic processes more complicated than the one described above. In this paper, we shall deal with two types of complications.

I. In addition to the transition process defined by $R(E'|E)$ we are given that $\pm \beta(E)\, dt$, β being always positive, is the deterministic change in the stochastic variable in interval dt if it has the value E at t. The positive sign before β denotes an increment and a negative sign, a loss. In such a case, the stochastic integro-differential equation is given by*

$$\frac{\partial \pi (E|E_0, t)}{\partial t} \pm \frac{\partial}{\partial E}\left\{\beta\pi (E|E_0, t)\right\} = -\pi (E|E_0, t) \int_0^E R (E'|E)\, dE' + \int_0^\infty \pi (E'|E_0, t)\, R (E|E')\, dE' \qquad (2)$$

We write $+\frac{\partial}{\partial E}[\beta\pi (E|E_0, t)]$ when we deal with an increment and $-\frac{\partial}{\partial E}[\beta\pi (E|E_0, t)]$ when we deal with a loss.

This process will be dealt with in Part I of this paper.

II. We assume $R (E'|E) = 0$ if $E < \nu_0$. In such a case, equation (1) splits up into two equations. This process, we shall consider in Part II of this paper.

Physical examples of the above process can be cited.

Equation (1) is useful in describing the energy loss of fast particles due to radiation. Equation (2) with $+\beta$ is the fundamental equation of the astrophysical problem of the fluctuations in brightness of the Milky Way. Equation (2) with $-\beta$ has not been solved explicitly and the main contribution in this paper is *to obtain the solution in this case from the solution for the case* $+\beta$. Such a process arises where we take ionisation loss also into account when fast particles passing through matter lose energy by radiation.

The complication of Type II arises in the case of the energy distribution of recoil atoms in liquids, a problem considered in Part II.

Before we deal with the actual problems, we shall obtain the formal solution of (2) [which includes that of (1)] assuming that (i) $R (E'|E)\, dE'$ can be written as $R (q)\, dq$ where $q = E'/E$, $R (q) = 0$ for $q > 1$ or $q < 0$, and (ii) $\beta (E) = \beta E^k H (E)$ where k is a non-negative integer and $H (E)$ is the Heaviside unit function, *i.e.*, $H (E) = 1$ if $E > 0$ and $H (E) = 0$ if $E < 0$. This only means that in our stochastic process E can take only non-negative values.

We now reduce (2) to a differential equation by the use of a Mellin's transformation.

* For the derivation of this equation see, for example, Alladi Ramakrishnan (1952).

Defining

$$p(s, t) = \int_0^\infty \pi(E|E_0, t)\, E^{s-1}\, dE,$$

$$\omega(s) = \int_0^1 R(q)\, q^{s-1}\, dq, \quad \alpha = \omega(1) = \int_0^1 R(q)\, dq \tag{3}$$

we get

$$\frac{\partial p(s, t)}{\partial t} = -\alpha p(s, t) + \omega(s)\, p(s, t) \pm \beta(s-1)\, p(s+k-1, t) \tag{4}$$

with the initial conditions,

$$\pi(E|E_0, 0) = \delta(E - E_0) \qquad i.e.,\ p(s, 0) = E_0^{s-1} \tag{5}$$

where δ is the Dirac delta-function.

When $\beta = 0$ the above equation can be directly solved.

$$p(s, t) = E_0^{s-1}\, e^{\{\omega(s)-\alpha\}t} \tag{6}$$

and $\pi(E|E_0, t)$ is obtained by inversion.

$$\pi(E|E_0, t) = \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^s e^{\{\omega(s)-\alpha\}t}\, ds. \tag{7}$$

When $k = 1$ and $\beta \neq 0$, the equation can be solved easily. The solution is

$$\pi(E|E_0, t) = \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{E}\right)^s e^{\{\omega(s)-\alpha\pm(s-1)\beta\}t}\, ds.$$

PART I

Let us take the case when $k = 0$, $\beta \neq 0$. Then we are faced with a difference equation in a complex variable which obviously cannot be solved by simple iteration. We shall show that in the special case of $R(q') = \delta(q' - q)$, the equation can be completely solved. The solution when βdt represents a deterministic increment was first obtained by Chandrasekhar and Munch (1950, 51) in connection with a stochastic problem in astrophysics and subsequently by one of us using different methods (Ramakrishnan, 1953). The solutions are strikingly different in the two cases of deterministic gain and loss and our object is to discuss the two, and study their behaviour as t tends to infinitity.

Before we do so, it is necessary to note that $\alpha = \int_0^1 R(q)\, dq$ is independent of q and αdt represents the probability that a discrete jump occurs in dt, The number of discrete jumps in an interval t has a Poisson distribution given by

$$e^{-at}(at)^n/n\,! \tag{8}$$

We can define $\pi_n(E|E_0, t)\,dE$ to be the joint probability that n events have occurred and the variable has a value between E and $E + dE$ given that it had a value E_0 at $t = 0$.

Then

$$\pi_n(E|E_0, t) = \frac{e^{-at}(at)^n}{n\,!}\,\pi(E|E_0, n, t) \tag{9}$$

where $\pi(E|E_0, n, t)$ is the conditional probability that the variable lies between E and $E + dE$ given that n discrete transitions have occurred. The stochastic variable in the case of $\pi_n(E|E_0, t)$ when βdt denotes an increment is obviously

$$\begin{aligned} E &= E_0q^n + x_1q^{n-1} + (x_2-x_1)\,q^{n-2} + \ldots + (x_n - x_{n-1})\,q + (t-x_n) \\ &= E_0q^n + X \end{aligned} \tag{10}$$

where we assume that the n discrete transitions occur between x_1 and $x_1 + dx_1$, x_2 and $x_2 + dx_2, \ldots, x_n$ and $x_n + dx_n$ respectively. X is the stochastic variable in $\pi_n(E|E_0, t)$ when $E_0 = 0$. The distribution function of X has been obtained by Chandrasekhar and Munch* (β can be put equal to unity without loss of generality).

$$\begin{aligned} \pi_n(E, t) &= -\frac{a^n e^{-at}}{(n-1)\,!}\sum_{k=0}^{m-1} A_k{}^n\,(E - tq^k)^{n-1}, \quad (tq^m < E < tq^{m-1}) \\ \pi_0(E, t) &= e^{-at}\,\delta(E - t) \end{aligned} \tag{11}$$

where

$$A_k{}^n = \frac{1}{(q^n-q^k)(q^{n-1}-q^k)\ldots(q^{k+1}-q^k)(q^{k-1}-q^k)(q^{k-2}-q^k)\ldots(1-q^k)}.$$

We can immediately extend the above solution to the case when $E_0 \neq 0$

$$\begin{aligned} \pi_n(E|E_0, t) &= -\frac{a^n e^{-at}}{(n-1)\,!}\sum_{k=0}^{m-1} A_k{}^n\,(E - E_0q^n - tq^k)^{n-1}, \\ &\qquad (tq^m < E - E_0q^n < tq^{m-1}) \end{aligned} \tag{12}$$

$$\pi_0(E|E_0, t) = e^{-at}\,\delta(E - E_0 - t); \quad \pi(E|E_0, t) = \sum_{n=0}^{\infty}\pi_n(E|E_0, t).$$

We note that the solution is continuous in the range $0 \leqslant E < E_0 + t$ and at $E = t + E_0$, there is a delta-function singularity.

---

* We give here the solution in the form obtained by one of us (R) in a subsequent paper: *Proc. Camb. Phil. Soc.*, 1953, 49, 473. When $E_0 = 0$, we write $\pi_n(E|E_0, t)$ as $\pi_n(E, t)$.

Now consider the case when βdt represents deterministic loss. Then our stochastic variable is obviously

$$\begin{aligned} E &= E_0 q^n - x_1 q^{n-1} - (x_2 - x_1) q^{n-2} - \cdots - (x_n - x_{n-1}) q - (t - x_n) \\ &= E_0 q^n - X \end{aligned} \tag{13}$$

Compare (13) with (10). While in (10) the variable is the sum of two stochastic variables which can assume only positive values, in (13) it is the difference between two variables which can assume only positive values. If the physical conditions of the problem require that E should be non-negative then the probability that E should be exactly zero is given by the probability that X is greater than $E_0 q^n$, *i.e.*, at $E = 0$, there is a delta-function singularity. This feature is not present in the case of deterministic gain. The difference will be apparent if we take two stochastic variables x and y which can assume only non-negative values and form the stochastic variable $z = x + y$. If $D_1(x)$, $D_2(y)$, $D_3(z)$ are the distribution functions of x, y and z respectively, then

$$D_3(z) = \int_0^z D_1(x) D_2(z - x)\, dx, \quad \bar{z} = \bar{x} + \bar{y} \tag{14}$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the mean values of x, y, z, respectively.

If $z = x - y$, then

$$D_3(z) = \int_0^\infty D_1(y + z) D_2(y)\, dy \tag{15}$$

and

$$\bar{z} = \bar{x} - \bar{y} \tag{16}$$

But if we impose the condition that z *must always be positive, then* $\bar{z} \neq \bar{x} - \bar{y}$ *and there is a delta-function singularity for* $D_3(z)$ at $z = 0$. The probability that z is equal to 0 is

$$\int_{-\infty}^0 D_3(z)\, dz$$

where $D_3(z)$ is given by (15)

$$\bar{z} = \bar{x} - \bar{y} + \int_0^\infty z D_3(-z)\, dz = \int_0^\infty z\, D_3(z)\, dz. \tag{17}$$

Hence our solution in the case of loss is given by

$$\pi_n(E|E_0, t) = -\frac{a^n e^{-at}}{(n-1)!} \sum_{k=0}^{m-1} A_k{}^n (E_0 q^n - E - tq^k)^{n-1},$$
$$(tq^m < E_0 q^n - E < tq^{m-1}) \tag{18}$$

The probability that $E = 0$ and n discrete transitions have occurred is given by†

$$P(0, n, t) = \int_{E_0 q^n}^{E_n} \pi_n(E, t)\, dE \quad \text{where } \pi_n(E, t) \text{ is given by (11)}$$

$$\mathbf{P}(0, t) = \sum_{n=0}^{\infty} P(0, n, t) \qquad (19)$$

where $\mathbf{P}(0, t)$ represents the probability that $E = 0$ at t.

An example of a stochastic process involving deterministic gain is the astrophysical problem of "Fluctuations in Brightness of the Milky Way" so fully discussed by Chandrasekhar and Munch. As t tends to infinity, $\pi(E|E_0, t)$ has a stationary distribution which can be shown to be

$$\pi(E|E_0, t) \text{ as } t \to \infty, = \pi(E, t) \text{ as } t \to \infty, = \pi(E)$$

where

$$\pi(E) = \frac{d}{dE}\left[K e^{-E} \sum_{n=0}^{\infty} Q_n\, e^{-(1-q^n)E/q^n}\right], \qquad (20)$$

$$K = \frac{1}{\prod_{i=1}^{\infty}(1 - q^i)}$$

and

$$Q_n = (-1)^n \prod_{r=1}^{n} \frac{q^r}{1 - q^r}$$

a solution obtained by Chandrasekhar and Munch. In the case of deterministic loss the stationary solution is trivial, and for all $t > E_0$, $\pi(E|E_0, t) = 0$ for all $E > 0$ and $\mathbf{P}(0, t) = 1$. This is the direct result of the restrictive condition that $E = E_0 q^n - X$ must always be positive.

A direct example of the above process is the energy loss of fast particles when passing through matter. The discrete loss occurs due to the radiation, *i.e.*, Bremmstrahlung, while the steady deterministic loss occurs due to ionisation. $R(q)\, dq$ is given by the Bethe-Heitler cross-sections, but here we have made the approximation $R(q') = \delta(q' - q)$. But (19) represents the rigorous solution based upon the above assumption and $\frac{\partial \mathbf{P}(0, t)}{\partial t} dt$ represents the probability that the energy of the particle assumes the value zero for the first time between t and $t + dt$. Hence $\frac{\partial \mathbf{P}(0, t)}{\partial t}$ is the distribution function of the range of the particle.

† The actual numerical values for $P(0, t)$ for various values of t and q will be given in a later note.

PART II

We shall take the case when $\beta = 0$ but impose the following condition on $R(E'|E)\,dE'$. While $R(E'|E)\,dE'$ can be expressed as $R(q)\,dq$ where $q = E'/E$, $R(E'|E) = 0$ if $E < \nu$, *i.e.*, the discrete process is "frozen" when $E < \nu$. This splits the equation (1) into two and defining

$$\pi^{(2)}(E|E_0, t) = \pi(E|E_0, t) \qquad \text{if } E > \nu,$$

$$\pi^{(1)}(E|E_0, t) = \pi(E|E_0, t) \qquad \text{if } E < \nu$$

we obtain the equations

$$\frac{\partial \pi^{(2)}(E|E_0, t)}{\partial t} = -\pi^{(2)}(E|E_0, t)\,\alpha(E) + \int_E^{E_0} \pi^{(2)}(E'|E_0, t)\,R(E|E')\,dE', \quad (E > \nu)$$

$$\frac{\partial \pi^{(1)}(E|E_0, t)}{\partial t} = \int_\nu^{E_0} \pi^{(2)}(E'|E_0, t)\,R(E|E')\,dE', \quad (E < \nu). \tag{21}$$

If

$$\mathbf{R}(t) = \int_0^\epsilon \pi^{(1)}(E|E_0, t)\,dE$$

$$= \int_0^\epsilon dE \int_\nu^{E_0} R(E|E')\,dE' \int_0^t \pi^{(2)}(E'|E_0, t)\,dt \tag{22}$$

$\mathbf{R}(t)$ represents the probability that the stochastic variable assumes a value less than ϵ in time t. Let us obtain $\mathbf{R}(t)$ as t tends to infinity. It is quite clear that as $t \to \infty$

$$\pi^{(2)}(E|E_0, t) \to 0, \qquad (E > \nu) \tag{23}$$

$$\int_0^\nu \pi^{(1)}(E|E_0, t)\,dE \to 1, \quad (E < \nu). \tag{24}$$

We define

$$\Psi(E) = \int_0^\infty \pi^{(2)}(E|E_0, t)\,dt \tag{25}$$

$$\pi^{(1)}(E|E_0, \infty) = \int_\nu^{E_0} \Psi(E')\,R(E|E')\,dE' \tag{26}$$

$$\mathbf{R} = \mathbf{R}(\infty) = \int_0^\epsilon \pi^{(1)}(E|E_0, \infty)\,dE \tag{27}$$

The Solution of the Problem

We shall solve equations (1) and (21) when $R(E|E')\,dE$ is homogeneous in E, E′, *i.e.*, $R(E|E')\,dE = R(q)\,dq$, $q = E/E'$ by defining the Mellin's transformations,

$$p(s,t) = \int_0^\infty \pi(E|E_0, t)\, E^{s-1}\, dE, \quad \omega_s = \int_0^1 R(q)\, q^{s-1}\, dq \tag{28}$$

Equation (1) reduces to (putting without loss of generality, $\alpha = 1$)

$$\frac{\partial p(s,t)}{\partial t} = -p(s,t)(1-\omega_s); \quad p(s,0) = E_0^{s-1} \tag{29}$$

$$p(s,t) = E_0^{s-1} e^{-(1-\omega_s)t} \tag{30}$$

$$\int_0^t p(s,t)\, dt = \frac{E_0^{s-1}}{1-\omega_s}\left(1 - e^{-(1-\omega_s)t}\right) \tag{31}$$

$$\int_0^\infty p(s,t)\, dt = \frac{E_0^{s-1}}{1-\omega_s} \tag{32}$$

$$\Psi(E) = \int_0^\infty \pi(E|E_0, t)\, dt = \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{(E_0/E)^{s-1}}{1-\omega_s}\, ds. \tag{33}$$

We shall now assume

$$R(E|E')\,dE = \frac{dE}{E'(1-r)}, \; R(E|E') = 0 \text{ only if } E' > E > E'r. \tag{34}$$

Thus

$$\omega_s = \int_r^1 q^{s-1} \frac{dq}{1-r} = \frac{1-r^s}{s(1-r)}. \tag{35}$$

Expressing $\dfrac{1}{1-\omega_s}$ as the infinite series $\sum_{m=0}^{\infty} \omega_s^m$ we get

$$\Psi(E) = \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(a^s \sum_{m=0}^{\infty} \omega_s^m \right) ds \text{ where } a = \frac{E_0}{E} \tag{36}$$

Substituting for ω_s from (35) and expanding $(1-r^s)^m$ as a binomial series,

$$\Psi(E) = \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left[a^s \sum_{m=0}^{\infty} \frac{1}{s^m (1-r)^m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} r^{sj} \right] ds \tag{37}$$

Now

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{a^s r^{sj}}{s^m}\,ds = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{s\log ar^j}}{s^m}\,ds \tag{38}$$

We know

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{sz}}{s^m}\,ds = \frac{z^{m-1}}{(m-1)\,!} \quad \text{if } z > 0 \tag{39}$$

and is equal to zero if $z < 0$,

so that the contour integral in (38) is equal to

$$\frac{(\log ar^j)^{m-1}}{(m-1)\,!} \quad \text{if } ar^j > 1,\ i.e.,\ \mathrm{E} < \mathrm{E}_0 r^j \tag{40}$$

and to zero if $\mathrm{E} > \mathrm{E}_0 r^j$.

Hence, if $\mathrm{E}_0 r^{l-1} > \mathrm{E} > \mathrm{E}_0 r^l$ the expression (37) for $\Psi(\mathrm{E})$, on integration, splits up into two parts, for the contour integral in it vanishes for $j \geqslant l$.

Thus

$$\Psi(\mathrm{E}) = \frac{1}{\mathrm{E}_0}\sum_{m=0}^{l-1}\sum_{j=0}^{m}\frac{(-1)^j\binom{m}{j}(\log ar^j)^{m-1}}{(m-1)\,!\,(1-r)^m}$$

$$+\frac{1}{\mathrm{E}_0}\sum_{m=l}^{\infty}\sum_{j=0}^{l-1}\frac{(-1)^j\binom{m}{j}(\log ar^j)^{m-1}}{(m-1)\,!\,(1-r)^m} \tag{41}$$

Putting $(\log ar^j)^{m-1}$ as $(\log a + j\log r)^{m-1}$ and expanding as a binomial series, the first part of expression (41) for $\Psi(\mathrm{E})$ can be put in such a form that by identity (A 16) (*see* Appendix) it can be shown to vanish.

Hence

$$\Psi(\mathrm{E}) = \frac{1}{\mathrm{E}_0}\sum_{m=l}^{\infty}\sum_{j=0}^{l-1}\frac{(-1)^j\binom{m}{j}\left(\log\frac{\mathrm{E}_0 r^j}{\mathrm{E}}\right)^{m-1}}{(m-1)\,!\,(1-r)^m}. \tag{42}$$

This expression can be substituted in (26)

$$\pi^{(1)}(\mathrm{E}|\mathrm{E}_0,\infty) = \int_{r}^{\mathrm{E}/r}\Psi(\mathrm{E}')\frac{d\mathrm{E}'}{\mathrm{E}'(1-r)}. \tag{43}$$

In this integration, because of the discontinuities of the function $\Psi(E)$ as given by (42), two distinct cases arise.

Let

$$E_0 r^{k-1} > \nu > E_0 r^k. \tag{44}$$

Case I. $E_0 r^k > E > E_0 r^{k+1}$

The whole range of integration ν to $\frac{E}{r}$ being in the same domain $\left(E_0 r^{k-1} > \frac{E}{r} > \nu > E_0 r^k\right)$ direct integration is possible, and we have

$$\pi^{(1)}(E|E_0, \infty) = \int_\nu^{E/r} \sum_{m=k}^{\infty} \sum_{j=0}^{k-1} \frac{(-1)^j \binom{m}{j} \left(\log \frac{E_0}{E'} r^j\right)^{m-1}}{E_0 (m-1)!(1-r)^m} \frac{dE'}{E'(1-r)}$$

$$= \frac{1}{E_0(1-r)} \sum_{j=0}^{k-1} \sum_{m=k}^{\infty} \frac{(-1)^j \binom{m}{j}}{m!(1-r)^m} \left[\left(\log \frac{E_0 r^j}{\nu}\right)^m - \left(\log \frac{E_0 r^{j+1}}{E}\right)^m\right], \quad (E_0 r^k > E > E_0 r^{k+1}). \tag{45}$$

Case II. $\nu > E > E_0 r^k$

The range of integration being in two different domains, the discontinuity of $\Psi(E)$ necessitates separate integration for the two parts.

Thus

$$\pi^{(1)}(E|E_0, \infty) = \int_\nu^{E_0 r^{k-1}} \sum_{j=0}^{k-1} \sum_{m=k}^{\infty} \frac{(-1)^j \binom{m}{j} \left(\log \frac{E_0 r^j}{E'}\right)^{m-1}}{E_0 (m-1)!(1-r)^m} \frac{dE'}{E'(1-r)}$$

$$+ \int_{E_0 r^{k-1}}^{E/r} \sum_{j=0}^{k-2} \sum_{m=k-1}^{\infty} \frac{(-1)^j \binom{m}{j} \left(\log \frac{E_0 r^j}{E'}\right)^{m-1}}{E_0 (m-1)!(1-r)^m} \times \frac{dE'}{E'(1-r)} \tag{46}$$

which gives

$$\pi^{(1)}(E|E_0, \infty) = \frac{1}{E_0(1-r)} \sum_{j=0}^{k-1} \sum_{m=k}^{\infty} \frac{(-1)^j \binom{m}{j}}{m!\,(1-r)^m} \left[\left(\log \frac{E_0 r^j}{\nu}\right)^m \right.$$

$$\left. - \left((j-k+1)\log r\right)^m\right] + \frac{1}{E_0(1-r)}$$

$$\times \sum_{j=0}^{k-2} \sum_{m=k-1}^{\infty} \frac{(-1)^j \binom{m}{j}}{m!\,(1-r)^m} \left[\left((j-k+1)\log r\right)^m \right.$$

$$\left. - \left(\log \frac{E_0 r^{j+1}}{E}\right)^m\right] \qquad (47)$$

This can be reduced to

$$\pi^{(1)}(E|E_0, \infty) = \frac{1}{E_0(1-r)} \sum_{j=0}^{k-1} \sum_{m=k}^{\infty} \frac{(-1)^j \binom{m}{j}}{m!\,(1-r)^m} \left(\log \frac{E_0 r^j}{\nu}\right)^m$$

$$- \frac{1}{E_0(1-r)} \sum_{j=0}^{k-2} \sum_{m=k-1}^{\infty} \frac{(-1)^j \binom{m}{j}}{m!\,(1-r)^m}$$

$$\times \left(\log \frac{E_0 r^{j+1}}{E}\right)^m + \frac{(-1)^{k-1}}{E_0(1-r)} \frac{(\log r)^{k-1}}{(1-r)^{k-1}},$$

$$(\nu > E > E_0 r^k). \qquad (48)$$

In making this reduction, identities (A 16) and (A 17) (*see* Appendix) have been used.

Equations (45) and (48) can be put in a more useful form by considering the infinite sum over m as an exponential function minus a finite sum. Then they become

$$\pi^{(1)}(E|E_0, \infty) = \frac{1}{E_0(1-r)} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \left(\frac{\log E_0 r^j/\nu}{1-r}\right)^j \left[\left(\frac{E_0 r^j}{\nu}\right)^{\frac{1}{1-r}} \right.$$

$$\left. - \sum_{m=0}^{k-j-1} \frac{1}{m!} \left(\frac{\log E_0 r^j/\nu}{1-r}\right)^m\right] - \frac{1}{E_0(1-r)}$$

$$\times \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \left(\frac{\log E_0 r^{j+1}/E}{1-r}\right)^j \Bigg[\left(\frac{E_0 r^{j+1}}{E}\right)^{\frac{1}{1-r}}$$

$$- \sum_{m=0}^{k-j-1} \frac{1}{m!} \left(\frac{\log E_0 r^{j+1}/E}{1-r}\right)^m \Bigg], \; (E_0 r^k > E > E_0 r^{k+1}) \tag{49}$$

and

$$\pi^{(1)}(E|E_0, \infty) = \frac{1}{E_0(1-r)} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \left(\frac{\log E_0 r^j/\nu}{1-r}\right)^j \Bigg[\left(\frac{E_0 r^j}{\nu}\right)^{\frac{1}{1-r}}$$

$$- \sum_{m=0}^{k-j-1} \frac{1}{m!} \left(\frac{\log E_0 r^j/\nu}{1-r}\right)^m \Bigg] - \frac{1}{E_0(1-r)}$$

$$\times \sum_{j=0}^{k-2} \frac{(-1)^j}{j!} \left(\frac{\log E_0 r^{j+1}/E}{1-r}\right)^j \Bigg[\left(\frac{E_0 r^{j+1}}{E}\right)^{\frac{1}{1-r}}$$

$$- \sum_{m=0}^{k-j-2} \frac{1}{m!} \left(\frac{\log E_0 r^{j+1}/E}{1-r}\right)^m \Bigg] + \frac{(-1)^{k-1}}{E_0(1-r)}$$

$$\times \frac{(\log r)^{k-1}}{(1-r)^{k-1}}, \; (\nu > E > E_0 r^k) \tag{50}$$

Again using identities (A 16) and (A 17) after putting the finite sums over m in the two expressions in a suitable form, we find that (49) and (50) reduce to

$$\pi^{(1)}(E|E_0, \infty) = \frac{1}{E_0(1-r)} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \left(\frac{E_0 r^j}{\nu}\right)^{\frac{1}{1-r}} \left(\frac{\log E_0 r^j/\nu}{1-r}\right)^j$$

$$- \frac{1}{E_0(1-r)} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \left(\frac{E_0 r^{j+1}}{E}\right)^{\frac{1}{1-r}}$$

$$\times \left(\frac{\log E_0 r^{j+1}/E}{1-r}\right)^j, \; (E_0 r^k > E > E_0 r^{k+1}) \tag{51}$$

and

$$\pi^{(1)}(E|E_0, \infty) = \frac{1}{E_0(1-r)} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \left(\frac{E_0 r^j}{\nu}\right)^{\frac{1}{1-r}} \left(\frac{\log E_0 r^j/\nu}{1-r}\right)^j$$

$$- \frac{1}{E_0(1-r)} \sum_{j=0}^{k-2} \frac{(-1)^j}{j!} \left(\frac{E_0 r^{j+1}}{E}\right)^{\frac{1}{1-r}}$$

$$\times \left(\frac{\log E_0 r^{j+1}/E}{1-r}\right)^j, \ (\nu > E > E_0 r^k). \quad (52)$$

This is the probability distribution of the stochastic variable after the stationary state is attained. From expressions (51) and (52), the probability that the variable assumes a value less than ϵ can be calculated simply by integrating these expressions over E between the limits $r\nu$ (energies below which cannot be reached) and ϵ; or by substracting from unity, the integral of $\pi^{(1)}(E|E_0, \infty)$ over E between the limits ϵ and ν. Both these give identical values since the integral of $\pi^{(1)}(E|E_0, \infty)$ over the whole range $r\nu$ to ν is unity; but for convenience, (51) is integrated by the former method (because $r\nu$ and ϵ being in the same domain, no discontinuities occur in the region of integration), while (52) is integrated by the latter way (here ϵ, ν are in the same domain).

The integration is done easily by making the substitution, $\frac{1}{1-r} \log \frac{E_0 r^{j+1}}{E} = y$ when the integral assumes the form $\int e^{ry} y^j dy$ which can easily be evaluated.

Thus we get the probability **R** in the two cases:

$$R = \frac{\epsilon - r\nu}{\nu(1-r)} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \left(\frac{E_0 r^j}{\nu}\right)^{\frac{r}{1-r}} \left\{\log \left(\frac{E_0 r^j}{\nu}\right)^{\frac{r}{1-r}}\right\}^j$$

$$+ \sum_{j=0}^{k-1} \sum_{l=0}^{j} \frac{(-1)^{j-l}}{(j-l)!} \left[\left(\frac{E_0 r^{j+1}}{\epsilon}\right)^{\frac{r}{1-r}} \left\{\log \left(\frac{E_0 r^{j+1}}{\epsilon}\right)^{\frac{r}{1-r}}\right\}^{j-l}\right.$$

$$\left. - \left(\frac{E_0 r^j}{\nu}\right)^{\frac{r}{1-r}} \left\{\log \left(\frac{E_0 r^j}{\nu}\right)^{\frac{r}{1-r}}\right\}^{j-l}\right], \ (E_0 r^k > \epsilon > E_0 r^{k+1}) \quad (53)$$

and

$$R = 1 - \frac{\nu - \epsilon}{\nu(1-r)} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} \left(\frac{E_0 r^j}{\nu}\right)^{\frac{r}{1-r}} \left\{\log\left(\frac{E_0 r^j}{\nu}\right)^{\frac{r}{1-r}}\right\}^j$$

$$- \sum_{j=0}^{k-2} \sum_{l=0}^{j} \frac{(-1)^{j-l}}{(j-l)!} \left[\left(\frac{E_0 r^{j+1}}{\nu}\right)^{\frac{r}{1-r}} \left\{\log\left(\frac{E_0 r^{j+1}}{\nu}\right)^{\frac{r}{1-r}}\right\}^{j-l}\right.$$

$$\left. - \left(\frac{E_0 r^{j+1}}{\epsilon}\right)^{\frac{r}{1-r}} \left\{\log\left(\frac{E_0 r^{j+1}}{\epsilon}\right)^{\frac{r}{1-r}}\right\}^{j-l}\right], (\nu > \epsilon > E_0 r^k) \qquad (54)$$

The above mathematical problem arose in connection with the theoretical treatment of the energy distribution of recoil atoms in liquids produced as a consequence of the emission of gamma-rays of substantial momentum due to the radio-active capture of neutrons by nuclei. The physical process, according to Libby, Miller and others (Libby, 1947; Miller, Gryder and Dodson, 1950; Capron and Oshima, 1952) can be briefly described as follows:

(i) Every (n, γ) process produces a free recoil atom with the same kinetic energy E_0. The kinetic energy of the atom is such that the atom cannot participate in stable chemical combination until it has lost energy.

(ii) Energy loss occurs by elastic "billiard ball collisions" with the solvent atoms.

(iii) If sufficient energy is transferred in a given collision the struck molecule will be dissociated into free radicals. In case the recoil atom has a kinetic energy less than a critical amount ϵ after this impact it will be trapped in a liquid cage of free radicals.

(iv) We assume that sufficient energy is not transferred if the energy is below a critical amount ν, $(\nu > \epsilon)$ to produce dissociation. *The effect of assumption (iv) is that only those atoms which 'drop' to an energy below ϵ from an energy state greater than ν, are 'trapped' while atoms with energy between ν and ϵ which drop to an energy below ϵ escape the cage and move freely.*

Our object is to find **R** the ratio of the trapped recoil atoms to the total number of recoil atoms in the stationary state. **R** is given by (53) and (54).

APPENDIX

We define coefficients A_k^r ($r \geqslant k$ always) obeying the relations

$$A_k^n = \frac{A_k^{n-1}}{q_n - q_k} \tag{A 1}$$

$$A_k^k = -[A_0^k + A_1^k + \ldots\ldots + A_{k-1}^k], \quad A_0^0 = 1 \tag{A 2}$$

From (A 1) and (A 2)

$$(-1)^k A_k^k = -\Big[\frac{A_0^0}{(q_0 - q_k)(q_0 - q_{k-1})\ldots\ldots(q_0 - q_1)} - \frac{A_1^1}{(q_1 - q_k)(q_1 - q_{k-1})\ldots\ldots(q_1 - q_2)} + \ldots\ldots + (-1)^{k-1}\frac{A_{k-1}^{k-1}}{q_{k-1} - q_k}\Big] \tag{A 3}$$

Substituting in (A 3) for A_{k-1}^{k-1} only, an expression similar to (A 3), and after reducing, substituting for A_{k-2}^{k-2} alone and so on, by successive reduction we get

$$(-1)^k A_k^k = \frac{1}{(q_k - q_{k-1})(q_k - q_{k-2})\ldots\ldots(q_k - q_0)} \tag{A 4}$$

and

$$(-1)^n A_k^n = \frac{1}{(q_k - q_n)(q_k - q_{n-1})\ldots(q_k - q_{k+1})(q_k - q_{k-1})(q_k - q_{k-2})\ldots(q_k - q_0)} \tag{A 5}$$

The sum

$$\sum_{k=0}^{n} A_k^n q_k^m = \sum_{k=0}^{n} A_k^n (q_k^{m-1})(q_k - q_n) + q_n \sum_{k=0}^{n} A_k^n q_k^{m-1} \tag{A 6}$$

i.e.,

$$-\sum_{k=0}^{n} A_k^n q_k^m = \sum_{k=0}^{n-1} A_k^{n-1} q_k^{m-1} - q_n \sum_{k=0}^{n} A_k^n q_k^{m-1}$$

$$= \sum_{k=0}^{n-1} A_k^{n-1} q_k^{m-1} + q_n \sum_{k=0}^{n-1} A_k^{n-1} q_k^{m-2} + \ldots\ldots + q_n^{m-1} \sum_{k=0}^{n-1} A_k^{n-1}$$

$$= \sum_{r=0}^{m-1} K_r^1 \sum_{k=0}^{n-1} A_k^{n-1} q_k^{m-1-r} \quad \text{where} \quad K_r^1 = q_n^r \tag{A 7}$$

Using relation (A 6) this can be reduced to

$$-\sum_{r=0}^{m-2} K_r^2 \sum_{k=0}^{n-2} A_k^{n-2} q_k^{m-2-r}, \quad \text{where} \quad K_r^2 = \sum_{p=0}^{r} K_p^1 q_{n-1}^{r-p} \quad \text{(A 7 } a\text{)}$$

This can be further reduced so that in general, we can write

$$(-1)^i \sum_{k=0}^{n} A_k^n q_k^m = \sum_{r=0}^{m-i} K_r^i \sum_{k=0}^{n-i} A_k^{n-i} q_k^{m-i-r} \quad \text{(A 8)}$$

where i may vary from 0 to m or n, whichever is smaller, and

$$K_r^i = K_r^{i-1} + K_{r-1}^i \, q_{n-i+1} \quad \text{(A 9)}$$

from which we get

$$K_r^i = \sum_{p=0}^{i} K_{r-1}^p q_{n+1-p} \quad \text{(A 10)}$$

and

$$K_r^i = \sum_{p=0}^{r} K_p^{i-1} \, q_{n-i+1}^{r-p} \quad \text{(A 11)}$$

From (A 8), we see that if $m < n$, putting $i = m$ in the right-hand side of (A 8) it reduces to $K_0^m \sum_{k=0}^{m-n} (-1)^{m-n} A_k^{m-n}$ which is zero by (A 2). Hence

$$\sum_{k=0}^{n} A_k^n q_k^m = 0, \quad (m < n) \quad \text{(A 12)}$$

When $m = n$, putting $i = m = n$ in (A 8), we get, since $K_0^i = 1$ for all i,

$$(-1)^n \sum_{k=0}^{n} A_k^n q_k^m = 1, \quad (m = n) \quad \text{(A 13)}$$

For $m > n$, we can put $i = n$ which then gives

$$(-1)^n \sum_{k=0}^{n} A_k^n q_k^m = \sum_{r=0}^{m-n} K_r^n q_0^{m-n-r}, \quad (m > n). \quad \text{(A 14)}$$

Relations (A 12) and (A 13) are of direct application in the problem we are considering, where the coefficients A_k^n are of a particular form, with

$$q_k = k \text{ so that } (-1)^n A_k^n = \frac{(-1)^{n-k}}{(n-k)!\,k!} \quad \text{(A 15)}$$

Thus we have as particular cases of (A 12) and (A 13),

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!\,k!} k^m = 0 \qquad (m < n) \tag{A 16}$$

and

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!\,k!} k^m = 1 \qquad (m = n) \tag{A 17}$$

(A 16) and (A 17) have been used directly in the problem.

REFERENCES

1. Alladi Ramakrishnan .. *Ap. J.*, 1952, **115**, 141.
2. ——— .. *Proc. Camb. Phil. Soc.*, 1953, **49**, 473.
3. Chandrasekhar, S. and Munch, G. .. *Ap. J.*, 1950, **112**, 380, 393.
4. ——— .. *Ibid.*, 1951, **114**, 110.
5. Libby, W. F. .. *J. Am. Chem. Soc.*, 1947, **69**, 2523.
6. Miller, Gryder and Dodson .. *J. Chem. Phys.*, 1950, **18**, 579.
7. Capron, P. C. and Oshima, Y. .. *Ibid.*, 1952, **20**, 1403.