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Principal bundles on the projective line

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Abstract. We classify principal G-bundles on the projective line over an arbitrary field k of characteristic $\neq 2$ or 3, where G is a reductive group. If such a bundle is trivial at a k-rational point, then the structure group can be reduced to a maximal torus.

Keywords. Reductive groups; principal bundles.

1. Let k be a field of characteristic $\neq 2, 3$ and let G be a reductive group defined over k. Let $\mathbb{P}^1_k$ denote the projective line defined over k. Let $\mathcal{O}(1)$ denote the line bundle of degree one over $\mathbb{P}^1_k$. For a one-parameter subgroup $\lambda : G_m \to G$ of G, we denote by $E_{\lambda,G}$ the principal G-bundle associated to $\mathcal{O}(1)$ (regarded as a G_m bundle) by the group homomorphism $\lambda : G_m \to G$. A principal G-bundle E on $\mathbb{P}^1_k$ is said to be trivial at the origin if the restriction of E to the k-rational point $0 \in \mathbb{P}^1_k$ is the trivial principal homogeneous space over Spec k. In this article, we show

Main theorem. *Let $E \to \mathbb{P}^1_k$ be a principal G-bundle on $\mathbb{P}^1_k$ which is trivial at the origin. Then E is isomorphic to $E_{\lambda,G}$ for some one-parameter subgroup $\lambda : G_m \to G$ defined over k.*

In particular, since every one-parameter subgroup lands inside a maximal torus of G we observe that any principal G bundle on $\mathbb{P}^1_k$ which is trivial at the origin has a reduction of structure group to a maximal torus of G. This result was proved by Grothendieck [3] when k is the field of complex numbers and by Harder [4] when G is split over k and the principal bundle is Zariski locally trivial on $\mathbb{P}^1_k$.

While we understand that this result is known (see [2], unpublished), we believe that our geometric method of deducing it from the properties of the Harder–Narasimhan filtration of principal bundles is new and of interest in its own right.

2. Let X be a complete nonsingular curve over the algebraic closure $\overline{k}$ of k and G a reductive group over $\overline{k}$. Let $E \to X$ be a principal G-bundle on X. E is said to be semistable if, for every reduction of structure group $E_P \subset E$ to a maximal parabolic subgroup P of G, we have

$$\text{degree } E_P(\mathfrak{p}) \leq 0,$$

where $\mathfrak{p}$ is the Lie algebra of P and $E_P(\mathfrak{p})$ is the Lie algebra bundle of E_P [9]. When $G = GL(n)$, a vector bundle $V \to X$ is semistable if for every subbundle $S \subset V$, we have

$$\mu(S) \leq \mu(V),$$

where for a vector bundle W, $\mu(W)$ denotes the slope degree(W)/rank(W).

If the principal bundle $E \to X$ is not semistable, then there is a canonical reduction $E_P \subset E$ to a parabolic P in G (also called the Harder–Narasimhan filtration) with the properties:

(i) degree $E_P(\mathfrak{p}) > 0$.
(ii) If U denotes the unipotent radical of P and $L = P/U$ a Levi factor of P, then the associated L bundle E_P/U is semistable.
(iii) U has a filtration $U_0 = U \supset U_1 \supset U_2 \supset \cdots \supset U_k = e$, such that U_i/U_{i+1} is a direct sum of irreducible L-modules for the natural action of L (see Theorem 2 in [12]), and if M is an irreducible L-module which occurs as a factor of U_i/U_{i+1} for some i, then degree $E_L(M) > 0$, where $E_L(M)$ is the vector bundle associated to the principal L-bundle E_L by the action of L on M.

For the existence of such a canonical reduction, see Behrend's paper [1], where degree $E_L(M)$ as above is called a numerical invariant associated to the reduction $E_P \subset E$, and the vector bundles $E_L(M)$ are called the elementary vector bundles. This reduction was first defined for vector bundles by Harder–Narasimhan in [5], and for principal bundles by Ramanathan [9].

For convenience of reference, we recall Thereom 4.1 of [6] here.

Theorem. *Let $E \to X$ be a principal G-bundle on X, where G is a reductive group. Suppose that $\mu_{\min}(TX) \geq 0$, where TX is the tangent bundle of X. Then we have,*

(i) *If $E \to X$ is semistable, so is $F^*E \to X$, where F is the absolute Frobenius of X.*
(ii) *If $E \to X$ is not semistable, and $E_P \to X$ the canonical reduction of structure group of E, $E(\mathfrak{g})$ the adjoint bundle of E, and $E(\mathfrak{p})$ the adjoint bundle of E_P, then*

$$\mu_{\max}(E(\mathfrak{g})/E(\mathfrak{p})) < 0$$

and, in particular

$$H^0(X, E(\mathfrak{g})/E(\mathfrak{p})) = 0.$$

We now let $X = \mathbb{P}^1$. Since $\mu(T\mathbb{P}^1) = 2 > 0$, where $T\mathbb{P}^1$ denotes the tangent bundle of $\mathbb{P}^1$, we can apply Theorem 4.1 of [6] in our context. Let $E \to \mathbb{P}^1_k$ be a principal G-bundle over $\mathbb{P}^1_k$, where G is a reductive group defined over k, and let $\overline{k}$ denote the algebraic closure of k and k^s denote the separable closure of k. Let $E_{\overline{k}}$ be the principal $G_{\overline{k}}$ bundle over $\mathbb{P}^1_{\overline{k}}$ obtained by base changing to $\overline{k}$, and suppose $E_{\overline{k}}$ is not semistable. Then by Theorem 4.1(ii) of [6], the canonical parabolic reduction $E_P \subset E_{\overline{k}}$ descends to E_{k^s}. Further, by [1], the canonical reduction to the parabolic is unique (actually, it is the associated parabolic group scheme $E_P(P)$ which is unique) and hence by Galois descent, the reduction $E_P \subset E$ is defined over k. In particular, the parabolic subgroup P which is *a priori* defined over $\overline{k}$, is actually defined over k.

3. We first consider torus bundles on $\mathbb{P}^1$. Let k be a field, and let T be a torus over k. Let $E_T \to \mathbb{P}^1_k$ denote a principal T-bundle on $\mathbb{P}^1_k$. We have

PROPOSITION 3.1

Let $E_T \to \mathbb{P}^1_k$ be a T-bundle on $\mathbb{P}^1_k$ which is trivial at the origin. Then there exists a one-parameter subgroup $\lambda : G_m \to T$ defined over k such that E_T is isomorphic to $E_{\lambda,T}$, where $E_{\lambda,T}$ denotes the principal T bundle associated to $\mathcal{O}_{\mathbb{P}^1}(1)$ (regarded as a G_m bundle) by the group homomorphism $\lambda : G_m \to T$.

Proof. Let $A_0 = \mathbb{P}^1_k - \{\infty\}$ and $A_\infty = \mathbb{P}^1_k - \{0\}$. The two open sets A_0 and A_∞ cover $\mathbb{P}^1_k$. By Proposition 2.7 in [7], the restriction of E_T to A_0 is obtained from a T bundle on Spec k by base change, and since we assume E_T is trivial at the origin, we obtain that $E_T|A_0$ is trivial. Now consider $E_T|A_\infty$. Again by Proposition 2.7 of [7], $E_T|A_\infty$ comes from Spec k. Since $E_T|A_0$ is trivial, there are k-rational points of $A_\infty \cap A_0$ at which E_T is trivial. It follows $E_T|A_\infty$ is also trivial. Hence there is a transition function $\mu : A_0 \cap A_\infty \to T$ which defines the principal T bundle E_T on $\mathbb{P}^1_k$. Since $A_0 \cap A_\infty = G_m$, we have thus obtained a morphism of schemes $\mu : G_m \to T$ (which is *a priori* not a morphism of group schemes). Let 1 denote the identity of G_m, which is a k-rational point of G_m, and let $\mu(1)$ be the k-rational point of T which is the image of 1. Let λ be the morphism $\lambda = \mu(1)^{-1}\mu : G_m \to T$ which is the composite of $\mu : G_m \to T$ and $\mu(1)^{-1} : T \to T$ (multiplication by $\mu(1)^{-1}$). Then it can be seen that $\lambda : G_m \to T$ is a group homomorphism. However, the T-bundle on $\mathbb{P}^1_k$ defined by λ is isomorphic to the bundle defined by μ, and hence the proposition. Q.E.D.

4. Let k be a field and G a reductive group defined over k. We have

Lemma 4.1. *Let $E \to \mathbb{P}^1_k$ be a principal G bundle over $\mathbb{P}^1_k$ and let $G \to H$ be a homomorphism of reductive groups over k which maps the centre of G to the centre of H. Let E_H be the associated H-bundle. If E is a semistable G bundle, then E_H is a semistable H bundle on $\mathbb{P}^1_k$.*

Proof. Let $F : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be the absolute Frobenius morphism on $\mathbb{P}^1_k$ (if char $k > 0$). Then by Theorem 4.1(i) in [6], $F^{n^*}E$ is semistable for all $n \geq 1$, where F^n denotes the nth iterate of the Frobenius F. Hence, by Theorems 3.18 and 3.23 of [8], it follows that E_H is semistable. Q.E.D.

Lemma 4.2. *Let H be a semisimple group over k and $E_H \to \mathbb{P}^1_k$ a principal H bundle on $\mathbb{P}^1_k$. If E_H is semistable, then E_H is the pullback of a principal homogeneous space on* Spec k *by the structure morphism $\mathbb{P}^1 \to$* Spec k. *In particular, if E_H is trivial at the origin* (*which is a k-rational point of $\mathbb{P}^1_k$*) *then E_H is trivial on $\mathbb{P}^1_k$.*

Proof. Let $\rho : H \to SL(N)$ be a faithful representation of H and let $E_{SL(N)}$ be the associated $SL(N)$ bundle. By Lemma 4.1 above, $E_{SL(N)}$ is a semistable $SL(N)$ bundle. Let V_ρ be the vector bundle whose frame bundle is $E_{SL(N)}$. Then V_ρ is a semistable vector bundle on $\mathbb{P}^1_k$ of degree zero. By an application of the Riemann–Roch theorem, it follows that V_ρ has N linearly independent sections. Since every nonzero section of a semistable vector bundle of degree zero on a projective curve is nowhere vanishing, it follows that V_ρ is the trivial vector bundle. Hence $E_{SL(N)}$ is the trivial $SL(N)$ bundle. Therefore the principal fibre space $E_{SL(N)}/H$ with fibre $SL(N)/H$ is also trivial, so

$$E_{SL(N)}/H = \mathbb{P}^1_k \times (SL(N)/H).$$

It follows that the reduction of structure group of $E_{SL(N)}$ to H is defined by a morphism $\mathbb{P}^1_k \to SL(N)/H$. Since $SL(N)/H$ is affine, the morphism $\mathbb{P}^1_k \to SL(N)/H$ actually factors through a morphism Spec $k \to SL(N)/H$. It therefore follows that the principal H bundle E_H is the pullback of a principal homogeneous space on Spec k by the structure morphism $\mathbb{P}^1_k \to$ Spec k. Q.E.D.

We can now prove

Theorem 4.3. *Let G be a reductive group over a field k of* char $\neq 2, 3$ *and let $E \to \mathbb{P}^1_k$ be a principal G bundle on $\mathbb{P}^1_k$, which is trivial at the origin. Then there is a reduction of structure group $E_T \subset E$ to a maximal torus T of G such that the T bundle E_T is trivial at the origin.*

Proof.

Case (i). Suppose E is a semistable G bundle. Let Z be the centre of G (which is defined over k). Let E/Z be the G/Z bundle associated to the homomorphism $G \to G/Z$. Since E is trivial at the origin, E/Z is also trivial at the origin. Also, by Lemma 4.1 above, E/Z is a semistable G/Z bundle. Since G/Z is a semisimple group, we can apply Lemma 4.2 above to conclude that E/Z is the trivial G/Z bundle, and hence has sections. Any section of E/Z defines a reduction of structure group $E_Z \subset E$ of E to Z. Since Z is contained in a maximal torus T, we obtain a reduction of structure group $E_T \subset E$ to T. Let $c \in H^1(k, T)$ be the class of the principal homogeneous T space obtained by restricting E_T to the origin of $\mathbb{P}^1_k$. Then c goes to zero under the map $H^1(k, T) \to H^1(k, G)$. We now twist E_T by the class c^{-1} to obtain a new T bundle E'_T which is trivial at the origin. The twisted T-bundle E'_T is contained in the twist of E by c^{-1}. However since the image of c in $H^1(k, G)$ is trivial, we see that the twist of E by c^{-1} is E itself. Hence we obtain a reduction of structure group $E'_T \subset E$ to a maximal torus T such that E'_T is trivial at its origin.

Case (ii). Now suppose E is not semistable. Then by §2, there is a canonical reduction $E_P \subset E$ to a parabolic P in G. We first observe that since the map $H^1(k, P) \to H^1(k, G)$ is injective (see Corollary 15.1.4 in [11]), the P bundle E_P is trivial at the origin. Now let L be a Levi factor of P. The L bundle E_L associated to E_P by the projection $P \to L$ is semistable (see §2 above) and since E_P is trivial at the origin, so is E_L. Let U be the unipotent radical of P. Let P act on U by the inner conjugation. Then the bundle $E_P(U)$ associated to E_P by this action on U is a group scheme over $\mathbb{P}^1$. Further, the P/L bundle E_P/L is a principal homogeneous space under the group scheme $E_P(U)$ over $\mathbb{P}^1$ (see Lemma 3.6 in [10]), and hence E_P/L determines an element of $H^1(\mathbb{P}^1, E_P(U))$. However, U has a filtration $U_0 = U \supset U_1 \supset U_2 \supset \cdots \supset U_k = e$ such that U_i/U_{i+1} is a direct sum of irreducible L-modules for the natural action of L, and if M is an irreducible L-module which occurs as a factor of U_i/U_{i+1} for some i, then degree $E_L(M) > 0$, where $E_L(M)$ is the vector bundle associated to E_L by the action of L on M (see §2 above). Since E_L is semistable, $E_L(M)$ is a semistable vector bundle by Lemma 4.1. Thus $E_L(M)$ is a semistable vector bundle on $\mathbb{P}^1$ of positive degree and hence $H^1(\mathbb{P}^1, E_L(M)) = 0$. Inducting on the filtration $U_0 = U_1 \supset U_2 \supset U_3 \cdots \supset U_k = e$, we obtain that $H^1(\mathbb{P}^1_k, E_P((U))$ is trivial, and hence E_P/L is the trivial principal homogeneous space. This defines a reduction of structure group $E'_L \subset E_P$ of E_P to the Levi factor L. However, since the composite $L \subset P \to P/U = L$ is the identity, it follows that E'_L is isomorphic to E_L. Thus we have obtained a reduction of structure group $E_L \subset E$, where E_L is a semistable L bundle which is trivial at the origin. Let Z denote the centre of L. Then arguing as in Case (i) above, we obtain a reduction of structure group $E_Z \subset E$ to the centre Z of L, such that E_Z is trivial at the origin. Since Z is contained in a maximal torus T of G, we have obtained a T bundle E_T trivial at the origin, and a reduction of structure group $E_T \subset E$ of E to the maximal torus T. Q.E.D.

Proof of main Theorem. This follows from Theorem 4.3 and Proposition 3.1 above.

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