STOCHASTIC PROCESSES ASSOCIATED WITH A SYMMETRIC OSCILLATORY POISSON PROCESS

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ABSTRACT

A symmetric oscillatory Poisson process is defined and its stochastic features studied. The process represented by the symbolic integral of this oscillatory Poisson process is then discussed in detail. The results obtained are applied to the well-known stochastic problem of multiple scattering of charged particles in their passage through matter.

INTRODUCTION

IN this paper we shall study processes which are associated with what we shall call a symmetric oscillatory Poisson process (hereinafter referred to as S.O.P.) defined as follows. Events occur in a Poisson manner, along the one-dimensional 't' axis, *i.e.*, if $n(t)$ is the stochastic variable representing the number of events in the interval (o, t), then the probability that $n(t) = n$ is given by $\pi(n, t) = e^{-\lambda t}(\lambda t)^n/n!$, where λ is the parameter of the Poisson distribution and λdt, the probability that an event occurs between t and $t + dt$. If we associate with the i-th event a stochastic variable x_i which assumes the values $+1$ and -1 with probability p and $(1 - p)$ respectively then $m(t) = \sum_{i=0}^{i=n(t)} x_i$ is a stochastic variable which can assume integer values from $-\infty$ to $+\infty$ and represents an oscillatory Poisson process (O.P.). If $p = \frac{1}{2}$ we call it, for obvious reasons, an S.O.P. The typical trajectory, that is, the realised curve of $n(\tau)$ in the interval o to t, τ representing a typical point on the t-axis, consists of lines parallel to the t-axis with jumps of unit magnitude (positive and negative) at points where the Poisson events occur.

We shall here deal with the following:

(1) The probability distribution of $m(t)$,

(2) Recurrence and first passage times of m,

(3) The processes represented by the symbolic integrals

$$\int_0^t m(\tau)\, d\tau \text{ and } \int_0^t m(t - \tau)\, d\tau,$$

(4) Application of these results to the problem of multiple scattering,

(5) Generalisation of m to the case $m = \sum_i \alpha_i$ where α_i is a stochastic variable associated with the i-th event, all the α_i's having the same probability frequency function. This problem is shown to be equivalent to that of Snyder and Scott (1949).

(6) Multiple scattering in multiplicative processes.

The object of the paper is twofold: (1) First is to explain, in physical terms, the importance of the concept of trajectory in the computation of first passage and recurrence times. The authors have not seen a systematic treatment of this concept and its relevance to the problems of first passage and recurrence from a physical point of view and have assumed that a summary of the present state of knowledge is contained in Bartlett's recent book (1955) on stochastic processes. The sections on recurrence times in this paper amount to an extension of the ideas in Bartlett's book but with additional emphasis on the concept of the trajectory of a stochastic process and consequently on the necessity of a distinction that has to be made between two "equivalent" processes (to be explained presently) when considering first passage and recurrence times. This aspect has not been stressed in Bartlett's book. The phenomenological difficulties relating to first passage and recurrence problems have been classified and discussed in a recent contribution by one of us (R, 1955) and the general considerations of that paper are applied to the solution of the problems here.

(2) The second object is to consider the problem of multiple scattering, as an integral of a basic random process and suggest an approximation which helps us to obtain exact solutions. The last section on multiple scattering and multiplicative processes deals with the beginnings of one of the most vexed problems of cascade theory in cosmic radiation, *i.e.*, the lateral spread of showers.

1. The Probability Distribution of $m(t)$ for an O.P.

The standard procedure is to express $\pi(m, t + dt)$* the probability that $m(t + dt) = m$ in terms of $\pi(m, t)$ that probability that $m(t) = m$.

The forward differential equation of the process is then given by

$$\frac{\partial \pi(m, t)}{\partial t} = -\lambda \pi(m, t) + \lambda\{p\pi(m-1, t) + (1-p)\pi(m+1, t)\},$$

$$\pi(m, 0) = 0, \quad (m \neq 0)$$

$$\pi(0, 0) = 1. \tag{1}$$

* Throughout this paper, π denotes any probability frequency function; distinction between two functions will be apparent from the context. Where Fourier transforms are defined for these functions, the same convention is adopted.

It is not easy to solve the above equation by simple iteration, since $\pi(m, t)$ is expressed in terms of both $\pi(m+1, t)$ and $\pi(m-1, t)$. The validity of defining a generating function

$$G(u, t) = \sum_{m} \pi(m, t) u^m \tag{2}$$

depends upon the uniform convergence of the series with positive and negative powers of u in a domain including $u = 1$. To avoid these difficulties we adopt the following argument.

The above process can be considered as a superposition of two Poisson processes, having the parameters λp and $\lambda(1-p)$ but while one of these has a value $+1$ associated with each event the other has the value -1 assigned to each event.

Using this argument we immediately obtain the frequency function $\pi(m, t)$ as

$$\pi(m, t) = \sum_{r=0 \text{ or } -m}^{r=\infty} e^{-\lambda pt} \frac{(\lambda pt)^{m+r}}{(m+r)!} e^{-\lambda(1-p)t} \frac{[\lambda(1-p)t]^r}{r!}. \tag{3}$$

In the summation, the lower limit of r is zero if m is positive and $-m$ if m is negative. In particular if we assume $p = \frac{1}{2}$, *i.e.*, $m(t)$ is an S.O.P.,

$$\pi(m, t) = e^{-\lambda t} I_m(\lambda t), \tag{4}$$

where I_m is the modified Bessel's function of the first kind of order m. Note that $\pi(m, t) = \pi(-m, t)$, *i.e.*, the distribution is symmetric about the value $m = 0$. Then $\epsilon\{m(t)\} = 0$† and $\epsilon\{m^2(t)\} = \lambda t$, the same as the mean square deviation of $n(t)$ with parameter λ.

2. Recurrence and First Passage Times of m

We now ask for the probability $F(m, t)\,dt$ that the variable $m(\tau)$ reaches the value m for the first time between t and $t + dt$. For any $m \neq 0$, $F(m, t)$ can be obtained by writing $\pi(m, t)$ in terms of $F(m, t)$ which is easy for any Markoff process. By simple arguments, we have

$$\pi(m, t) = \int_0^t F(m, \tau)\, \pi(m \mid m, t-\tau)\, d\tau \tag{5}$$

where $\pi(m|m, t)$ is the probability of having a state m at t, given that the state at 0 is m. In our case as the variable $m(\tau)$ is purely additive, this probability is identical with $\pi(o, t)$ and hence we can immediately obtain a solution for $F(m, t)$ by resorting to the Laplace transform technique. Defining

† Throughout this paper ϵ denotes the expectation value.

$p(m, s)$ and $f(m, s)$, as the Laplace transforms of $\pi(m, t)$ and $F(m, t)$ respectively, we have

$$p(m, s) = f(m, s)\, p(o, s) \tag{6}$$

or

$$f(m, s) = p(m, s)/p(o, s) = f(-m, s). \tag{7}$$

Now from (1)

$$p(m, s) = \frac{\{s + \lambda - \sqrt{(s+\lambda)^2 - \lambda^2}\}^m}{\lambda^m \sqrt{(s+\lambda)^2 - \lambda^2}}. \tag{8}$$

Hence

$$f(m, s) = f(-m, s) = \frac{\{s + \lambda - \sqrt{(s+\lambda)^2 - \lambda^2}\}^m}{\lambda^m}, \tag{9}$$

which yields

$$F(m, t) = F(-m, t) = \frac{m\, I_m(\lambda t)\, e^{-\lambda t}}{t}. \tag{10}$$

This expression is valid for all m except $m = 0$. In the latter case a first passage is said to occur only if the state has changed from o at some point between o and t and then returned to o for the first time at t and this is more appropriately called a "recurrence". Continuance in the state o from o to $t + dt$ does not constitute a recurrence between t and $t + dt$. On the basis of this definition, we find that the state o may be realised at t either by a first passage to o at some $\tau < t$ and a subsequent realisation of the state o at t or directly by a continuance of the state o throughout the interval o to t. We are thus led to the equation

$$\pi(o, t) = \int_0^t F(o, \tau)\, \pi(o, t - \tau)\, d\tau + e^{-\lambda t} \tag{11}$$

so that

$$p(o, s) = f(o, s)\, p(o, s) + \frac{1}{\lambda + s}.$$

On using (5) we obtain

$$f(o, s) = \frac{s + \lambda - \sqrt{(s+\lambda)^2 - \lambda^2}}{s + \lambda} \tag{12}$$

and hence

$$F(o, t) = \lambda e^{-\lambda t} \int_0^t \frac{I_1(\lambda \tau)}{\tau}\, d\tau. \tag{13}$$

F $(o, t)\, dt$ gives the probability that the state o recurs between t and $t + dt$ and F (o, t) therefore gives the probability frequency function of the "Recurrence Time".

If we try to obtain the mean value of the first passage time by multiplying the R.H.S. of (10) by t and integrating with respect to t or directly from $f(m, s)$, we are led to the interesting result that the first, and consequently higher, moments of the first passage times are infinite for any m.

3. THE PROCESS REPRESENTED BY THE SYMBOLIC INTEGRAL $\int_0^t m(\tau)\, d\tau$

To study the process symbolically represented by the integral $\int_0^t m(\tau)\, d\tau$ we adopt the phenomenological interpretation of integrals of random functions given recently by one of us (R., 1955 *b*, *c*). A typical realised trajectory of $m(\tau)$ is given in Fig. (1).

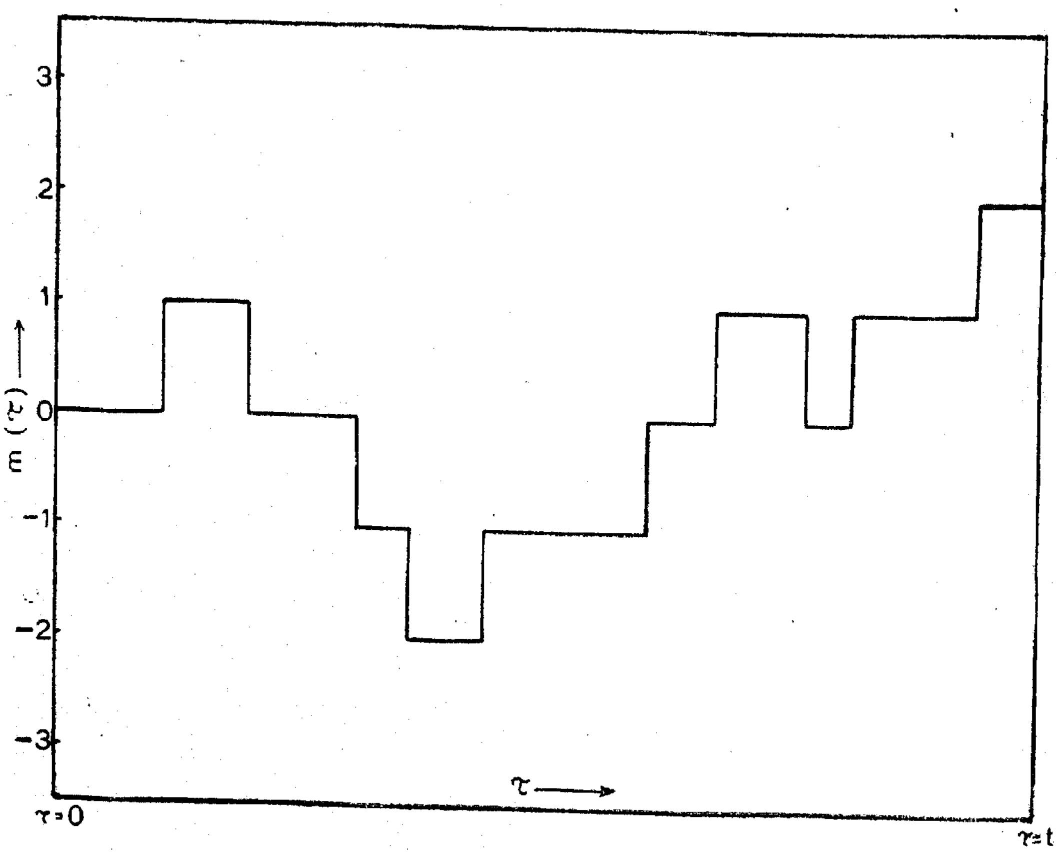

FIG. 1. Typical trajectory of $m(\tau)$ in the interval (o, t).

For a given realisation of random points along the t axis in the interval o to t, the corresponding realised value of $y(\tau)$, which we shall denote by

$y^{R}(\tau)$, is equal to $\int_{0}^{\tau} m^{R}(\tau')\, d\tau'$ where $m^{R}(\tau')$ is the realised value of $m(\tau')$. $y^{R}(\tau)$ represents the area enclosed by the realised curve of the stochastic variable $m(\tau')$ as τ' varies from o to τ. In computing the area, the area below the τ axis has to be taken as negative. Thus we plot the realised curve of $y(\tau)$ in the interval o to t; see Fig. 2.

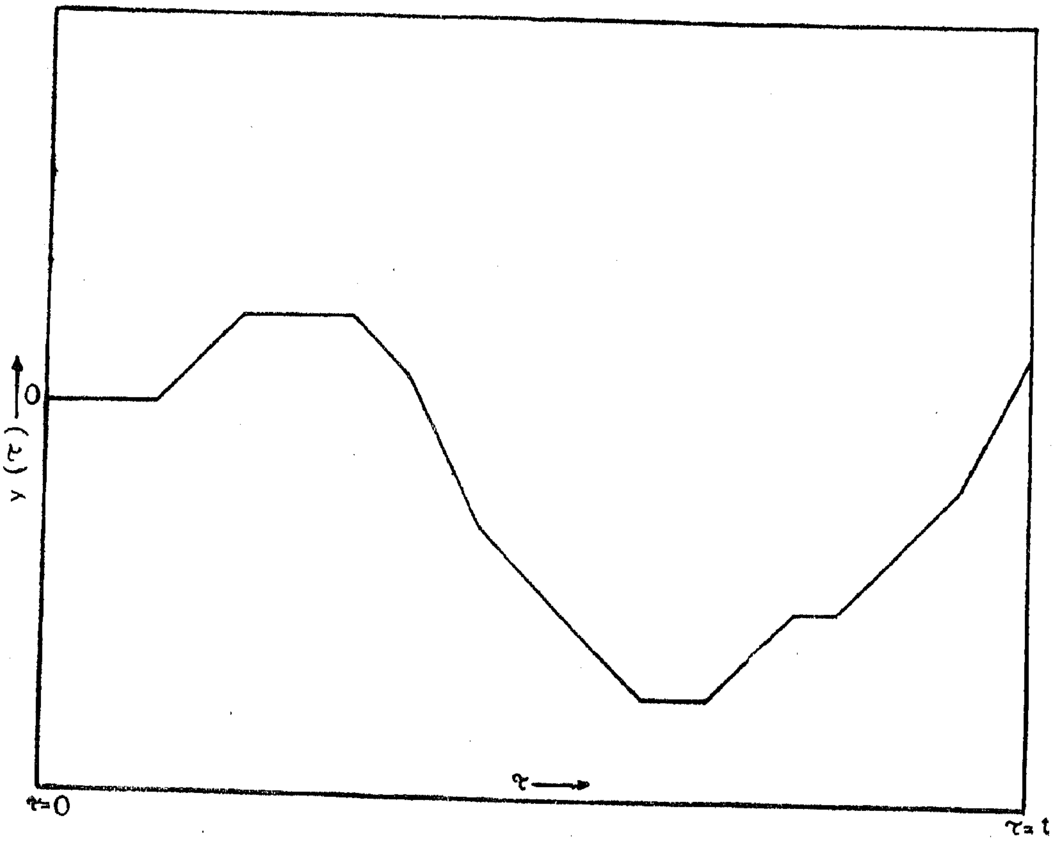

FIG. 2. Typical trajectory of $y(\tau) = \int_{0}^{\tau} m(\tau')\, d\tau'$ in the interval (o, t).

We note that it is continuous and its slope at any point τ is the realised value of $m^{R}(\tau)$ at that point. y therefore is a continuous stochastic variate and we can define the probability $\pi(y, t)\, dy$ that $y(t)$ lies between y and $y + dy$. The obvious method of obtaining the equation for $\pi(y, t)$ is to study the variation of the area referred to above by increasing t to $t + dt$. Since the increase in area is $m(t)\, dt$ we note that the process $y(t)$ is non-Markovian and hence we are constrained to introduce the function $\pi(m, y)\, dy$, the joint probability that $m(t) = m$ and $y(t)$ lies between y and $y + dy$. The process defined by the pair $m(t)$ and $y(t)$ is Markovian and by simple arguments we obtain

$$\frac{\partial \pi(m, y, t)}{\partial t} = -\lambda \pi(m, y, t) + \frac{\lambda}{2}\{\pi(m-1, y, t) + \pi(m+1, y, t)\} - m\frac{\partial \pi(m, y, t)}{\partial y}, \tag{14}$$

and

$$\pi(y, t) = \sum_{-\infty}^{+\infty} \pi(m, y, t). \tag{15}$$

It is difficult to solve this equation in view of the occurrence of m and to avoid this complication we take recourse to the method of regeneration points which yields an integral equation for $\pi(y, t)$ directly.

$$\pi(y, t) = \int_0^t e^{-\lambda\tau} \frac{\lambda}{2}\{\pi(y-(t-\tau), t-\tau) + \pi(y+(t-\tau), t-\tau)\} d\tau + \delta(y) e^{-\lambda t}. \tag{16}$$

This equation is obtained by using the now familiar arguments of the regeneration point method that the first Poisson point occurs between τ and $\tau + d\tau$ with probability $e^{-\lambda\tau} \lambda d\tau$ and the realised value of y when events are counted from τ differs from that of y when counted from $\tau = 0$ by $\pm(t-\tau)$ according as we assign $+1$ or -1 to the Poisson point at τ. Differentiating the above equation we obtain what is known as the backward differential equation of the p.f.f. of $y(t)$:

$$\frac{\partial \pi(y, t)}{\partial t} = -\lambda \pi(y, t) + \frac{\lambda}{2}\{\pi(y-t, t) + \pi(y+t, t)\}. \tag{17}$$

This equation is easier to solve. The equivalence of (14) and (17) is difficult to prove. But therein lies the advantage of dealing with equation (17)! Defining the Fourier transform of $\pi(y, t)$ as

$$p(u, t) = \int_{-\infty}^{\infty} \pi(y, t) e^{-iuy} dy, \tag{18}$$

equation (16) reduces to

$$p(u, t) = \int_0^t e^{-\lambda\tau} \frac{\lambda}{2}\{e^{-iu(t-\tau)} + e^{iu(t-\tau)}\} p(u, t-\tau) d\tau + e^{-\lambda t}. \tag{19}$$

Differentiating this with respect to t we obtain

$$\frac{\partial p(u, t)}{\partial t} = -\lambda p(u, t) + \frac{\lambda}{2}\{e^{-iut} + e^{-iut}\} p(u, t). \tag{20}$$

Hence

$$p(u, t) = \exp \left\{ - \lambda t + \frac{\lambda}{2iu} (e^{iut} - e^{-iut}) \right\}. \tag{21}$$

Inversion of this expression yields

$$\pi(y, t) = e^{-\lambda t} \delta(y) + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{\lambda^n}{2^n n!} \sum_{r=0}^{n}{}' \binom{n}{r} (-1)^r \frac{\{y + (n-2r)t\}^{n-1}}{(n-1)!} \mathrm{H}(y + \overline{n - 2r}\, t). \tag{22}$$

The mean value of y of course is zero. Its variance may be obtained directly from (21); we find $\sigma^2 = \lambda t^3/3$.

3 (*a*). The Process Represented by the Symbolic Integral $\int_0^t m\ (t - \tau)\, d\tau$

On examining equation (17), we observe that it can be interpreted as the forward differential equation of another stochastic process which unlike $\int_0^t m(\tau)\, d\tau$ is a quasi-Markovian (the significance of the qualification " quasi " will be explained presently) basic random process characterised by discrete jumps. For if we define a stochastic process by attributing a random variable x_i to a point occurring at τ_i where x_i can take the values $\pm\, \tau_i$ with equal probability $\frac{1}{2}$, then $y^*(t) = \Sigma\, x_i$ defines a stochastic process progressing with t whose forward differential equation is given by (17). Its typical trajectory is given in Fig. 3.

According to the theory of equivalent processes developed by one of us (R.), $y^*(t)$ can be represented by the symbolic integral

$$y^*(t) = \int_0^t m(t - \tau)\, d\tau \tag{23}$$

i.e., the integral is associated with the *inverse trajectory of* $m(\tau)$. The concept of an inverse trajectory and the equivalent processes arising therefrom has been fully discussed by one of us in a series of three papers (R., 1955 *b, c, d*) and no purpose would be served here by a repetition of that discussion. Suffice it to say, that $y^*(t)$ and $y(t)$ are two distinct processes, which progress with t in strikingly different ways as is revealed by the nature of their trajectories. But their probability distributions are identical *at every t*. $y(t)$ is non-Markovian as has been explained in the derivation of (14) and (15). $y^*(t)$ is quasi-Markovian (as distinguished from Markovian), *i.e.*, the state of $y^*(t)$ at $t + dt$ can be predicted, if we know the state at t *provided we assume we also know the origin of the process lies at* $t = 0$, an information

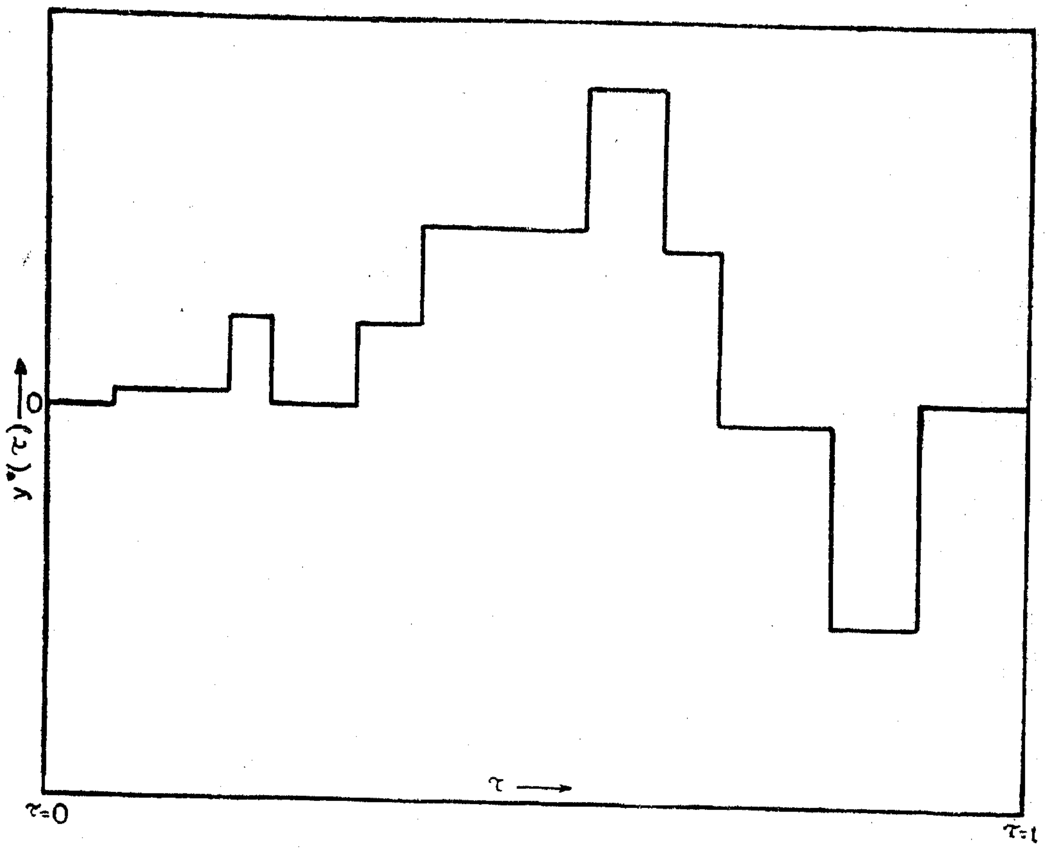

FIG. 3. Typical trajectory of $y^*(\tau) = \int_0^\tau m(\tau - \tau')\, d\tau'$ in the interval (o, t). The ordinates have been scaled down for convenience of representation.

unnecessary in Markovian processes. Otherwise, $y^*(t)$ has all the characteristics of the Markovian process in that no information between 0 and t is needed, provided the state at t is known. It is obvious from the definition of $y^*(t) = \sum_i x_i$ that its backward differential equation is just the forward differential equation of $y(t)$. Hence the solution for $\pi(y, t)$ gives the distribution function of $y^*(t)$ also.

First Passage and Recurrence Times of $y(t)$ and $y^*(t)$

The above distinction between $y(t)$ and $y^*(t)$ may seem academic if we are interested only in the distribution function. But as has been stressed by one of us, the distinction is fundamental, since the integral of a $y(t)$ process is different from that of a $y^*(t)$ process and they are not even equivalent. The distinction is as vital when we consider first passage and recurrence times of the two processes. We shall consider the problems separately.

$y(t)$.—Since the trajectory of $y(t)$ is continuous as shown by R., we have to define a function $F(y, t)$, where $F(y, t)\, dt$ represents the probability that $y(t)$ *crosses* the value y, between t and $t+dt$, or in other words, will assume exactly the value y, sometime between t and $t+dt$. $\int_0^\infty F(y, t)\, dt = 1$ for all values of y. The problem of determining $F(y, t)$ is fraught with very great difficulties which have not been adequately resolved till now. But R. has given a simple expression for the mean number of recurrences in an interval o to t. We say that y recurs between t and $t+dt$ when $y(t)$ crosses y in the interval dt if it has crossed the same value earlier. It is of course assumed that continuance in the value y (this happens only when m is o in some interval) does not constitute crossing. As regards the problem of recurrence, we can only obtain, according to R., the mean number of recurrences in the interval (o, t).

If the stochastic variable representing the number of recurrences of the state y is denoted by $R(y, t)$, its mean value is given by

$$\epsilon\{R(y, t)\} = \sum_{o}^{\infty} m\pi(m, y, t). \tag{24}$$

Denoting the interval between the n-th and $(n+1)$-th recurrences by T_n we note that T_n is a stochastic variate and the distribution functions of the various T_n's are different for different values of n. So there is no meaning in speaking in general of a mean recurrence time. When the mean number of recurrences is very large, $t/\epsilon\{R(y; t)\}$ can be taken to be roughly an estimate of the mean recurrence time.

$y^*(t)$.—Though trajectory of $y^*(t)$ is characterised by discrete jumps, as the magnitude of the jumps is a continuous variate, $y^*(t)$ is also a continuous variate. It represents, of course, a basic random process. R. has shown that in the case of such processes we have to define $F(y, t)\, dydt$, as the probability that $y(t)$ jumps into a value between y and $y+dy$, in the interval dt for the first time. Note the necessity of attaching an infinitesimal dy to the function F, to give it the significance of a probability magnitude. R. has shown that for basic random processes represented by a continuous stochastic variate, every state between y and $y+dy$ is entered only once—a rather surprising result, which on a closer examination is found to be satisfactory both mathematically and phenomenologically; $F(y, t)$ is therefore given by

$$F(y, t) = \frac{\lambda}{2}\{\pi(y-t, t) + \pi(y+t, t)\}. \tag{25}$$

Since every state is entered only once, the probability $\pi(y, t)$ can be expressed neatly in terms of $F(y, t)$.

$$\pi(y, t) = \int_0^t F(y, \tau) e^{-\lambda(t-\tau)} d\tau. \tag{26}$$

$e^{-\lambda(t-\tau)}$ represents the probability of continuance in the state y.

$$\frac{\partial \pi(y, t)}{\partial t} = -\lambda \pi(y, t) + F(y, t). \tag{27}$$

Substituting the value of $F(y, t)$ as given by (25), we obtain the equation (17) thus proving the correctness of our expression for $F(y, t)$. The problem of recurrence has no meaning in this process, since every state can be entered only once.

Generalisation of the Process $\int_0^t m(\tau) d\tau$.

We were concerned so far with an S.O.P. in which with each realised Poisson event, a value $+1$ or -1 was associated with equal probability $\frac{1}{2}$. We can generalise this process by associating a value α with each event, α itself being random, with a probability frequency function $\phi(\alpha)$ which is symmetrical about $\alpha = 0$. For such a process we ask for the probability distribution of the sum θ of the values associated with the Poisson points realised in an interval (o, t). It is easy to obtain the equation satisfied by the probability frequency function $\pi(\theta, t)$ of this sum.

$$\pi(\theta, t) = \int_0^t e^{-\lambda\tau} \lambda d\tau \int_\alpha \phi(\alpha) \pi(\theta - \alpha, t - \tau) d\alpha + e^{-\lambda t} \delta(\theta). \tag{28}$$

To solve this we define $p(u, t)$ and $\eta(u)$ as the Fourier transforms of $\pi(\theta, t)$ and $\phi(\alpha)$ respectively. Then

$$p(u, t) = \int_0^t e^{-\lambda\tau} \lambda d\tau\, p(u, t - \tau) \eta(u) + e^{-\lambda t}. \tag{29}$$

A further Laplace transform with respect to t yields the L.T. of $p(u, t)$ as

$$\rho(u, r) = \frac{\lambda}{\lambda + r} \{\rho(u, r) \eta(u)\} + \frac{1}{\lambda + r}, \tag{30}$$

so that

$$\rho(u, r) = \frac{1}{r + \lambda\{1 - \eta(u)\}}. \tag{31}$$

Hence

$$p(u, t) = \exp[-\lambda(1 - \eta(u)) t] \tag{32}$$

and

$$\pi(\theta, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iu\theta} \exp\,[-\lambda(1-\eta(u))\,t]\; du. \tag{33}$$

Thus the form of $\pi(\theta, t)$ can be determined either analytically or numerically, when the form of $\phi(\alpha)$ is known. Again, denoting $\int_0^t \theta dt$ by y, the probability distribution of y can be found by the use of the regeneration point method:

$$\pi(y, t) = \int_0^t e^{-\lambda\tau}\,\lambda\, d\tau \int_\alpha \phi(\alpha)\, \pi(y-\alpha(t-\tau);\, t-\tau)\, d\alpha + e^{-\lambda t}\,\delta(y) \tag{34}$$

and the Fourier transform

$p(u, t)$ of $\pi(y, t)$

is given by

$$p(u, t) = \int_0^t e^{-\lambda\tau}\,\lambda d\tau \int_\alpha e^{-iu\alpha\,(t-\tau)}\; p(u, t-\tau)\, d\alpha + e^{-\lambda t}. \tag{35}$$

Differentiating the above equation, we obtain

$$\frac{\partial p(u, t)}{\partial t} = -\lambda p(u, t) + \lambda p(u, t)\,\eta(ut). \tag{36}$$

Hence

$$\begin{aligned} p(u, t) &= \exp.\,[\lambda \int_0^t \{\eta(ut) - 1\}\, dt] \\ &= \exp.\left[-\frac{\lambda h(ut)}{u}\right], \end{aligned} \tag{37}$$

where

$$h(u) = \int_0^u \{1 - \eta(u')\}\, du'. \tag{38}$$

The "delta" approximation

Though we have introduced the function $\phi(\alpha)$ we shall now show that the following approximation is possible and yields good results if $\phi(\alpha)$ is a function which rapidly falls off to zero as α differs from o. To put it more precisely, assuming $\phi(\alpha)$ to be a symmetric function, $\int_{-\infty}^{+\infty} \alpha^{2n+1}\;\phi(\alpha)\, d\alpha = 0$ and $\int_{-\infty}^{+\infty} \alpha^{2n}\,\phi(\alpha)\, d\alpha = \epsilon\{\alpha^{2n}\}$ exists, but we shall adopt the approximation that $\epsilon\{\alpha^2\}$ is very small, and its higher powers can be neglected. Since $\epsilon\{\alpha^{2n}\}$

is of the order of $[\epsilon\{\alpha^2\}]^n$ we shall assume all the moments are negligible except $\epsilon\{\alpha^2\}$. The same results can be obtained, if we assume

$$\phi(\alpha) = \tfrac{1}{2}\{\delta(\alpha - \alpha_0) + \delta(\alpha + \alpha_0)\}, \tag{39}$$

where $\alpha_0 = \sqrt{\epsilon\{\alpha^2\}}$ and δ is the Dirac-delta function. Hence in equation (34), if we adopt this approximation the generalised process reduces to the simpler type given by equation (16) which is explicitly solved. We shall call the above approximation the δ-approximation remembering that it is not $\phi(\alpha)$ that it is approximated to a δ-function, which gives the trivial result that there is no dispersion at all, but that $\phi(\alpha)$ is a sum of two δ-functions as given in equation (39).

4. Application to Multiple Scattering

The above theory can be directly applied to the well-known multiple scattering problem. When a charged particle passes through matter, it suffers collisions with the atoms of the matter. If we interpret the Poisson events to refer to these collisions and α to the angle of scattering in a single collision then $y(t) = \int_0^t \theta(\tau)\, d\tau$ is the lateral displacement of the particle provided we assume that $\sum_i \alpha_i = \theta$ the total angular scattering, is small and that the particle pursues almost a straight path with slight lateral displacement in the same plane (" $t - y$ " plane); t is the thickness of matter traversed. This problem has been elaborately investigated by Snyder and Scott and our equation (37) is identical with theirs. On the δ-approximation we get the explicit solution (22).

5. Multiple Scattering in Multiplicative Processes

The problem of multiple scattering of a single particle is simple when compared to the multiple scattering in multiplicative processes. Such a situation arises for example in a cascade, *i.e.*, if we admit the possibility of new particles being produced on collision. The essential complication consists in the randomness of the points of birth of new particles and the directions in which they are projected at the time of birth, since the lateral displacement of any particle is dependent upon the above factors. Many attempts have been made to deal with the three or two dimensional cascade spread. It is not difficult to write down formal integral equations using the regeneration point method. But these equations are intractable, and are not amenable to numerical computation. We here suggest three simple models whose stochastic features will be discussed in a later paper supported by numerical work.

Model 1.—We shall assume $\mu\, dt$ is the probability that a particle collides and produces another new particle and $\lambda\, dt$ is the probability that it collides and gets merely scattered. When it produces a particle, the 'parent' and the 'offspring' continue to move in the same direction as that of the parent before collision. Defining $P(n, y, t)$ as the probability that there are n particles with displacement less than y, using the standard arguments of the regeneration point method, we can at once write the integral equation

$$
\begin{aligned}
P(n, y, t) = \int_0^t e^{-(\lambda+\mu)\tau} \Big[\lambda \int_\theta P(n, y - \theta(t-\tau), t-\tau)\, \phi(\theta)\, d\theta \\
+ \sum_{n_1+n_2=n} \mu\, P(n_1, y, t-\tau)\, P(n_2, y, t-\tau)\Big]\, d\tau \\
+ e^{-(\lambda+\mu)t}\, H(y)\, \delta_1^{\,n},
\end{aligned} \tag{40}
$$

where $\delta_1^{\,n} = 0$ if $n \neq 1$ and $\delta_1^{\,1} = 1$ and $H(y) = 0$ if $y < 0$ and $H(y) = 1$ if $y > 0$. The mean number of particles with displacements less than y can be obtained easily as

$$
e^{\mu t}\, P(y, t) \text{ where } P(y, t) = \int_{-\infty}^{y} \pi(y', t)\, dy'
$$

and $\pi(y, t)$ is the solution of equation (34). It must however be stated that the fluctuation about this mean number cannot be obtained so simply in terms of $\pi(y, t)$.

Model 2.—We shall assume the 'parent' to move in the same direction and the 'offspring' to be projected at an angle β to the direction of the 'parent', where β has the probability frequency function $\phi(\beta)$ symmetric about zero. The corresponding integral equation of $P(n, y, t)$ is given by

$$
\begin{aligned}
P(n, y, t) = \int_0^t e^{-(\lambda+\mu)\tau}\, \lambda\, d\tau \Big[\int_\theta P(n, y - \theta(t-\tau), t-\tau)\, \phi(\theta)\, d\theta \\
+ \sum_{n_1+n_2=n} \int_\beta P(n_1, y - \beta(t-\tau), t-\tau)\, P(n_2, y, t-\tau)\, \mu\, \phi(\beta)\, d\beta\Big] \\
+ e^{-(\lambda+\mu)t}\, H(y)\, \delta_1^{\,n}.
\end{aligned} \tag{41}
$$

Model 3.—The 'parent' and 'offspring' are projected at angles β_1 and β_2 with respect to the original direction of the 'parent' and β_1 and β_2 have the joint probability frequency function $\phi(\beta_1, \beta_2)$. The integral equation for this model is given by

$$
\begin{aligned}
P(n, y, t) = \int_0^t e^{-(\lambda+\mu)\tau}\, \lambda\, d\tau \Big[\int_\theta P(n, y - \theta(t-\tau), t-\tau)\, \phi(\theta)\, d\theta \\
+ \sum_{n_1+n_2=n} \int_{\beta_1}\int_{\beta_2} P(n_1, y - \beta_1(t-\tau), t-\tau)\, P(n_2, y - \beta_2(t-\tau), t-\tau) \\
\times\, \mu\phi(\beta_1, \beta_2)\, d\beta_1 d\beta_2\Big]
\end{aligned}
$$