ORDINARY LINEAR DIFFERENTIAL EQUATIONS INVOLVING RANDOM FUNCTIONS

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Abstract

Physical processes which can be represented by symbolic differential equations involving random functions are cited and studied. The solutions of these equations are obtained using Ramakrishnan's recent phenomenological interpretation of integrals of random functions.

In this paper, we study various physical processes that can be represented by ordinary differential equations involving random functions of the parameter with respect to which the differential coefficients are defined. Such a study has been made possible in view of Ramakrishnan's recent phenomenological interpretation of integrals of random functions (Ramakrishnan, 1955).

The present method, based on that interpretation, is applicable to processes represented by stochastic variables which satisfy differential equations involving random functions, provided the solution of the equations can be put in the form of simple or iterated integrals of these random functions. Such iterated integrals have already been discussed earlier by Ramakrishnan (1955) and the authors (1955) without any special reference to differential equations. In spite of the risk of possible repetition in some cases, it is considered worthwhile to deal with these processes defined by solutions of linear differential equations.

General Method of Solution

Let us consider the linear differential equation of m-th order,

$$\frac{d^m \mathrm{Y}}{dt^m} + \lambda_1(t)\frac{d^{m-1}\mathrm{Y}}{dt^{m-1}} + \ldots + \lambda_m(t)\,\mathrm{Y} = \kappa(t)\,\mathrm{X}(t), \tag{1}$$

where $\lambda_i(t)$'s and $\kappa(t)$ are fully determinate functions of t, and $\mathrm{X}(t)$ is a random function of t. The problem of studying the distribution of Y given the distribution of X is indeed a very difficult one. However, the problem can be treated quite satisfactorily, if we are able to obtain the solution of (1) in the form

$$Y = \phi_m(t) \int_0^t \phi_{m-1}(t_{m-1})\, dt_{m-1} \int_0^{t_{m-1}} \phi_{m-2}(t_{m-2})\, dt_{m-2} \ldots \int_0^{t_1} \phi_0(t_0)\, \kappa(t_0)\, X(t_0)\, dt_0. \tag{2}$$

The passage from (1) to (2) for any general m is by itself a difficult problem except in the special case when all the λ's are constants and in this paper an attempt is made for the case $m = 2$ (see Appendix A). It is interesting to note that the solutions of a number of equations of type (1) representing physical processes are capable of being expressed in the form (2).

Y as given by (2) is an m-th iterated integral of a random function, and the normal interpretation of such integrals given recently by Ramakrishnan is based on the concept of the " trajectory " or curve of growth of a stochastic process. In any typical realisation of the process $X(\tau)$, we ' plot ' the value of the random function $X(\tau)$ in the interval (o, t) and thus obtain the realised trajectory. By a simple extension of the theory of Riemann integration, the realised value of the symbolic integral $\int_0^t X(\tau)\, d\tau$ corresponding to the given realisation of $X(\tau)$ is now defined as the area enclosed by the curve $X(\tau)$, the τ-axis and the ordinates at $\tau = 0$ and $\tau = t$. Iterated integrals of $X(\tau)$ can be defined similarly. If we wish to obtain the probability frequency function of $\int_0^t X(\tau)\, d\tau$ [or of iterated integrals of $X(\tau)$] we have to assign a probability measure to the trajectory of $X(\tau)$ in the interval (o, t) and this is known to be a very difficult mathematical problem. However, if we confine ourselves to a simple random function $X(\tau)$ which represents a " basic random process "—a Markoff process, homogeneous with respect to τ, whose typical trajectory is characterised by a finite number of discrete transitions, the trajectory remaining parallel to the τ axis between two transitions—it is possible to assign a measure to its trajectory. Many physical processes whose trajectories are continuous can therefore be studied by considering the stochastic variables involved as iterated integrals of a basic random process or by approximating the variables to such integrals. Even if $X(t)$ is not a basic random process, we can readily obtain the moments and correlation functions of iterated integrals of $X(t)$ by the use of a theorem due to Ramakrishnan (1954 *a*).

The random variable Y as defined in (2) has been dealt with by the authors (1955) for the special case, $X(\tau) = dn(\tau)/d\tau$ where $n(\tau)$ represents a Poisson process, *i.e.*, $n(\tau)$ has value n with probability $e^{-\lambda\tau}(\lambda\tau)^n/n!$

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where λ is the parameter of the process. $dn(\tau)/d\tau$ in such a case has a specialised meaning which will be explained presently. The Laplace transform solution of the p.f.f. of Y has been explicitly obtained in that paper and here we shall make use of the solution to obtain the moments in particular cases of physical interest.

LINEAR DIFFERENTIAL EQUATION OF THE FIRST ORDER

Let us consider a stochastic variable $Y(t)$ (t is the parameter with respect to which the physical process we have to deal with progresses) defined by

$$\frac{dy}{dt} + \lambda_1(t)\,Y = \kappa(t)\,X(t), \tag{3}$$

where $\lambda_1(t)$ and $\kappa(t)$ are fully determinate functions of t, and $\kappa(t)$ is a random variable representing a process progressing with t. Our object is to obtain the p.f.f. of $Y(t)$, or its moments—at least the first few—given the nature of the process $X(t)$. The first step consists in formally writing the solution of (3) as

$$Y(t) - Y_0 e^{-\mu(t)} = Y'(t) = \int_0^t e^{-\{\mu(t)-\mu(\tau)\}}\,\kappa(\tau)\,X(\tau)\,d\tau, \tag{4}$$

as if $X(t)$ were a determinate function. $\mu(t)$ is given by

$$\mu(t) = \int_0^t \lambda_1(\tau)\,d\tau \tag{5}$$

The initial condition is defined as $Y(t) \equiv Y_0$ at $t = 0$.

We shall now consider particular cases of $X(t)$.

(i) $X(t) = \dfrac{dn(t)}{dt}$ *where $n(t)$ represents a Poisson Process.*

It is well known that corresponding to a realisation of events at $t_1, t_2, \ldots, t_n$ along the t-axis, the realised value of $dn(t)/dt$ is given by

$$\sum_i \delta(t - t_i)$$

where δ is the Dirac-delta function. The distribution of $Y'(t) = Y - Y_0 e^{-\mu(t)}$ has been studied by the authors (1955).

Defining the Laplace transform $p(s;t)$ of $\pi(Y';t)$, the probability frequency function of $Y'(t)$, as

$$p(s;t) = \int_0^\infty e^{-sy'}\,\pi(Y';t)\,dY', \tag{6}$$

the solution for $p(s;t)$ was obtained there as

$$p(s;t) = \exp\left[-\lambda t + \lambda \int_0^t \exp\{-s\kappa(\tau)\, e^{-[\mu(t)-\mu(\tau)]}\}\, d\tau\right]. \tag{7}$$

Moments of $Y - Y_0 e^{\mu(t)}$ and hence those of Y can be obtained from (7).

An example of such a process is provided by the fluctuation of voltage at the anode of a thermionic valve due to the fluctuations in the number of electrons per unit time emitted by the cathode (Moyal, 1950). If we assume the probability of emission of an electron per unit time to be constant, we may reasonably expect the number $n(t)$ of electrons emitted in the interval $(0, t)$ to be governed by the Poisson distribution. The fluctuating voltage $V(t)$ across C satisfies the equation

$$\frac{dV}{dt} + \frac{V}{RC} = -\frac{\epsilon}{C}\frac{dn}{dt} \tag{8}$$

where we have assumed that the circuit between anode and earth is equivalent to a resistance R in parallel with a capacity C. ϵ is the charge of the electron. Substituting the values of $\lambda(t)$, $\kappa(t)$ (which are constants in this particular case) in (7) the Laplace transform solution of $\pi(V;t)$ the p.f.f. of V is obtained as

$$p(s;t) = \exp\left[-n_0 t + n_0 \int_0^t \exp\left\{\frac{s\epsilon}{C}\, e^{-(t-\tau)/RC}\right\} d\tau\right], \tag{9}$$

where n_0 is the mean number of electrons emitted per unit time.

The moments are given by

$$E\{V(t) - V_0 e^{-t/RC}\} = -n_0 \epsilon R(1 - e^{-t/RC}),^* \tag{10}$$

$$E\{[V(t) - V_0 e^{-t/RC}]^2\} = n_0^2 \epsilon^2 R^2 (1 - e^{-t/RC})^2 + \frac{n_0 \epsilon^2 R}{2C}(1 - e^{-2t/RC}), \tag{11}$$

$$E\{[V(t) - V_0 e^{-t/RC}]^m\} = n_0 \sum_{r=0}^{m-1} \left(-\frac{\epsilon}{C}\right)^{m-r} E\{[V(t) - V_0 e^{-t/RC}]^r\} \frac{(1 - e^{t(m-r)/RC})RC(m-1)!}{(m-r-1)!\, r!} \tag{12}$$

(ii) $X(t)$ represents a 'Q' process.

* Throughout this paper E denotes expectation value.

A 'Q' process is defined as follows: Consider the distribution of random points on a line representing the t-axis, according to a Poisson distribution with parameter λ. Let the value q_i be associated with the i-th point, all the q_i's having the same probability frequency function $\phi(q)$. Then the Q-process is defined by the stochastic variable

$$Q(t) = q_1 q_2 \ldots . q_n \tag{13}$$

which is associated with the interval $(0, t)$ when n points are realised in $(0, t)$.

By the application of the theorem established by Ramakrishnan (1954 *a*), the n-th moment of $Y - Y_0 e^{-\mu(t)}$ as defined in (4) when $X(t) \equiv Q(t)$ is given by

$$E\{(Y - Y_0 e^{-\mu(t)})^n\} = \int_0^t \int_0^t \ldots . \int_0^t E\{Q(\tau_1) Q(\tau_2) \ldots . Q(\tau_n)\}$$

$$\kappa(\tau_1) \kappa(\tau_2) \ldots . \kappa(\tau_n) \exp[-n\mu(t) + \mu(\tau_1) + \mu(\tau_1) + \ldots . +$$

$$+ \mu(\tau_n)] \, d\tau_1 \, d\tau_2 \ldots . d\tau_n, \tag{14}$$

where

$$E\{Q(\tau_1) Q(\tau_2) \ldots . Q(\tau_n)\} = \exp[-\lambda\{(1 - q_n)\tau_1 + (1 - q_{n-1})$$

$$(\tau_2 - \tau_1) + \ldots . + (1 - q_1)(\tau_n - \tau_{n-1})\}],$$

$$(0 < \tau_1 < \tau_2 < \ldots . < \tau_n), \tag{15*}$$

$$q_n = E\{q^n\} = \int_q q^n \phi(q) \, dq. \tag{16}$$

Such a Q process occurs for example in statistical astronomy. If we assume that interstellar matter consists of discrete clouds, q being the transparency factor of a single cloud, then Q is the cumulative transparency factor of an aggregate of clouds occurring in a Poisson manner. If it is further assumed that the amount of light radiation from an element of length $d\tau$ at τ along the line of sight of an observer at $t = 0$ is $d\tau$ (*i.e.*, a uniform distribution of sources exists), then $Y(t)$, the total intensity reaching the observer from the system extending from 0 to t satisfies equation (3), with $\lambda_1(t) = 0$, $\kappa(t) = 1$ and $Y(0) \equiv 0$. The moments of Y have been obtained by Ramakrishnan (1954 *b*). The correlation function of Y of degree n is given by

$$E\{Y(t_1) Y(t_2) \ldots . Y(t_n)\}$$

$$= \int_0^{t_1} \int_0^{t_2} \ldots . \int_0^{t_n} E\{Q(\tau_1) Q(\tau_2) \ldots . Q(\tau_n)\} \, d\tau_1 \, d\tau_2 \ldots . d\tau_n. \tag{17}$$

* For a derivation of this result, see Alladi Ramakrishnan (1954 *b*).

We note that the explicit expression (15) for the correlation function can be written down only after the variables are ordered. In the integration of the expression (17), we have to take into consideration the relative order of the n variables $\tau_1, \tau_2, \ldots, \tau_n$. Hence in the process of integration, the range of integration of each of the variables is split up into intervals $(0, t_1)$, $(t_1, t_2), \ldots, (t_{n-1}, t_n)$; $(t_1 < t_2 < \ldots < t_n)$, and the resulting n-fold integrals computed in the ranges one by one. Use is also made of the following well-known lemma on ordering of the variables of integration.

$$\int_0^t \int_0^t \ldots \int_0^t f(X_1, X_2, \ldots, X_n)\, dX_1\, dX_2 \ldots dX_n$$

$$= n!\int_0^t dX_1 \int_{X_1}^t dX_2 \ldots \int_{X_{n-1}}^t f(X_1, X_2, \ldots, X_n)\, dX_n, \tag{18}$$

where f is symmetrical in all the variables $X_1, X_2, \ldots, X_n$. The correlation functions of degree two and three have been evaluated by this method. These are given in a paper by Alladi Ramakrishnan and S. K. Srinivasan (1956) for the particular case when $\phi(q') = \delta(q' - q)$, but the same expressions are valid for any general $\phi(q)$ if q^r is replaced by q_r everywhere in them.

(iii) *$X(t)$ represents a fluctuating density field : (F.D.F.).*

Recently, the concept of a wildly fluctuating field has been introduced by Chandrasekhar and Munch (1952) and studied in great detail by Ramakrishnan (1954). Here we recall some of the results obtained by Ramakrishnan. Let $\pi(X' \mid X; |t' - t|)dX'$ be the conditional probability that the density $X(t)$ (considered as a stochastic variable) has a value X' at t' given that it had a value X at t; we assume that $\pi(X' \mid X; |t' - t|) \to R(X' \mid X) \cdot |t' - t|$ as $|t' - t| \to 0$ and also that $R(X' \mid X)$ is a function of X' alone, and write $R(X' \mid X) = R(X')$. Ramakrishnan has proved that

$$\pi(X' \mid X; |t' - t|) = \frac{R(X')}{a}\left(1 - e^{-a|t'-t|}\right) + \delta(X' - X)\, e^{-a|t'-t|}, \tag{19}$$

where

$$a = \int_{X'} R(X')\, dX' \tag{20}$$

Note as $|t' - t| \to \infty$, $\pi(X' \mid X; |t' - t|) \to \dfrac{R(X')}{a} = \psi(X')$. (21)

The concept of wild fluctuation is introduced by making a very large. In this case $\pi \to \psi$ for $|t' - t| >> 1/a$ and hence the distribution at t' is effectively independent of the distribution at t. But it is to be noted that however large a may be, the densities are not independent in the range $0 < |t' - t| < 1/a$. Hence in any process of integration over the t-space (this has to be carried out if we desire to obtain the moments of integrals associated with X) a correlation exists and contributes a correction term of the order of $1/a$ when a is very large. The correction term is taken care of by replacing $e^{-a(t'-t)}$ by $1/a\ \delta(t' - t)$ in (19).

We now consider physical processes represented by (3) when $X(t)$ represents an F.D.F. For a free particle (of mass m, velocity v) the equation of motion is

$$m \frac{dv}{dt} + fv = F(t), \tag{22}$$

where f is the frictional force and $F(t)$ is an F.D.F. defining the random force (Wang and Uhlenbeck, 1945). Again, the current i in an R — L circuit satisfies a similar equation. If a body of thermal capacity C is connected to its surroundings by a thermal resistance R, the temperature difference θ between the body and the surroundings (which are supposed to form a "heat-bath") satisfies the equation (MacDonald, 1948–49)

$$C \frac{d\theta}{dt} + \frac{1}{R}\ \theta = H(t), \tag{23}$$

where $H(t)$ is an F.D.F. defining the flow of heat.

Investigations of these processes by the workers mentioned above and by others are confined only to the first two moments in the stationary case (*i.e.*, $t \to \infty$).

All these processes will now be studied by taking $\lambda_1(t)$ and $\kappa(t)$ as constants in (3). Then (4) takes the form

$$Y - Y_0 e^{-\lambda_1 t} = \kappa \int_0^t e^{-\lambda_1 (t-\tau)} X(\tau)\, d\tau. \tag{24}$$

As we are interested in the stationary case, *i.e.*, $t \to \infty$, the second term on the LHS can be neglected. The n-th moment of Y is given by

$$E\{Y^n\} = \lim_{t\to\infty} \kappa^n \int_0^t \int_0^t \ldots\ldots \int_0^t e^{-\lambda_1 (nt-\tau_1-\tau_2-\ldots-\tau_n)}$$

$$E\{X(\tau_1)\, X(\tau_2) \ldots\ldots X(\tau_n)\}\, d\tau_1\, d\tau_2 \ldots d\tau_n \tag{25}$$

By definition,

$$E\{X(\tau_1)X(\tau_2)\ldots X(\tau_n)\} = \int_{X_n}\int_{X_{n-1}}\ldots\int_{X_1} X_1 X_2 \ldots X_n$$

$$\pi(X_n, X_{n-1}, \ldots, X_1 \mid X_0\,;\, \tau_n, \tau_{n-1}, \ldots, \tau_1)\, dX_1\, dX_2 \ldots dX_n$$

$$= \int_{X_n}\int_{X_{n-1}}\ldots\int_{X_1} X_1 X_2 \ldots X_n\, \pi(X_n \mid X_{n-1};\, \tau_n - \tau_{n-1})$$

$$\pi(X_{n-1} \mid X_{n-2};\, \tau_{n-1} - \tau_{n-2}) \ldots$$

$$\ldots \pi(X_1 \mid X_0;\, \tau_1)\, dX_1\, dX_2 \ldots dX_n. \qquad (26)^*$$

Now, the exponential factor in the integral in (25) vanishes as $t \to \infty$ except for values of $\tau_1, \tau_2, \ldots, \tau_n$ which keep $(nt - \tau_1 - \tau_2 - \ldots - \tau_n)$ finite; hence the whole contribution to the integral comes from the region where values of $\tau_1, \tau_2, \ldots, \tau_n$ all tend to infinity. The differences $\tau_i - \tau_{i-1}$ $(i = 1, 2, 3, \ldots, n)$ are all however considered finite. Consequently in the expression (26) for $E\{X(\tau_1)X(\tau_2)\ldots X(\tau_n)\}$ which is to be fed into (25), we put

$$\pi(X_1 \mid X_0;\, \tau_1) = \psi(X_1), \qquad (27)$$

since only values of τ_1 tending to infinity are significant for our purpose, as explained above, while for the remaining factors $\pi(x_i \mid x_{i-1};\, \tau_i - \tau_{i-1})$ we have to substitute from (19) because $\tau_i - \tau_{i-1}$ is finite.

We then have,

$$E\{X(\tau_1)X(\tau_2)\ldots X(\tau_n)\} = \int_{X_1}\int_{X_2}\ldots\int_{X_n} X_1X_2\ldots X_n \Big[\psi(X_1)\,\psi(X_2)\ldots$$

$$\ldots\psi(X_n)\Big\{1 - \frac{1}{a}\sum_i \delta(\tau_i - \tau_{i-1})\Big\} + \frac{1}{a}\sum_i \frac{\psi(X_1)\,\psi(X_2)\ldots\psi(X_n)}{\psi(X_i)}$$

$$\delta(X_i - X_{i-1})\,\delta(\tau_i - \tau_{i-1}) + O\left(\frac{1}{a^2}\right)\Big]\, dX_1\, dX_2 \ldots dX_n. \qquad (28)$$

Introducing the expression (28) into the right hand side of (25), we obtain after some calculation

$$E\{Y^n\} = \Big[E\{Y\}\Big]^n + \Big[E\{Y\}\Big]^{n-2} \frac{n(n-1)\kappa}{2a\lambda_1}\Big[E\{X^2\} - (E\{X\})^2\Big] \qquad (29)$$

where

$$E\{X^n\} = \int_X X^n\, \psi(X)\, dX. \qquad (30)$$

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* In this paper the symbol $\int_X$ means integration over the whole range of the variable X.

In particular when $n = 2$,

$$E\{Y^2\} = \left[E\{Y\}\right]^2 + \frac{\kappa}{a\lambda_1}\left[E\{X^2\} - (E\{X\})^2\right]. \tag{31}$$

(29) can be written as

$$E\{Y^n\} = \left[E\{X\}\right]^n + \frac{n(n-1)}{2}\left[E\{Y^2\} - (E\{Y\})^2\right](E\{Y\})^{n-2}. \tag{32}$$

LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER

We proceed to discuss the case where $Y(t)$ satisfies the second order linear differential equation

$$\frac{d^2Y}{dt^2} + \lambda_1(t)\frac{dY}{dt} + \lambda_2(t)\,Y = \kappa(t)\,X(t), \tag{33}$$

where $\lambda_1(t)$, $\lambda_2(t)$ and $\kappa(t)$ are fully determinate functions of t and $X(t)$ is a random function as before. The solution of (33) can always be put in the form (see Appendix A)

$$Y = e^{-\frac{1}{2}\mu(t)}\int_0^t e^{-\{\alpha_2(t)-\alpha_2(\tau)\}}\,d\tau\int_0^\tau e^{+\{\alpha_2(\tau)-\alpha_2(\tau')\}}\,\kappa(\tau')\,X(\tau')\,d\tau' \tag{34}$$

under proper initial conditions.

We will now consider some particular cases of the function $X(t)$.

(i) $X(t) = \dfrac{dn(t)}{dt}$ *where $n(t)$ is a random variable representing a Poisson Process.*

This has been dealt with by the authors earlier (1955); the Laplace transform of $\pi(Y; t)$ is given by

$$p(s; t) = \exp\left[-\lambda t + \lambda\int_0^t \exp\{-s f(t,\tau)\,\kappa(\tau)\}\,d\tau\right] \tag{35}$$

where

$$f(t,\tau) = \exp\left[-\tfrac{1}{2}\mu(t) - \alpha_2(t) - \alpha_2(\tau)\right]\int_\tau^t \exp\left[2\alpha_2(\tau')\right]d\tau'. \tag{36}$$

The moments of Y of any order can be readily obtained from (35).

(ii) $X(t)$ *represents a 'Q' process,* i.e., $X(t) = Q(t)$.

It is convenient to express Y as a single iterated integral. This can be achieved by changing the order of integration of τ and τ' on the R.H.S. of (34) and performing the integration over τ. Thus we obtain

$$Y = \int_0^t \kappa(\tau) f(t, \tau) X(\tau)\, d\tau, \tag{37}$$

where $f(t, \tau)$ is defined in (36).

The n-th moment of Y is given by an expression similar to (14).

A special case of interest is when $\lambda_1(t) \equiv \lambda_2(t) \equiv 0$ and $\kappa(t) = 1$, *i.e.*, $f(t, \tau) = t - \tau$. The first few moments of Y are then given by

$$E\{Y(t)\} = \frac{t}{\lambda(1-q_1)} - \frac{1 - e^{-\lambda(1-q_1)t}}{\lambda^2(1-q_1)^2}, \tag{38}$$

$$\begin{aligned} E\{Y^2(t)\} = 2\Big[&\frac{1}{\lambda(1-q_1)}\left\{\frac{t^2}{\lambda(1-q_2)} - \frac{2t}{\lambda^2(1-q_2)^2} + 2\cdot\frac{1-e^{-\lambda(1-q_2)t}}{\lambda^3(1-q_2)^3}\right\} \\ &- \frac{1}{\lambda^2(1-q_1)^2}\left\{\frac{t}{\lambda(1-q_2)} - \frac{1-e^{-\lambda(1-q_2)t}}{\lambda^2(1-q_2)^2}\right\} \\ &+ \frac{1}{\lambda^2(1-q_1)^2}\left\{\frac{t\,e^{-\lambda(1-q_1)t}}{\lambda(q_1-q_2)} - \frac{e^{-\lambda(1-q_1)t}}{\lambda^2(q_1-q_2)^2} + \frac{e^{-\lambda(1-q_2)t}}{\lambda^2(q_1-q_2)^2}\right\}\Big], \end{aligned} \tag{39}$$

$$\begin{aligned} E\{Y^3(t)\} = 6\Big[&\frac{1}{\lambda^2(1-q_1)(1-q_2)}\left\{\frac{t^3}{\lambda(1-q_3)} - \frac{3t^2}{\lambda^2(1-q_3)^2} + \frac{6t}{\lambda^3(1-q_3)^3} - 6\cdot\frac{1-e^{-\lambda(1-q_3)t}}{\lambda^4(1-q_3)^4}\right\} \\ &- \left\{\frac{1}{\lambda^3(1-q_1)^2(1-q_2)} + \frac{2}{\lambda^3(1-q_1)(1-q_2)^2}\right\} \\ &\left\{\frac{t^2}{\lambda(1-q_3)} - \frac{2t}{\lambda^2(1-q_3)^2} + 2\cdot\frac{1-e^{-\lambda(1-q_3)t}}{\lambda(1-q_3)^3}\right\} + \frac{1}{\lambda^3(1-q_1)^2(q_1-q_2)} \\ &\left\{\frac{t^2 e^{-\lambda(1-q_1)t}}{\lambda(q_1-q_3)} - \frac{2t\,e^{-\lambda(1-q_1)t}}{\lambda^2(q_1-q_3)^2} + 2\frac{e^{-\lambda(1-q_1)t}}{\lambda^3(q_1-q_3)^3} - 2\frac{e^{-\lambda(1-q_3)t}}{\lambda^3(q_1-q_3)^3}\right\} \\ &+ \left\{\frac{2}{\lambda^4(1-q_1)(1-q_2)^3} + \frac{1}{\lambda^4(1-q_1)^2(1-q_2)^2}\right\}\left\{\frac{t}{\lambda(1-q_3)} - \frac{1-e^{-\lambda(1-q_3)t}}{\lambda^2(1-q_3)^2}\right\} \\ &- \left\{\frac{2}{\lambda^4(1-q_1)(1-q_2)^3} + \frac{1}{\lambda^4(1-q_1)^2(1-q_2)^2} - \frac{1}{\lambda^4(1-q_1)^2(q_1-q_2)^2}\right\} \\ &\left\{\frac{t\,e^{-\lambda(1-q_2)t}}{\lambda(q_2-q_3)} - \frac{e^{-\lambda(1-q_2)t} - e^{-\lambda(1-q_3)t}}{\lambda^2(q_2-q_3)^2}\right\} - \frac{1}{\lambda^4(1-q_1)^2(q_1-q_2)^2} \\ &\left\{\frac{t\,e^{-\lambda(1-q_1)t}}{\lambda(q_1-q_3)} - \frac{e^{-\lambda(1-q_1)t} - e^{-\lambda(1-q_3)t}}{\lambda^2(q_1-q_3)^2}\right\}\Big]. \end{aligned} \tag{40}$$

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Ramakrishnan (1955) has proved that in this case $\pi(Y; t)$, the p.f.f. of Y satisfies the equation

$$\frac{\partial \pi(Y; t)}{\partial t} = -\lambda \pi(Y; t) + \lambda \int \pi\left(\frac{Y}{q}; t\right) \frac{\phi(q)\, dq}{q} - t \frac{\partial \pi(Y; t)}{\partial Y}. \quad (41)$$

If $\rho(s; t)$ be the Mellin's transform of $\pi(Y; t)$ with respect to Y then $\rho(s; t)$ satisfies the equation

$$\frac{\partial \rho(s; t)}{\partial t} = -\lambda \rho(s; t) + \lambda \rho(s; t) q^{s-1} + t(s-1)\rho(s-1; t). \quad (42)$$

The n-th moment of Y is given by $\rho(n+1; t)$. The first three moments of Y have also been calculated from (42) and are found to be identical with (38), (39) and (40) as expected.

(iii) $X(t)$ *represents an F.D.F.*

We can use (28) and the n-th moment of Y is given by an expression similar to (25). Physical processes represented by (33) when $X(t)$ is an F.D.F. can easily be cited. The distance s traversed by a Brownian particle (of mass m) is determined by the equation (Wang and Uhlenbeck, 1945)

$$m \frac{d^2 s}{dt^2} + f \frac{ds}{dt} = F(t), \quad (43)$$

or, considering an example from electricity, the total charge K in an R-L circuit satisfies the equation (McCombie, 1953)

$$L \frac{d^2 K}{dt^2} + R \frac{dK}{dt} = E(t). \quad (44)$$

Again let us consider the fluctuation of θ, the torsion of a suspended coil galvanometer. The equations governing the process are [Jones and McCombie (1952)]

$$I \frac{d^2\theta}{dt^2} + k \frac{d\theta}{dt} + c\theta = F(t) + Gi, \quad (45)$$

$$L \frac{di}{dt} + Ri = E(t) - G \frac{d\theta}{dt}, \quad (46)$$

where I is the moment of inertia of the suspended system, k and c are the damping and torsional constants and G is the flux linkage of the coil. $F(t)$ and $E(t)$ in (45) and (46) are F.D.F.'s defining the random couple and random voltage respectively. If the inductance is negligible, we obtain

$$I \frac{d^2\theta}{dt^2} + \left(k + \frac{G^2}{R}\right) \frac{d\theta}{dt} + c\theta = F(t) + \frac{G}{R} E(t). \quad (47)$$

It is assumed that the random couple is independent of that arising from the random e.m.f. in the coil so that E (t) and F (t) are uncorrelated. Hence (47) is of the same type as (33).

We can study (43), (44) and (47) by assuming that λ_1, λ_2 and κ are constants. Note that the procedure adopted in the Appendix is not necessary. If $-a_1$ and $-a_2$ are the roots of the equation

$$X^2 + \lambda_1 X + \lambda_2 = 0, \tag{48}$$

then

$$Y = \frac{\kappa}{a_1 - a_2} \int_0^t X(\tau)\left(e^{-a_1(t-\tau)} - e^{-a_2(t-\tau)}\right) d\tau. \tag{49}$$

The real parts of a_1, a_2 are all positive in the physical examples considered above. (Otherwise the differential equation would imply instability).

The moments of Y can be obtained by the same technique as we have adopted before. In this case, it is not possible to obtain an explicit expression for the n-th moment. Nevertheless the first few moments can always be obtained without much difficulty and we give below the second and third moments.

$$E\{Y^2\} = \left[E\{Y\}\right]^2 + \frac{\kappa^2}{a}\,\frac{E\{X^2\} - [E\{X\}]^2}{\lambda_2\lambda_1}, \tag{50}$$

$$E\{Y^3\} = \left[E\{Y\}\right]^3 + 3\kappa^2 E\{Y\}\,\frac{E\{X^2\} - [E\{X\}]^2}{a\lambda_2\lambda_1}$$

$$= [E\{Y\}]^3 + 3E\{Y\}(E\{Y^2\} - [E\{Y\}]^2). \tag{51}$$

To obtain the correlation function of Y of degree two we write $a_1 = \xi + i\eta$, $a_2 = \xi - i\eta$, ξ being positive. Then (49) becomes

$$Y(t) = \frac{1}{\eta}\int_0^t X(\tau)\, e^{-\xi(t-\tau)} \sin\eta(t-\tau)\, d\tau. \tag{52}$$

We wish to obtain $E\{Y(t_1)\,Y(t_2)\}$ when $t_1 \to \infty$ and $t_2 \to \infty$ in such a manner that $t_2 - t_1 =$ a constant $\equiv b$.

$E\{Y(t_1)\,Y(t_2)\}$ is given by

$$E\{Y(t_1)\,Y(t_2)\} = \frac{1}{\eta^2}\int_0^{t_1}\int_0^{t_2} e^{-\xi\{(t_1-\tau_1)+(t_2-\tau_2)\}} \sin\eta(t_1-\tau_1)$$

$$\sin\eta(t_2-\tau_2)\, E\{X(\tau_1)\,X(\tau_2)\}\, d\tau_1 d\tau_2. \tag{53}$$

Ramakrishnan (1954 *a*) has shown that

$$E\{X(\tau_1)\,X(\tau_2)\} = [E\{X\}]^2\left[1 - \frac{1}{a_1}\,\delta(|\,\tau_2 - \tau_1\,|)\right] + \frac{1}{a}\,E\{X^2\}\,\delta(|\,\tau_2 - \tau_1\,|). \tag{54}$$

[This is indeed an immediate deduction from (28)]. If we now denote the R.H.S. of (54) by $R(\,|\,\tau_2 - \tau_1\,|\,)$ and introduce it into the R.H.S. of (53), the double integral can be reduced to a single integral (see Appendix B), and we have,

$$\lim_{\substack{t_1\to\infty,\ t_2\to\infty \\ t_2-t_1=b}} E\{Y(t_1)\,Y(t_2)\} = \frac{1}{\xi\eta\sqrt{\xi^2+\eta^2}}\int_{-\infty}^{\infty} R(b-\tau)\,e^{-\xi|\tau|} \sin\left(\eta\,|\,\tau\,| + \tan^{-1}\frac{\eta}{\xi}\right)d\tau. \tag{55}$$

Using the expression (54) for R, we obtain

$$\lim E\{Y(t_1)\,Y(t_2)\} = \frac{E\{X^2\} - [E\{X\}]^2}{a\,\xi\eta\,(\xi^2+\eta^2)}\,e^{-\xi b}\,(\eta\cos b\eta + \xi\sin b\eta), \tag{56}$$

a result obtained by Wang and Uhlenbeck using spectral theory.

LINEAR DIFFERENTIAL EQUATION OF *m*-TH ORDER

We shall finally consider the *m*-th order differential equation given by (1). We shall be concerned with the case where the solution is given by (2). When $X(t) = dn(t)/dt$ explicit Laplace Transform solution of the p.f.f. of Y has been obtained by the authors. Note that the R.H.S. of (2) can be telescoped into a single integral so that (2) becomes

$$Y = \phi_m(t)\int_0^t \phi_0(t_0)\,\kappa(t_0)\,X(t_0)\,f(t, t_0)\,dt_0, \tag{57}$$

where

$$f(t, t_0) = \int_{t_0}^{t}\phi_1(t_1)\,dt_1\int_{t_1}^{t}\phi_2(t_2)\,dt_2\ldots\int_{t_{m-2}}^{t}\phi_{m-1}(t_{m-1})\,dt_{m-1}. \tag{58}$$

Written in this form, it is quite easy to see how the problem of obtaining the moments of Y could be dealt with. A formal expression for the *n*-th moment of Y is given by

$$E\{Y^n\} = [\phi_m(t)]^n \int_0^t \int_0^t \ldots \int_0^t \phi_0(\tau_1)\,\phi_0(\tau_2)\ldots\phi_0(\tau_n)\,\kappa(\tau_1)\,\kappa(\tau_2)\ldots$$
$$\ldots\kappa(\tau_n) f(t, \tau_1) f(t, \tau_2)\ldots f(t, \tau_n)\, E\{X(\tau_1)\,X(\tau_2)\ldots X(\tau_n)\}$$
$$d\tau_1\, d\tau_2 \ldots d\tau_n. \qquad (59)$$

In the theory of servo-mechanisms, the output Y and the input X are related by (2). If the input is *white noise*, *i.e.*, F.D.F. (as is usually the case) the moments of Y can be readily obtained. Jones and McCombie have recently discussed the Brownian fluctuation of galvanometers and galvanometer amplifiers. These processes are represented by differential equations of the type (1) and order higher than two. All the results of Jones and McCombie are capable of extension by the present method of approach.

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References

1. Chandrasekhar, S. .. *Rev. Mod. Phys.*, 1943, **15**, 1.
2. ——— and Munch, G. .. *Astrophys. J.*, 1952, **115**, 103.
3. Jones, R. W. and McCombie, C. W. *Phil. Trans.*, 1952, **244 A**, 208.
4. Mathews, P. M. and Srinivasan, S. K. *Proc. Nat. Inst. Sci., India*, 1955 (in Press).
5. McDonald, D. K. C. .. *Rep. Prog. Phys.*, 1948–49, **12**, 56.
6. Moyal, J. E. .. *J. Roy. Statist. Soc.*, 1949, **11 B**, 150.
7. Ramakrishnan, Alladi .. *Proc. Kon. Ned. Akad. van Wet.*, 1955 (in Press).
8. ——— .. *Astrophys. J.*, 1954 *a*, **119**, 443.
9. ——— .. *Ibid.*, 1954 *b*, **119**, 682.
10. ——— and Srinivasan, S. K. *Ibid.*, 1956 (in Press).
11. Wang, Ming Chen and Uhlenbeck, G. E. *Rev. Mod. Phys.*, 1945, **17**, 323.

APPENDIX A

Our object is to resolve the L.H.S. of (33) into operational factors. We make the transformation

$$Y = Z\, e^{-\frac{1}{2}\mu(t)} \tag{A–1}$$

where $\mu(t) = \int_0^t \lambda_1(\tau)\, d\tau$, (33) becomes

$$\frac{d^2 z}{dt^2} + P(t)\, z = \kappa(t)\, X(t), \tag{A–2}$$

where $P(t)$ is given by

$$P(t) = -\frac{1}{4}\left[\lambda_1(t)\right]^2 - \frac{1}{2}\frac{d\lambda_1(t)}{dt} + \lambda_2(t). \tag{A–3}$$

Next we use the identity

$$\left[\frac{d}{dt} - \alpha_1(t)\right]\left[\frac{d}{dt} + \alpha_1(t)\right] z$$

$$\equiv \frac{d^2 Z}{dt^2} + Z\{\alpha_1'(t) - [\alpha_1(t)]^2\}. \tag{A–4}$$

Hence if α_1 is a solution of the Riccati equation

$$\frac{d\alpha}{dt} - \alpha^2 = P(t), \tag{A–5}$$

we can rewrite (A–2) as

$$\left[\frac{d}{dt} - \alpha_1(t)\right]\left[\frac{d}{dt} + \alpha_1(t)\right] Z = \kappa(t)\, X(t). \tag{A–6}$$

The solution is therefore given by

$$Z = \int_0^t e^{-\alpha_2(t) + \alpha_2(\tau)}\, d\tau \int_0^\tau e^{+\{\alpha_2(\tau) - \alpha_2(\tau')\}}\, \kappa(\tau')\, X(\tau')\, d\tau' \tag{A–7}$$

under proper initial conditions. Thus Y is given by

$$Y = e^{-\frac{1}{2}\mu(t)} \int_0^t e^{-\alpha_2(t) + \alpha_2(\tau)}\, d\tau \int_0^\tau e^{+\{\alpha_2(\tau) - \alpha_2(\tau')\}}\, \kappa(\tau')\, X(\tau')\, d\tau'. \tag{A–8}$$

In equations (A–7) and (A–8) above, $\alpha_2(t) = \int_0^t \alpha_1(\tau)\, d\tau$.

APPENDIX B

We wish to reduce the R.H.S. of (53) to a single integral when $t_1 \to \infty$ and $t_2 \to \infty$ in such a manner that $t_2 - t_1 =$ a constant $= b$. For finite τ_1 and τ_2, $e^{-\xi\{(t_1-\tau_1)+(t_2-\tau_2)\}}$ tends to zero as t_1 and t_2 tend to infinity, and hence the whole contribution to the integral comes from values of τ_1 and τ_2 for which $t_1 - \tau_1 + t_2 - \tau_2$ is finite as $t_1, t_2 \to \infty$. The integral can be split up into two parts, in one of which $\tau_2 > \tau_1$, while in the other $\tau_2 < \tau_1$. We then obtain,

$$E\{Y(t_1)\,Y(t_2)\} = A + B \tag{B-1}$$

where

$$A = \frac{\kappa^2}{\eta^2}\int_0^{t_1} d\tau_1 \int_{\tau_1}^{t_2} e^{-\xi(t_1-\tau_1+t_2-\tau_2)} \sin\eta(t_1-\tau_1) \sin\eta(t_2-\tau_2)\,R(\tau_2-\tau_1)\,d\tau_2 \tag{B-2}$$

and

$$B = \frac{\kappa^2}{\eta^2}\int_0^{t_1} d\tau_2 \int_{\tau_2}^{t_1} e^{-\xi(t_1-\tau_1+t_2-\tau_2)} \sin\eta(t_1-\tau_1) \sin\eta(t_2-\tau_2)\,R(\tau_1-\tau_2)\,d\tau_1 \tag{B-3}$$

These double integrals can be reduced to single integrals by changing the order of integration. We can then write A as

$$A = \tfrac{1}{2}\int_0^{b} d\tau \int_0^{t_1} e^{-\xi(t_1+t_2-2\tau_1-\tau)}\,R(\tau)[\cos\eta(\tau-b) - \cos\eta(t_1+t_2-2\tau_1-\tau)]$$

$$d\tau_1 + \tfrac{1}{2}\int_b^{t_2} d\tau \int_0^{t_2-\tau} e^{-\xi(t_1+t_2-2\tau_1-\tau)}\,R(\tau)\,[\cos\eta(\tau-b)$$

$$- \cos\eta\,(t_1+t_2-2\tau_1-\tau_1)]\,d\tau. \tag{B-4}$$

Performing the integration we obtain,

$$A = \frac{1}{2}\int_0^{b} R(\tau)\Big[\cos\eta(\tau-b)\cdot\frac{1}{2\xi}\{e^{-\xi(b-\tau)} - e^{-\xi(t_1+t_2-\tau)}\}$$

$$- \frac{1}{4(\xi^2+\eta^2)}\,e^{-\xi(b-\tau)}\,\{2\xi\cos\eta(b-\tau) - 2\eta\sin\eta\,(t_1+t_2-\tau)\}$$

$$+ \frac{1}{4(\xi^2+\eta^2)}\,e^{-\xi(t_1+t_2-\tau)}\,\{2\xi\cos\eta(t_1+t_2-\tau) - 2\eta$$

$$\sin\eta\,(t_1+t_2-\tau)\}\Big]\,d\tau$$

$$+\frac{1}{2}\int_{b}^{t_2} R(\tau)\Big[\cos\eta(t-b)\cdot\frac{1}{2\xi}\{e^{-\xi(\tau-b)}-e^{-\xi(t_1+t_2-\tau)}\}$$

$$-\frac{1}{4(\xi^2+\eta^2)}e^{-\xi(\tau-b)}\{2\xi\cos\eta(\tau-b)-2\eta\sin\eta(\tau-b)\}$$

$$+\frac{1}{4(\xi^2+\eta^2)}e^{-\xi(t_1+t_2-\tau)}\{2\xi\cos\eta(t_1+t_2-\tau)-2\eta$$

$$\sin\eta(t_1+t_2-\tau)\}\Big]\,d\tau. \qquad \text{(B-5)}$$

Passing to the limit, we have

$$\lim A = \frac{\eta}{4\xi\sqrt{\xi^2+\eta^2}}\int_0^b R(\tau)\sin\left\{\eta(b-\tau)+\sin^{-1}\frac{\xi}{\sqrt{\xi^2+\eta^2}}\right\}e^{-\xi(b-\tau)}\,d\tau$$

$$+\frac{\eta}{4\xi\sqrt{\xi^2+\eta^2}}\int_b^{\infty} R(\tau)\sin\left\{\eta(\tau-b)+\sin^{-1}\frac{\eta}{\sqrt{\xi^2+\eta^2}}\right\}e^{-\xi(\tau-b)}\,d\tau. \qquad \text{(B-6)}$$

Repeating the same process for B, we obtain

$$\lim B = \lim\frac{1}{2}\int_0^{t_1} R(\tau)\Big[\cos\eta(b+\tau)\cdot\frac{1}{2\xi}\{e^{-\xi(b+\tau)}-e^{-\xi(t_1+t_2-\tau)}\}$$

$$-\frac{1}{4(\xi^2+\eta^2)}e^{-\xi(b+\tau)}\{2\xi\cos\eta(b+\tau)-2\eta\sin\eta(b+\tau)\}$$

$$+\frac{1}{4(\xi^2+\eta^2)}e^{-\xi(t_1+t_2-\tau)}\{2\xi\cos\eta(t_1+t_2-\tau)-2\eta$$

$$\sin(t_1+t_2-\tau)\}\Big]\,d\tau$$

$$=\frac{\eta}{4\xi\sqrt{\xi^2+\eta^2}}\int_0^{\infty} R(\tau)\,e^{-\xi(b+\tau)}\sin\left\{\eta(b+\tau)+\sin^{-1}\frac{\eta}{\sqrt{\xi^2+\eta^2}}\right\}d\tau. \qquad \text{(B-7)}$$

Hence on adding (B-6) and (B-7), we have

$$\lim E\{Y(t_1)\,Y(t_2)\} = \frac{\kappa^2}{4\xi\eta\sqrt{\xi^2+\eta^2}}\int_{-\infty}^{+\infty} R(b-\tau)\,e^{-\xi|\tau|}\sin\left\{\eta|\tau|+\sin^{-1}\frac{\eta}{\sqrt{\xi^2+\eta^2}}\right\}d\tau. \qquad \text{(B-8)}$$