# STRAGGLING OF THE RANGE OF FAST PARTICLES AS A STOCHASTIC PROCESS*

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## Statement of the Problem

We shall adopt the following stochastic model to represent the well-known phenomenon of straggling of the range of fast particles in passing through matter.

Considering only one-dimensional penetration of matter by fast particles we assume that a particle of energy E at $t$ ($t$ being the thickness of matter traversed) loses energy in two ways:—

1. Deterministically by an amount $\beta(E)\, dt$ in passing through matter of thickness $dt$. Such a loss is attributed to ionisation of the atoms of matter.

2. By the random process of radiation of photons (Bremmstrahlung). A particle of energy E drops to an energy interval $(E', E' + dE')$ by radiating a photon of energy $E - E'$ in passing through matter of thickness $dt$ with probability $R(E' | E)\, dE'\, dt$.

We now define absorption in the following manner. If a particle drops to an energy below a critical energy $E_c$ between $t$ and $t + dt$, we say it is absorbed and its range lies between $t$ and $t + dt$. Our problem is to determine the probability distribution of the range, given the initial energy (*i.e.*, at $t = 0$) of the particle.

## Formal Solution of the Problem

The first step is to write the stochastic equation for the probability $\pi(E | E_0;\, t)\, dE$ of the energy of the particle having a value between E and $E + dE$ at $t$, given that the initial energy was $E_0$. This has been already done by the authors in an earlier contribution to these *Proceedings* (1953). The equation is

$$\frac{\partial \pi(E | E_0;\, t)}{\partial t} = -\pi(E | E_0;\, t) \int_0^E R(E' | E)\, dE' + \int_E^{E_0} \pi(E' | E_0;\, t)\, R(E | E')\, dE'. \tag{1}$$

* Read at the Annual Meeting of the Indian Academy of Sciences at Belgaum in December 1954.

Using our definition of absorption we find that the probability that the particle is absorbed within a distance $t$ can be formally written as

$$\mathrm{P}(\mathrm{E}_c;\ t) = \int_0^{\mathrm{E}_c} \pi(\mathrm{E}|\ \mathrm{E}_0;\ t)\, d\mathrm{E} = 1 - \int_{\mathrm{E}_c}^{\mathrm{E}_0} \pi(\mathrm{E}|\ \mathrm{E}_0;\ t)\, d\mathrm{E} \tag{2}$$

and it immediately follows that the probability that the particle is absorbed between $t$ and $t + dt$ is $\frac{\partial \mathrm{P}(\mathrm{E}_c;t)}{\partial t}\, dt$ *i.e.*, $\frac{\partial \mathrm{P}\ (\mathrm{E}_c;\ t)}{\partial t}$ is the probability frequency function of the range of the particle.

To proceed further and obtain the explicit solution of equation (1) and to determine the distribution of the range, it is necessary to make simplifying assumptions regarding the form of the cross-section $\mathrm{R}(\mathrm{E}'|\ \mathrm{E})$ and of the function $\beta(\mathrm{E})$. In the case when $\mathrm{R}(\mathrm{E}'|\ \mathrm{E})\, d\mathrm{E}'$ can be expressed as $\mathrm{R}(q)\, dq$, $(q = \mathrm{E}'|\mathrm{E})$ and $\beta(\mathrm{E})$ is of the form $\beta \mathrm{E}^k$, equation (1) has been reduced to a difference-differential equation by the application of the Mellin's transform technique. However the solution of the transform equation is still difficult and therefore the authors made the further assumptions that $\mathrm{R}(q') = \delta(q' - q)$, $\delta$ being the Dirac delta function, and $\beta(\mathrm{E}) = \beta$, a constant for $\mathrm{E} > 0$ and $\beta(\mathrm{E}) = 0$ for $\mathrm{E} = 0$. $p(s;\ t)$, the Mellin's transform of $\pi$ satisfies the equation

$$\frac{\partial p(s;\ t)}{\partial t} = -\,a\, p(s;\ t) + \omega(s)\, p(s;\ t) + \beta(s-1)\, p(s-1;\ t) \tag{3}$$

where

$$a = \int_0^1 \mathrm{R}(q)\, dq,\quad \omega(s) = \int_0^1 \mathrm{R}(q)\, q^{s-1}\, dq$$

The solution of this equation was obtained earlier by the authors by an indirect procedure. The equation with a negative sign for the last term on the right was first solved, since in this case the stochastic variate cannot assume the value zero and therefore no complication due to the delta function singularity at zero energy arises. This result was used to obtain the solution for equation (1) (see Ramakrishnan and Mathews, 1953). However Srinivasan** has shown recently that the solution of the integro-differential equation for $\pi$ may be obtained directly if a Laplace transformation instead of Mellin's is employed, though it would appear to be unsuitable at first sight. The solution for $\pi$ is

$$\pi(\mathrm{E}|\ \mathrm{E}_0;\ t) = \sum_{n=0}^{\infty} \pi_n(\mathrm{E}|\ \mathrm{E}_0;\ t) \tag{4}$$

** Private communication. A brief outline of his derivation is given in the Appendix.

where

$$\pi_0(\mathrm{E}|\,\mathrm{E}_0;\ t) = e^{-at}\delta(\mathrm{E} - \mathrm{E}_0 + t)$$

and

$$\pi_n(\mathrm{E}|\,\mathrm{E}_0;\ t) = -\frac{a^n e^{-at}}{(n-1)!}\sum_{k=0}^{m-1} \mathrm{A}_k{}^n (\mathrm{E}_0 q^n - \mathrm{E} - tq^k)^{n-1},$$

$$n \geqslant 1,\ \mathrm{E} > 0,\ tq^m < \mathrm{E}_0 q^n - \mathrm{E} < tq^{m-1}. \tag{5}$$

We have put $\beta$ equal to unity without loss of generality by suitably choosing the unit of E and $\mathrm{A}_k{}^n$ is given by

$$\mathrm{A}_k{}^n = \prod_{\substack{j=0\\ j\neq k}}^{n} (q^j - q^k)^{-1} \tag{6}$$

Since we have assumed $\beta$ to be independent of E for $\mathrm{E} > 0$, it is clear from the very nature of the problem that there is a finite probability of the particle having an energy exactly equal to zero [*i.e.*, there exists a delta function singularity for $\pi$ $(\mathrm{E}|\,\mathrm{E}_0;\ t)$ at $\mathrm{E} = 0$], and this is the probability that the particle is absorbed, provided we consider absorption to take place as soon as the energy is reduced to zero‡. This probability has been shown to be

$$\mathrm{P}\ (0;\ t) = \sum_{n=0}^{\infty} \mathrm{P}\,(0,\ n;\ t),$$

$$\mathrm{P}\,(0,\ 0;\ t) = e^{-t}\ \mathrm{H}\ (t - \mathrm{E}_0)$$

$$\mathrm{P}\,(0,\ n;\ t) = \frac{e^{-t}\,t^n}{n!}\sum_{k=0}^{r-1} \mathrm{A}_k{}^n\ (q^n\,\mathrm{E}_0/t - q^k)^n,\ \ (tq^r < \mathrm{E}_0 q^n < tq^{r-1}) \tag{7}$$

where $a$ also has been put equal to unity by a proper choice of the unit of $t$.

Numerical evaluation of this expression has been done for various values of the initial energy $\mathrm{E}_0$, with $q = \frac{1}{2}$. In the energy loss of fast electrons, the fraction of energy lost by the electron in each radiative collision is usually between $\frac{1}{3}$ and $\frac{2}{3}$ (see for example, Heitler, 1943). So the value we have chosen corresponds fairly to physical facts. The results are tabulated below and also illustrated in a graph.

‡ It is necessary to assume a non-zero value for $\mathrm{E}_c$ if the mechanism of energy loss is such that the energy of the particle approaches zero only asymptotically, for example when $\beta(\mathrm{E})$ is of the form $\beta \mathrm{E}^k$, $k > 0$. But in our model the constant rate of loss causes the energy to drop to zero within a distance $\mathrm{E}_0/\beta$, and so the range can be defined as the distance in which the energy drops to zero.

TABLE I

P (0, *t*)

| $t$ \ $E_0$ | 5 | 10 | 20 | 50 | 100 | 500 | 1000 | 10000 |
|---|---|---|---|---|---|---|---|---|
| 1 | ·024 | ·003 | | | | | | |
| 2 | ·381 | ·139 | ·038 | ·005 | ·001 | | | |
| 3 | ·814 | ·489 | ·232 | ·066 | ·021 | ·001 | | |
| 4 | ·960 | ·784 | ·523 | ·232 | ·106 | ·011 | ·003 | |
| 5 | | ·924 | ·757 | ·463 | ·271 | ·050 | ·020 | ·001 |
| 6 | | ·980 | ·894 | ·673 | ·474 | ·137 | ·068 | ·004 |
| 8 | | ·999 | ·985 | ·912 | ·799 | ·433 | ·289 | ·044 |
| 10 | | | ·998 | ·982 | ·944 | ·724 | ·585 | ·181 |

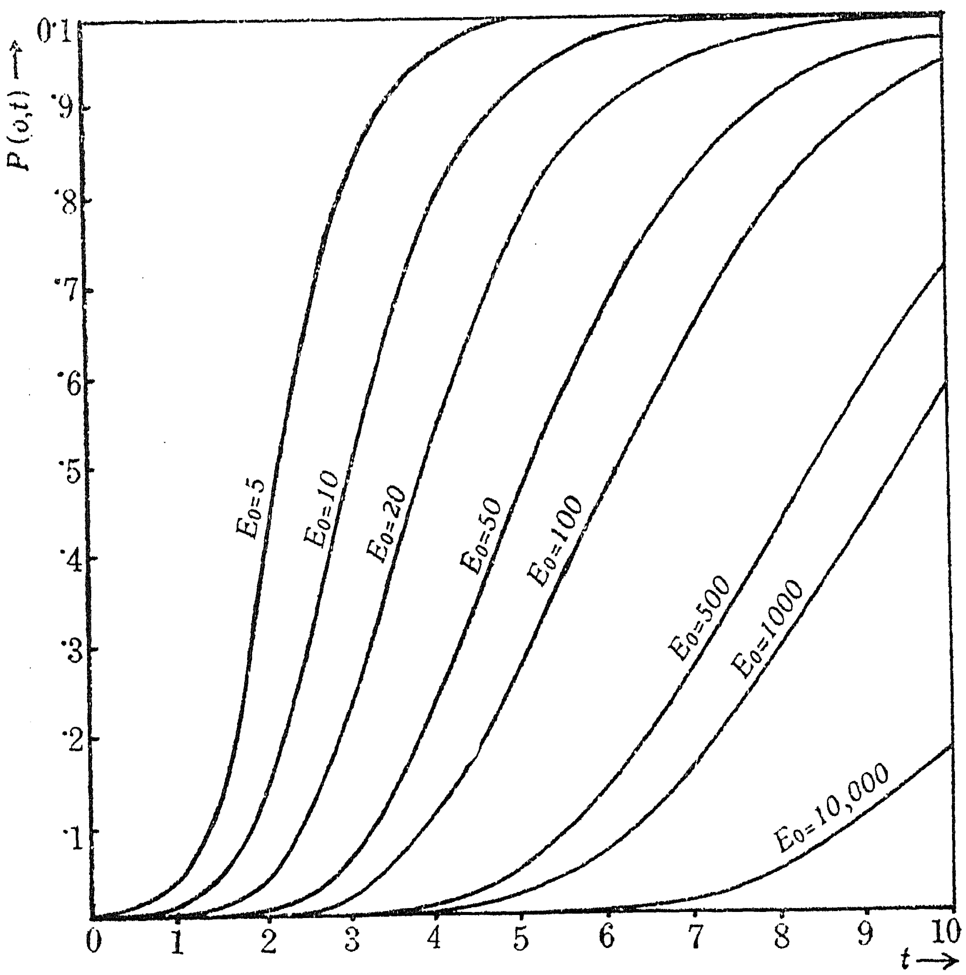


Fig. 1. P (0; *t*), the probability that the particle has been absorbed between 0 and *t*.

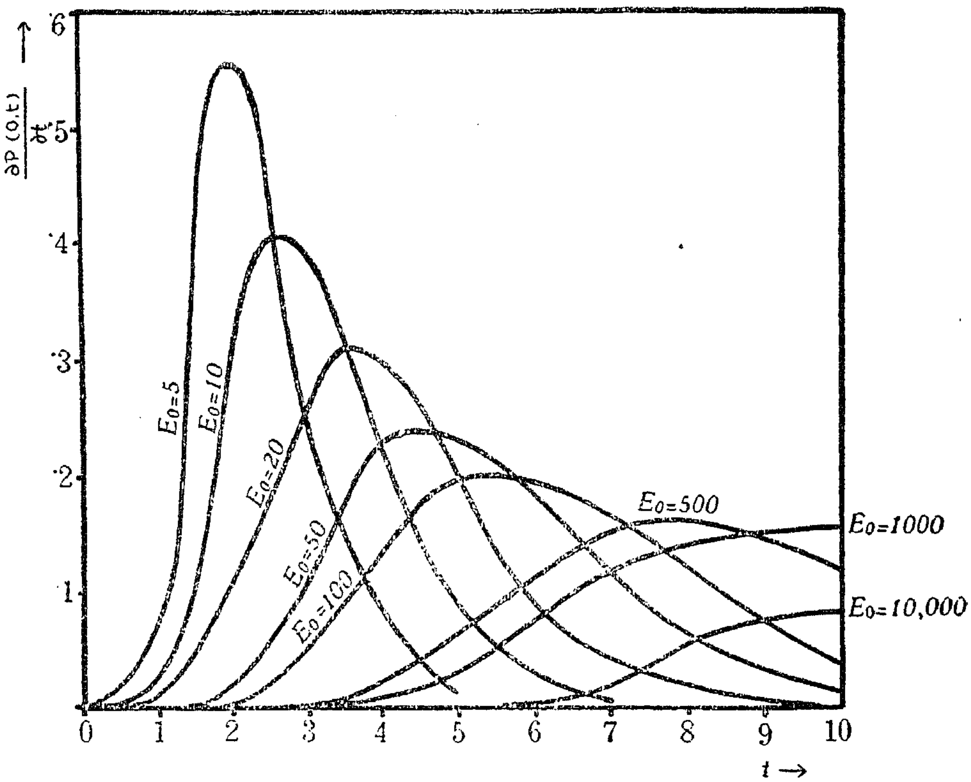


Fig. 2. $\partial P(0;\ t)/dt$, the probability frequency function of the range $t$.

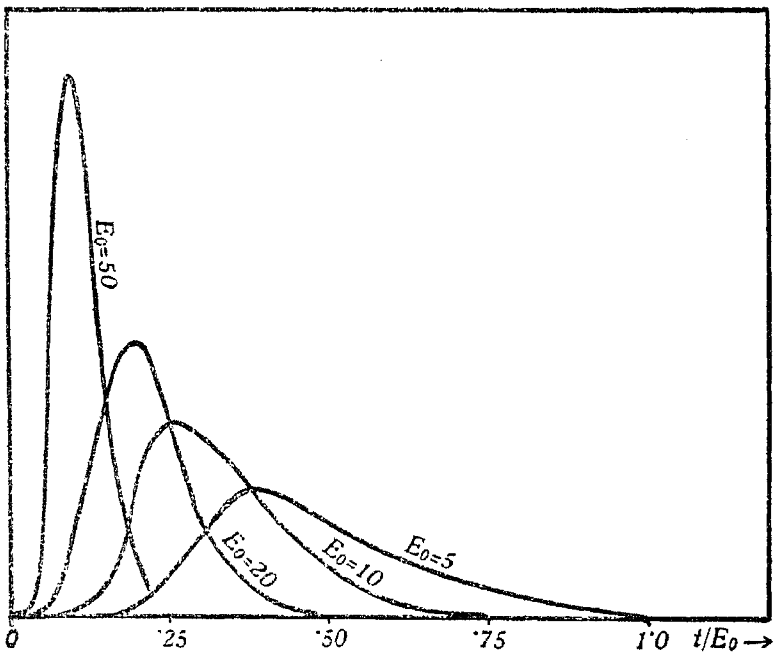


Fig. 3. The probability frequency function of the random variable $t/E_0$, $t$ being the range.

The probability frequency function of the range, as mentioned before, is $\partial P/\partial t$ and the moments of the range R of the particle are given by

$$\epsilon\{R^r\} = \int_0^{E_0} t^r \frac{\partial P}{\partial t}\, dt, \quad r = 1, 2, \ldots. \tag{8}$$

We have not computed any of the moments, as the distribution itself is known.§ The graph of $\partial P/\partial t$, obtained by graphical differentiation of the $P - t$ curves is given to show the form of the range distribution. If $\partial P/\partial t$ is directly plotted against $t$, the curve for a given initial value of the energy $E_0$ extends from $t = 0$ to $t = E_0$, and the curves for the larger values of $E_0$ appear to be flatter than those for small values. But a plot of the distribution of $t/E_0$ gives a much better idea of the sharpness of the distribution. It is clearly seen that the distribution rapidly becomes sharp for increasing values of the initial energy $E_0$. One feature that is easily noticed from the curves is that the mean range $\bar{R}$ does not increase in proportion to the energy. In fact, $\bar{R}/E_0$ decreases as $E_0$ increases.

## Janossy's Simple Alternative Stochastic Model of Straggling

It is interesting to compare our model with a simpler alternative model discussed by Janossy (1950) in his book on *Cosmic Rays*.

A particle undergoes a collision with probability $\alpha dt$ in traversing material of thickness $dt$ and loses energy of magnitude $\epsilon$ per collision. Thus if $E_0$ is its initial energy, it is absorbed if it undergoes N collisions where N is an integer such that $(N - 1)\,\epsilon < E_0 \leqslant N\epsilon$. The probability that the Nth collision occurs between $t$ and $t + dt$ is given by the Poisson law.

$$P(N;\, t)\, dt = e^{-\alpha t} \frac{(\alpha t)^{N-1}}{(N-1)!}\, \alpha dt \tag{9}$$

and this is the distribution of the range. It is interesting to note that for very large N the distribution tends to a Gaussian one with respect to $t$ in the neighbourhood of the maximum which is very sharp and occurs at $t = (N-1)/\alpha$.

Normally, when we speak of a Gaussian approximation to a Poisson distribution we mean that $e^{-\alpha t}(\alpha t)^N/N\,!$ assumes a Gaussian form with respect to N if $\alpha t$ is very large. Here in (9) we are speaking of a distribution

§ Usually in stochastic problems the expressions for the first few moments are easily obtained in simple form, while the frequency function itself may be very complicated. In such cases, the moments are computed to get an approximate idea of the distribution. But the present case is among the few where the expressions for the moments are no simpler than the expression for the frequency function.

with respect to $t$ and hence it seems worthwhile to indicate the derivation of the asymptotic Gaussian form for large N.

Putting at

$$\alpha t = N - 1 + \alpha\tau$$

we have

$$\frac{e^{-\alpha t}(\alpha t)^{N-1}}{(N-1)!} = e^{-(N-1)}\frac{(N-1)^{N-1}}{(N-1)!}\left(1 + \frac{\alpha\tau}{N-1}\right)^{N-1} e^{-\alpha\tau} \tag{10}$$

We immediately note that for large N,

$$\frac{e^{-(N-1)}(N-1)^{N-1}}{(N-1)!} \sim \frac{1}{\sqrt{2\pi}(N-1)}$$

and

$$e^{-\alpha\tau}\left(1 + \frac{\alpha\tau}{N-1}\right)^{N-1} \sim e^{-(\alpha\tau)^2/2(N-1)}$$

in the region $\alpha\tau << N$. When $\alpha\tau$ is comparable to $N - 1$ (10) becomes vanishingly small. Thus (9) tends to a Gaussian distribution with the mean $(N - 1)/\alpha$.

## APPENDIX

In a private communication, S. K. Srinivasan has shown that it is possible to obtain $\pi(\mathrm{E}|\,\mathrm{E}_0;\, t)$ directly by applying a Laplace transformation to the differential equation satisfied by $\pi_n(\mathrm{E}|\,\mathrm{E}_0;\, t)$ where $\pi_n(\mathrm{E}|\,\mathrm{E}_0;\, t)\; d\mathrm{E}$ is the joint probability that the particle has undergone $n$ radiative collisions in the interval $(0, t)$ and its energy is between E and E $+ d$E at $t$. The equation for $\pi_n$ is

$$\frac{\partial \pi_n(\mathrm{E};\, t)}{\partial t} = -\, a\pi_n(\mathrm{E};\, t) + \frac{a}{q}\pi_{n-1}\left(\frac{\mathrm{E}}{q};\, t\right) + \frac{\partial}{\partial \mathrm{E}}\{\beta(\mathrm{E})\,\pi_n(\mathrm{E};\, t)\} \tag{11}$$

with $\beta(\mathrm{E}) = \beta = 1$ for $\mathrm{E} > 0$ and $\beta(\mathrm{E}) = 0$ for $\mathrm{E} = 0$,

and

$$\pi_0(\mathrm{E};\, t) = e^{-at}\,\delta(\mathrm{E} - \mathrm{E}_0 + t).$$

A Laplace transformation now yields

$$\frac{\partial p_n(s;\, t)}{\partial t} = -\, ap_n(s;\, t) + ap_{n-1}(sq;\, t) + s\, p_n(s;\, t) \tag{12}$$

when it is borne in mind that $\beta(\mathrm{E}) = 0$ at $\mathrm{E} = 0$, so that the term $[\beta(\mathrm{E})\,\pi_n(\mathrm{E};\, t)e^{-s\mathrm{E}}]_{\mathrm{E}=0}$ which occurs in the Laplace transform of the last term of equation (11) vanishes. It is interesting to note that it is this peculiar feature of the function $\beta(\mathrm{E})$ that renders the solution possible. Otherwise this extra term introduces complications which cannot easily be resolved.

A further Laplace Transform with respect to $t$ reduces the equation to a simple iterative relation

$$\rho_n(s;\, r) = \frac{a}{a - s + r}\,\rho_{n-1}(sq;\, r) \tag{13}$$

which immediately gives

$$\rho_n(s;\, r) = a^n\, e^{-s\mathrm{E}_0 q^n} \prod_{k=0}^{n} \frac{1}{a + r - sq^k} \tag{14}$$

The solution for $\pi$ can be obtained from this at once by inversion.

### References


1. Heitler, W. .. *Quantum Theory of Radiation* (Oxford, 1943).
2. Janossy, L. .. *Cosmic Rays* (Oxford, 1950).
3. Alladi Ramakrishnan and P. M. Mathews .. *Proc. Ind. Acad. Sci.*, 1953, **38**, 450.