RESTRICTION THEOREMS FOR HOMOGENEOUS BUNDLES

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ABSTRACT. We prove that for an irreducible representation $\tau : GL(n) \to GL(W)$, the associated homogeneous $\mathbf{P}^n_k$-vector bundle $\mathbb{W}_\tau$ is strongly semistable when restricted to any smooth quadric or to any smooth cubic in $\mathbf{P}^n_k$, where k is an algebraically closed field of characteristic $\neq 2, 3$ respectively. In particular $\mathbb{W}_\tau$ is semistable when restricted to general hypersurfaces of degree ≥ 2 and is strongly semistable when restricted to the k-generic hypersurface of degree ≥ 2.

1. INTRODUCTION

In this paper we study the semistable restriction theorem for the homogeneous vector bundles on $\mathbf{P}^n_k$ which come from irreducible $GL(n)$-representations.

In general suppose G is a reductive algebraic group over an algebraically closed field k and $P \subset G$ is a parabolic group. Then there is an equivalence between the category of homogeneous G-bundles over G/P and the category of P-representations, where a P-representation $\rho : P \to GL(V)$ on a k-vector space V induces a homogeneous G-bundle $\mathbb{V}_\rho$ on G/P given by

$$\mathbb{V}_\rho = \frac{G \times V}{P} = \frac{G \times V}{\{(g, v) \cong (gh, h^{-1}v) \mid g \in G, v \in V, h \in P\}}.$$

Now for the rest of the paper we fix the following

Notation 1.1. The field k is an algebraically closed field and $G = SL(n+1, k)$, and P is the maximal parabolic subgroup of G given by

$$P = \left\{ \begin{bmatrix} g_{11} & * \\ 0 & A \end{bmatrix} \in SL(n+1), \text{ where } A \in GL(n) \right\}$$

and $G/P \simeq \mathbf{P}^n_k$ is a canonical isomorphism.

Now, if $\sigma : GL(n) \to GL(V)$ is an irreducible $GL(n)$-representation then it induces an irreducible P-representation $\rho : P \to GL(V)$ given by

$$\begin{bmatrix} g_{11} & * \\ 0 & A \end{bmatrix} \mapsto \sigma(A), \tag{1.1}$$

which gives a G-homogeneous bundle on $G/P = \mathbf{P}^n_k$. Conversely, any G-homogeneous bundle $\mathbb{V}$, given by an irreducible P-representation $\rho : P \to GL(V)$, is in fact induced by an irreducible $GL(n)$-representation (upto tensoring by $\mathcal{O}_{\mathbf{P}^n_k}(r)$, for some r).

In this paper we prove the following

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Theorem 1.2. *Let $\tau : GL(n) \to GL(W)$ be an irreducible $GL(n)$-representation, where W is a k-vector space. Let $\mathbb{W}_\tau$ be the associated G-homogeneous bundle on $G/P = \mathbf{P}^n_k$. Let*

(1) X *= smooth quadric, if* char $k \neq 2$*, or*
(2) X *= smooth cubic, if* char $k \neq 3$*,*

Then the bundle $\mathbb{W}_\tau \mid_X$ is strongly semistable.

We note that Theorem 1.2 implies $\mathbb{W}_\tau$ itself is semistable on $\mathbf{P}^n_k$. However this result, in much more general form, has been proved in [R], [U], [MR1] and [B].

Theorem 1.2 implies (see Corollory 5.4) that, provided char $k \neq 2, 3$, the bundle $\mathbb{W}_\tau \mid_H$ is *semistable*, for a general hypersurface H of degree ≥ 2 in $\mathbf{P}^n_k$, and $\mathbb{W}_\tau \mid_{H_0}$ is *strongly semistable* for generic hypersurface H_0 of degree $d \geq 2$. This is equivalent to the statement that, given $s \geq 0$, the s^{th} Frobenius pull back $F^{s*}\mathbb{W}_\tau \mid_H$ is semistable for a general hypersurface H of degree ≥ 2 in $\mathbf{P}^n_k$. Moreover when the bundle $\mathbb{W}_\tau$ comes from the standard representation, *i.e.*, $\mathbb{W}_\tau$ is the tangent bundle (upto a twist by a line bundle) of $\mathbf{P}^n_k$, where $n \geq 4$, then we can prove a stronger statement, by replacing the word 'semistable' by 'stable' everywhere in Theorem 1.2 and Corollory 5.4.

In this context we recall that, Mehta-Ramanathan [MR2] have proved that if E is a semistable sheaf on a smooth projective variety (over a field of arbitrary characteristic) then E restricted to a general hypersurface of degree a (where a is any sufficiently large integer) is semistable. On the other hand, Flenner [F] proved this assertion, where the degree a of the hypersurface depends only on the rank of E and degree of the variety X, provided the characteristic is 0.

The paper is organised as follows: In Section 2, we recall some general facts about smooth quadrics. Then we discuss the vector bundle $\mathbb{V}_\sigma = \mathcal{T}_{\mathbf{P}^n_k}(-1)$ associated to the standard representation $\sigma : GL(n) \longrightarrow GL(V)$ and its restriction to smooth quadrics. In particular, for a smooth quadric $Q \subset \mathbf{P}^n_k$, we show that $\mathbb{V}_\sigma \mid_Q$ has a unique $SO(n+1)$-homogeneous proper subbundle, if $n \geq 4$, (see remark 3.4 for details).

In Section 3, we prove that if char $k \neq 2$ then $\mathcal{T}_{\mathbf{P}^n_k} \mid_Q$ is strongly stable if $n \geq 3$, and is strongly semistable if $n = 2$. Moreover the tangent bundle $\mathcal{T}_Q$ of Q is semistable and is of positive slope.

In Section 4 we prove that, if char $k \neq 3$ and $X \subset \mathbf{P}^n_k$ is an arbitrary smooth cubic hypersurface then $\mathcal{T}_{\mathbf{P}^n_k} \mid_X$ is strongly stable if $n \geq 4$ and strongly semistable if $n = 2$ or $n = 3$. Moreover the tangent bundle T_X of X is either stable if $n \neq 3$, or $\mu_{\min}(\mathcal{T}_X) \geq 0$ if $n = 3$. In fact, we show that the argument given in [PW], to prove stablity of $\mathcal{T}_X$, for a smooth hypersurface of $\deg d \geq 3$, $n \geq 4$ and $k = \mathbb{C}$, can be modified so as to work over any algebraically closed field of characteristic coprime to d (this hypothesis is needed so that the cup product with $c_1(\mathcal{O}_{\mathbf{P}^n_k}(d))$ is an injective map).

Finally in Section 5, we show (see Theorem 1.2) that, if $\mathbb{V}_\sigma \mid_X$ is semistable and $\mu_{\min}(\mathbb{V}_\sigma \mid_X) \geq 0$, where X is a smooth hypersurface in $\mathbf{P}^n_k$ then the bundle $\mathbb{W}_\tau \mid_X$ is strongly semistable for any irreducible representation $\tau : GL(n) \longrightarrow GL(W)$.

2. Some general facts about quadrics

2.1. **Embedding of quadrics in $\mathbf{P}_k^n$.** Let V be a vector-space of dimension $n+1$ over k (characteristic $k \neq 2$). Let us choose a basis $\{e_1, \ldots, e_{n+1}\}$ of V. Represent a point $v \in V$ by

$$v = (x_1, \ldots, x_{n/2}, z, y_1, \ldots, y_{n/2}), \text{ if } n \text{ is even,}$$

$$v = (x_1, \ldots, x_{(n+1)/2}, y_1, \ldots, y_{(n+1)/2}), \text{ if } n \text{ is odd,}$$

with respect to the basis $\{e_1, \ldots, e_{n+1}\}$. Without loss of generality, one can assume that any fixed smooth quadric $Q \subset \mathbf{P}_k^n$ is given by the quadratic form

$$\widetilde{Q}(v) = z^2 + 2(x_1 y_{n/2} + \cdots + x_{n/2} y_1), \quad \text{if } n \text{ is even and}$$

$$\widetilde{Q}(v) = x_1 y_{(n+1)/2} + \cdots + x_{(n+1)/2} y_1), \quad \text{if } n \text{ is odd.}$$

Let

$$\begin{array}{rcl} SO(n+1) & = & \{A \in SL(n+1) \mid \widetilde{Q}(Av) = \widetilde{Q}(v) \text{ for all } v \in V\} \\ & = & \{A \in SL(n+1) \mid A^t J A = J\} \end{array},$$

where

$$J = \begin{bmatrix} 0 & \cdots & 1 \\ & & \cdot \\ 0 & \cdot & 0 \\ \cdot & & \\ 1 & \cdots & 0 \end{bmatrix} \in GL(n+1).$$

Notation 2.1. Let $P_1 = P \cap SO(n+1)$ denote the maximal parabolic group in $SO(n+1)$ such that

$$\left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a_{11}^{-1} \end{bmatrix}, \text{ where } A \in SO(n-1), a_{11} \in k^* \right\} \subseteq P_1, \text{ and}$$

$$P_1 \subseteq \left\{ \begin{bmatrix} a_{11} & * & * \\ 0 & A & * \\ 0 & 0 & a_{11}^{-1} \end{bmatrix}, \text{ where } A \in SO(n-1), a_{11} \in k^* \right\}.$$

Then we have the canonical identification

$$\begin{array}{ccc} \mathbf{P}_k^n & \simeq & SL(n+1)/P \\ \uparrow & & \uparrow \\ Q & \simeq & SO(n+1)/P_1. \end{array}$$

2.2. **Standard representation of $GL(n)$.** Consider the canonical short exact sequence of sheaves of $\mathcal{O}_{\mathbf{P}^n_k}$-modules

$$0 \longrightarrow \Omega^1_{\mathbf{P}^n_k}(1) \longrightarrow H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}) \otimes \mathcal{O}_{\mathbf{P}^n_k} \longrightarrow \mathcal{O}_{\mathbf{P}^n_k}(1) \longrightarrow 0.$$

The dual sequence is

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n_k}(-1) \longrightarrow H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}) \otimes \mathcal{O}_{\mathbf{P}^n_k} \longrightarrow \mathcal{T}_{\mathbf{P}^n_k}(-1) \longrightarrow 0, \tag{2.1}$$

where $\mathcal{T}_{\mathbf{P}^n_k}$ is the tangent sheaf of $\mathbf{P}^n_k$. Now this sequence is also a short exact sequence of G-homogeneous bundles on $G/P = \mathbf{P}^n_k$ (see 1.1). Hence there exists a corresponding short exact sequence of P-modules

$$0 \longrightarrow V_2 \xrightarrow{f} V_1 \xrightarrow{\eta} V \longrightarrow 0,$$

where the P-module structure is given as follows.

Let V_1, V and V_2 be $n+1$, n and 1 dimensional k-vector spaces respectively, with fixed bases. Let $f : (c) \mapsto (c, 0, \ldots, 0)$ and let

$$\eta : (a_1, \ldots, a_{n+1}) \mapsto (0, a_2, \ldots, a_{n+1}).$$

Now representing the elements of the vector spaces as coloumn vectors and expressing any $g \in P$ as

$$g = \begin{bmatrix} g_{11} & * \\ 0 & B \end{bmatrix}, \text{ where } B \in GL(n),$$

we define the representations as follows:

The representation $\rho_1 : P \longrightarrow GL(V_1)$ is given by

$$\rho_1(g) \begin{bmatrix} a_1 \\ \vdots \\ a_{n+1} \end{bmatrix} = [g] \begin{bmatrix} a_1 \\ \vdots \\ a_{n+1} \end{bmatrix}.$$

The representation $\rho_2 : P \longrightarrow GL(V_2)$ is given by

$$\rho_2(g)[c] = [g_{11}][c]$$

and the representation $\sigma : P \longrightarrow GL(V)$ is given by

$$\sigma(g) \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = [B] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

which is the standard representation $\sigma : GL(n) \longrightarrow GL(V)$. Thus

$$\mathcal{T}_{\mathbf{P}^n_k}(-1) = \mathbb{V}_\sigma$$

is the homogeneous bundle on G/P associated to the standard representation σ. One can easily check that the maps f and η are compatible with the P-module structure of V_2, V_1 and V.

We write the sequence (2.1) as

$$0 \longrightarrow \mathbb{V}_{\rho_2} \longrightarrow \mathbb{V}_{\rho_1} \longrightarrow \mathbb{V}_\sigma \longrightarrow 0.$$

2.3. **Restriction of $\mathbb{V}_\sigma$ to the quadric $Q \subset \mathbf{P}^n_k$.** The bundle $\mathbb{V}_\sigma = \mathcal{T}_{\mathbf{P}^n_k}(-1)$, when restricted to Q, fits into an extension

$$0 \longrightarrow \mathcal{T}_Q(-1) \longrightarrow \mathcal{T}_{\mathbf{P}^n_k}(-1) \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{O}_Q \longrightarrow \mathcal{N}_{Q/\mathbf{P}^n_k}(-1) \longrightarrow 0, \tag{2.2}$$

where $\mathcal{T}_Q$ and $\mathcal{N}_{Q/\mathbf{P}^n_k}$ denote the tangent sheaf and the normal sheaf of $Q \subset \mathbf{P}^n_k$. Note that this is also a short exact sequence of $SO(n+1)$-homogeneous bundles on $Q = SO(n+1)/P_1$ (see 2.1), hence there exists the corresponding short exact sequence of P_1-modules

$$0 \longrightarrow U_1 \xrightarrow{\widetilde{f}} V \xrightarrow{\widetilde{g}} U_3 \longrightarrow 0, \tag{2.3}$$

where U_1 and U_3 are k-vector spaces of dimensions $n-1$ and 1 respectively. We define

$$\widetilde{f} : (b_1, \dots, b_{n-1}) \to (b_1, \dots, b_{n-1}, 0)$$

and

$$\widetilde{g} : (a_1, \dots, a_n) \to (a_n).$$

Now any $g \in P_1$ can be written as

$$g = \begin{bmatrix} a_{11} & * & * \\ 0 & A & * \\ 0 & 0 & a_{11}^{-1} \end{bmatrix}$$

where $A \in SO(n-1)$ and $a_{11} \in k \setminus \{0\}$. The representation $\widetilde{\sigma} : P_1 \longrightarrow GL(V)$ is given by

$$\widetilde{\sigma}(g) \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} A & * \\ 0 & a_{11}^{-1} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \tag{2.4}$$

The representation $\rho_3 : P_1 \longrightarrow GL(U_3)$ is given by

$$\rho_3(g)[x] = [a_{11}^{-1}][x]$$

and the representation $\sigma_1 : P_1 \longrightarrow GL(U_1)$ is given by

$$\sigma_1(g) \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = [A] \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \tag{2.5}$$

We write the sequence (2.3) as

$$0 \longrightarrow \mathbb{U}_1 \longrightarrow \mathbb{V}_{\widetilde{\sigma}} \longrightarrow \mathbb{U}_3 \longrightarrow 0.$$

Remark 2.2. Note that $\sigma_1 : P_1 \longrightarrow GL(U_1)$ factors through the standard representation $\widetilde{\sigma}_1 : SO(n-1) \longrightarrow GL(U_1)$ and hence is irreducible, for $n \neq 3$. This implies that the tangent bundle T_Q is semistable. For $n = 3$, the representation σ_1 is not irreducible and U_1 is a direct sum of two P_1-submodules, namely $k(1,0,0) \subset V$ and $k(0,1,0) \subset V$ respectively. In fact one can check easily that the only P_1-submodules of V are given by $k(1,0,0)$, $k(0,1,0)$, U_1 and V itself. In particular, all the homogeneous subbundles of $\mathbb{V}_{\widetilde{\sigma}}$ are given by these four P_1-submodules.

A smooth quadric $Q \subset \mathbf{P}^3_k$ is isomorphic to $\mathbf{P}^1_k \times \mathbf{P}^1_k$ and therefore the tangent bundle T_Q is a direct sum of line bundles of same degree. Hence the tangent bundle $\mathcal{T}_Q$ is always a semistable vector bundle for a smooth quadric Q. Moreover, by (2.2), one can compute that $\mu(\mathcal{T}_Q) > 0$, if $n \geq 2$.

3. Stablity of $\mathcal{T}_{\mathbf{P}^n_k} \mid_{\text{smooth quadric}}$

Proposition 3.1. *Let $\sigma : GL(n) \longrightarrow GL(V)$ be the standard representation (*i.e.,* $\sigma(g) = g$). Let $\mathbb{V}_\sigma$ be the associated G-homogeneous bundle on $G/P = \mathbf{P}^n_k$. Then for characteristic $k \neq 2$, the restriction of the bundle $\mathbb{V}_\sigma = \mathcal{T}_{\mathbf{P}^n_k}(-1)$ to any smooth quadric $Q \subset \mathbf{P}^n_k$ is semistable.*

Remark This result in characteristic 0 is proved by [F]. In fact later we prove a stronger version of the above proposition (see Proposition 3.6).

For the proof of the proposition we need the following two lemmas.

Lemma 3.2. *Let $\mathbb{U}_1$ and $\mathbb{V}_{\widetilde{\sigma}}$ denote the $SO(n+1)$-homogeneous bundles, associated to the σ_1 and $\widetilde{\sigma}$ respectively (as given in Section 2), on $Q = SO(n+1)/P_1$. Then*

$$\mu(\mathbb{U}_1) < \mu(\mathbb{V}_{\widetilde{\sigma}}).$$

Proof. We are given that

$$\mathbb{V}_{\widetilde{\sigma}} = \mathbb{V}_\sigma \mid_Q = \mathcal{T}_{\mathbf{P}^n_k}(-1) \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{O}_Q$$

and $\mathbb{U}_1 = \mathcal{T}_Q(-1)$. Now

$$\deg \mathcal{T}_{\mathbf{P}^n_k}(-1) \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{O}_Q = 2 \deg \mathcal{T}_{\mathbf{P}^n_k}(-1) = 2(\deg H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}) \otimes \mathcal{O}_{\mathbf{P}^n_k} - \deg \mathcal{O}_{\mathbf{P}^n_k}(-1)) = 2,$$

where the second last equality follows from (2.1). As

$$\mathcal{N}_{Q/\mathbf{P}^n_k} \simeq (\mathcal{I}/\mathcal{I}^2)^\vee = \mathcal{O}_{\mathbf{P}^n_k}(-2)^\vee \mid_Q = \mathcal{O}_{\mathbf{P}^n_k}(2) \mid_Q,$$

where $\mathcal{I}$ is the ideal sheaf of $Q \subset \mathbf{P}^n_k$, we have

$$\deg \mathcal{N}_{Q/\mathbf{P}^n_k}(-1) = \deg \mathcal{O}_{\mathbf{P}^n_k}(1) \mid_Q = 2.$$

Therefore

$$\deg \mathbb{U}_1 = \deg \mathcal{T}_Q(-1) = \deg \mathcal{T}_{\mathbf{P}^n_k}(-1) - \deg \mathcal{N}_{Q/\mathbf{P}^n_k}(-1) = 0.$$

Hence $\mu(\mathbb{U}_1) = 0 < \mu(\mathbb{V}_{\widetilde{\sigma}}) = 2/n$. This proves the lemma. □

Lemma 3.3. *The sequence (2.3)*

$$0 \longrightarrow U_1 \xrightarrow{\widetilde{f}} V \xrightarrow{\widetilde{g}} U_3 \longrightarrow 0,$$

defined as above, of P_1-representations does not split.

Proof. It is enough to prove that the short exact sequence (2.2) does not split as sheaves of $\mathcal{O}_Q$-modules. Suppose it does, then so does

$$0 \longrightarrow \mathcal{T}_Q(-2) \longrightarrow \mathcal{T}_{\mathbf{P}^n_k}(-2) \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{O}_Q \longrightarrow \mathcal{N}_{Q/\mathbf{P}^n_k}(-2) \longrightarrow 0,$$

where we know that $\mathcal{N}_{Q/\mathbf{P}^n_k}(-2) \simeq \mathcal{O}_Q$. This implies that $H^0(Q, \mathcal{T}_{\mathbf{P}^n_k}(-2) \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{O}_Q) \neq 0$. However we have

$$(3.1) \qquad 0 \longrightarrow \mathcal{T}_{\mathbf{P}^n_k}(-4) \longrightarrow \mathcal{T}_{\mathbf{P}^n_k}(-2) \longrightarrow \mathcal{T}_{\mathbf{P}^n_k}(-2) \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{O}_Q \longrightarrow 0,$$

where the first map is multiplication by the quadratic equation defining $Q \subset \mathbf{P}^n_k$. If we assume the following

Claim. $H^0(\mathbf{P}^n_k, \mathcal{T}_{\mathbf{P}^n_k}(-2)) = 0 = H^1(\mathbf{P}^n_k, \mathcal{T}_{\mathbf{P}^n_k}(-4))$.

Then (3.1) implies that $H^0(Q, \mathcal{T}_{\mathbf{P}^n_k}(-2) \otimes_{\mathcal{O}_{\mathbf{P}^n_k}} \mathcal{O}_Q) = 0$, which contradicts the hypothesis. Now we give the

Proof of the claim. Consider the following short exact sequence (which is derived from (2.1))

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n_k}(-2) \longrightarrow \mathcal{O}_{\mathbf{P}^n_k}(-1)^{n+1} \longrightarrow \mathcal{T}_{\mathbf{P}^n_k}(-2) \longrightarrow 0.$$

As $n \geq 2$, we have $H^1(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(-2)) = H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(-1)) = 0$, which implies $H^0(\mathbf{P}^n_k, \mathcal{T}_{\mathbf{P}^n_k}(-2)) = 0$. The above sequence also gives the long exact sequence

$$\longrightarrow \oplus^{n+1} H^1(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(-3)) \longrightarrow H^1(\mathbf{P}^n_k, \mathcal{T}_{\mathbf{P}^n_k}(-4)) \longrightarrow H^2(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(-4)) \longrightarrow$$
$$\longrightarrow \oplus^{n+1} H^2(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(-3)) \longrightarrow$$

(1) If $n \geq 3$ then $H^1(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(-3)) = H^2(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(-4)) = 0$, which implies $H^1(\mathbf{P}^n_k, \mathcal{T}_{\mathbf{P}^n_k}(-4)) = 0$.

(2) If $n = 2$ then $H^1(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(-3)) = 0$. Moreover the map
$$H^2(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(-4)) \longrightarrow \oplus^3 H^2(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(-3))$$
is dual to
$$\oplus^3 H^0(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}) \longrightarrow H^0(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(1))$$
which is an isomorphism as it comes from the evaluation map
$$H^0(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(1)) \otimes \mathcal{O}_{\mathbf{P}^2_k} \longrightarrow \mathcal{O}_{\mathbf{P}^2_k}(1).$$
This implies $H^1(\mathbf{P}^n_k, \mathcal{T}_{\mathbf{P}^n_k}(-4)) = 0$.

This proves the claim and hence the lemma. □

Proof of Proposition 3.1. Now suppose the $SO(n+1)$-homogeneous bundle $\mathbb{V}_{\widetilde{\sigma}}$ on Q is not semistable. Then it has a Harder-Narasimhan filtration

$$0 \subset \mathbb{V}_1 \subset \cdots \subset \mathbb{V}_k = \mathbb{V}_{\widetilde{\sigma}}$$

where $\mu(\mathbb{V}_1) > \mu(\mathbb{V}_{\widetilde{\sigma}})$. Now the uniqueness of the HN filtration implies that $\mathbb{V}_1$ is a $SO(n+1)$-homogeneous subbundle of $\mathbb{V}_{\widetilde{\sigma}}$. Therefore there exists a corresponding P_1-representation, say, $\rho_4 : P_1 \longrightarrow GL(\widetilde{V}_1)$ and an inclusion of P_1-modules $\widetilde{V}_1 \hookrightarrow V$ corresponding to the inclusion $\mathbb{V}_1 \hookrightarrow \mathbb{V}_{\widetilde{\sigma}}$.

Claim. $U_1 \subset \widetilde{V}_1$, where $\sigma_1 : P_1 \rightarrow GL(U_1)$ is the P_1-representation as defined in (2.5).

We assume the claim for the moment. Since V/U_1 is an irreducible P_1-module, we have either $\widetilde{V}_1 = U_1$ or $\widetilde{V}_1 = V$, *i.e.*, $\mathbb{V}_1 = \mathbb{U}_1$ or $\mathbb{V}_1 = \mathbb{V}_{\widetilde{\sigma}}$. By Lemma 3.2, in both the cases $\mu(\mathbb{V}_1) \leq \mu(\mathbb{V}_{\widetilde{\sigma}})$, which contradicts the fact that $\mathbb{V}_1$ is a term of the HN filtration of $\mathbb{V}_{\widetilde{\sigma}}$. Hence we conclude that the $\mathbb{V}_{\widetilde{\sigma}}$ is semistable.

Now we give

Proof of the claim. Suppose $\widetilde{V}_1 \cap U_1 = 0$. Then the composition map

$$\widetilde{V}_1 = \frac{\widetilde{V}_1}{\widetilde{V}_1 \cap U_1} \hookrightarrow \frac{V}{U_1} \hookrightarrow U_3,$$

gives an isomorphism $\widetilde{V}_1 \longrightarrow U_3$, which implies that (2.3) splits as a sequence of P_1-modules; by Lemma 3.3, this is a contradiction.

Hence $\widetilde{V}_1 \cap U_1 \neq 0$. If $n \neq 3$ then U_1 is an irreducible P_1-module (see Remark 2.2), which implies that $U_1 \subset \widetilde{V}_1$. Let $n = 3$ and $U_1 \not\subset \widetilde{V}_1$. Then Remark 2.2 implies that $V_1 \subset U_1$ as a P_1-submodule of rank 1 and therefore $\mu(\mathbb{V}_1) = \mu(\mathbb{U}_1) < \mu(\mathbb{V}_{\widetilde{\sigma}})$, which is a contradiction. Therefore $U_1 \subseteq \widetilde{V}_1$. Hence the claim. This proves the proposition. $\square$

Remark 3.4. The argument in the above proposition implies that the only $SO(n+1)$-homogeneous subbundle of $\mathcal{T}_{\mathbf{P}^n_k}(-1) \mid_Q = \mathbb{V}_{\widetilde{\sigma}}$ is either $\mathbb{U}_1$ or $\mathbb{V}_{\widetilde{\sigma}}$ itself, if $n \neq 3$. If $n = 3$ then the homogeneous subbundle of $\mathbb{V}_{\widetilde{\sigma}}$ is one of the two homogeneous line subbundles of $\mathbb{U}_1$ (as given in Remark 2.2) or $\mathbb{U}_1$ or $\mathbb{V}_{\widetilde{\sigma}}$ itself.

Remark 3.5. For $n = 3$, we can give another proof of the stability of $\mathbb{V}_{\widetilde{\sigma}}$ by reversing the role of cubic and quadric in the proof of Lemma 4.5.

Now we can strengthen Proposition 3.1 as follows.

Proposition 3.6. *With the notations as in Proposition 3.1, for $n \geq 3$, the restriction of the $\mathbf{P}^n_k$-bundle, $\mathbb{V}_\sigma$ to any smooth quadric $Q \subset \mathbf{P}^n_k$ is stable. If $n = 2$ then $\mathbb{V}_\sigma \mid_Q$ is a direct sum of two copies of a line bundle on Q.*

Before coming to the proof of this proposition we need the following lemma (which, perhaps, is already known to the experts). For this we recall some general facts. Let H be a reductive algebraic group over k and $P' \subset H$ be a parabolic group. Let $\mathbb{V}_\rho$ be a homogeneous H-bundle on $X = H/P'$ induced by a P'-representation $\rho : P' \longrightarrow GL(V)$ on a k-vector space V. Let the H action on $\mathbb{V}_\rho$ be given by the map $L : H \times \mathbb{V}_\rho \longrightarrow \mathbb{V}_\rho$, where we write $L(g, v) = L_g(v)$, for $g \in H$ and $v \in \mathbb{V}_\rho$. This induces the canonical H-action on the dual of $\mathbb{V}_\rho$, which makes $\mathbb{V}_\rho^\vee$ and $\mathbb{V}_\rho \otimes \mathbb{V}_\rho^\vee$ into H-homogeneous bundles such that the map

$$\begin{array}{ccc} \mathcal{E}nd_{\mathcal{O}_X}(\mathbb{V}_\rho) \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathbb{V}_\rho) & \longrightarrow & \mathcal{E}nd_{\mathcal{O}_X}(\mathbb{V}_\rho) \\ \downarrow_{\simeq} & & \downarrow_{\simeq} \\ (\mathbb{V}_\rho \otimes_{\mathcal{O}_X} \mathbb{V}_\rho^\vee) \otimes_{\mathcal{O}_X} (\mathbb{V}_\rho \otimes_{\mathcal{O}_X} \mathbb{V}_\rho^\vee) & \longrightarrow & (\mathbb{V}_\rho \otimes_{\mathcal{O}_X} \mathbb{V}_\rho^\vee). \end{array}$$

given by

$$(v_1 \otimes \phi_1) \otimes (v_2 \otimes \phi_2) \mapsto \phi_1(v_2)(v_1 \otimes \phi_2).$$

is H-equivariant. Hence $\mathrm{End}_{\mathcal{O}_X}(\mathbb{V}_\rho) = H^0(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathbb{V}_\rho))$ is a H-module such that H respects the algebra structure on it. This gives the homomorphism

$$\bar{L} : H \longrightarrow \mathrm{Aut}(\mathrm{End}_{\mathcal{O}_X}(\mathbb{V}_\rho)),$$

given by $\bar{L}(g)(\phi) = L_g \cdot \phi \cdot L_{g^{-1}}$, where

$$\mathrm{Aut}(\mathrm{End}_{\mathcal{O}_X}(\mathbb{V}_\rho)) = \text{ the set of ring automorphism on } \mathrm{End}_{\mathcal{O}_X}(\mathbb{V}_\rho).$$

Lemma 3.7. *With the above notations, assume that the map $\bar{L}$, defined as above, is the trivial map. Then any subbundle of $\mathbb{V}_\rho$ on X, which is also a direct summand of $\mathbb{V}_\rho$, is H-homogeneous vector subbundle.*

Proof. Now let $\mathbb{V}_\rho = \mathbb{U}^1 \oplus \mathbb{U}^2$ be the direct sum of subbundles $\mathbb{U}^1$ and $\mathbb{U}^2$. Let $\phi \in \mathrm{End}_{\mathcal{O}_X}(\mathbb{V}_\rho)$ be given by

$$\phi\mid_{\mathbb{U}_1} = \mathrm{Id} \text{ and } \phi\mid_{\mathbb{U}_2} = 0.$$

Now, since $\bar{L}$ is trivial, we have

$$\bar{L}(g)(\phi) = \phi \text{ for all } g \in G.$$

i.e.,

$$L_g \cdot \phi \cdot L_{g^{-1}} = \phi. \tag{3.2}$$

Let $(\mathbb{V}_\rho)_x$ be the fiber of $\mathbb{V}_\rho$ over $x \in X$. Then, by (3.2), we have the following commutative diagram

$$\begin{array}{ccc} (\mathbb{V}_\rho)_x & \xrightarrow{L_{g^{-1}}} & (\mathbb{V}_\rho)_{g^{-1}x} \\ \downarrow\phi & & \downarrow\phi_{g^{-1}} \\ (\mathbb{V}_\rho)_x & \xrightarrow{L_{g^{-1}}} & (\mathbb{V}_\rho)_{g^{-1}x}, \end{array}$$

for each $x \in X$. This may be written as

$$\begin{array}{ccc} \mathbb{U}^1_x \oplus \mathbb{U}^2_x & \xrightarrow{L_{g^{-1}}} & \mathbb{U}^1_{g^{-1}x} \oplus \mathbb{U}^2_{g^{-1}x} \\ \downarrow\phi_x & & \downarrow\phi_{g^{-1}x} \\ \mathbb{U}^1_x \oplus \mathbb{U}^2_x & \xrightarrow{L_{g^{-1}}} & \mathbb{U}^1_{g^{-1}x} \oplus \mathbb{U}^2_{g^{-1}x}. \end{array}$$

Now

$$\mathbb{U}^2_x \subseteq \ker \phi_x \implies \mathbb{U}^2_x \subseteq \ker(L_g \cdot \phi_{g^{-1}x} \cdot L_{g^{-1}}) = \ker(\phi_{g^{-1}x} \cdot L_{g^{-1}}).$$

This implies

$$L_{g^{-1}}(\mathbb{U}^2_x) \subseteq \ker\ \phi_{g^{-1}x} = \mathbb{U}^2_{g^{-1}x}.$$

Hence $L_{g^{-1}}(\mathbb{U}^2) \subseteq \mathbb{U}^2$, *i.e.*, $\mathbb{U}^2$ is a H-homogeneous subbundle of $\mathbb{V}_\rho$. This proves the lemma. □

Proof of Proposition 3.6. By Proposition 3.1, for a quadric $Q \subset \mathbf{P}^n_k$, the bundle $\mathbb{V}_\sigma\mid_Q \simeq V_{\tilde{\sigma}}$ is semistable. Hence there exists a nontrivial socle $\mathcal{F} \subseteq \mathbb{V}_{\tilde{\sigma}}$ such that $\mu(\mathcal{F}) = \mu(\mathbb{V}_{\tilde{\sigma}})$ and $\mathcal{F}$ is the maximal polystable subsheaf. Hence, by the uniqueness of maximal polystable sheaf, it follows that it is an $SO(n+1)$-homogeneous subbundle of $\mathbb{V}_{\tilde{\sigma}}$. Therefore, by Remark 3.4, either $\mathcal{F} = \mathbb{U}_1$ or $\mathcal{F} = \mathbb{V}_{\tilde{\sigma}}$. But $\mu(\mathcal{F}) = \mu(\mathbb{V}_{\tilde{\sigma}}) > \mu(\mathbb{U}_1)$, which implies $\mathcal{F} = \mathbb{V}_{\tilde{\sigma}}$. Therefore we can write

$$\mathbb{V}_{\tilde{\sigma}} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_r,$$

where $\mathcal{F}_i$ is a direct sum of isomorphic stable sheaves, and the stable summands of distinct $\mathcal{F}_i$ are non-isomorphic. But each $\mathcal{F}_i$ is an $SO(n+1)$-homogeneous subbundle of $\mathbb{V}_{\widetilde{\sigma}}$ and is of the same slope as of $\mathbb{V}_{\widetilde{\sigma}}$. Hence $r=1$ and $\mathbb{V}_{\widetilde{\sigma}}$ is a direct sum of isomorphic stable sub-bundles, *i.e.*

$$\mathbb{V}_{\widetilde{\sigma}} = \oplus^t \mathbb{U}, \text{ where } \mu(\mathbb{U}) = \mu(\mathbb{V}_{\widetilde{\sigma}}).$$

By Equation (2.1), we have

$$2 = \deg \mathbb{V}_{\widetilde{\sigma}} = t \cdot \deg \mathbb{U}.$$

Hence $t=1$ or $t=2$.

Suppose $n=2$. Then $Q \simeq \mathbf{P}^1_k$, hence $\mathbb{V}_{\widetilde{\sigma}}$ being rank 2 vector bundle on Q splits as a direct sum of two line bundles. Therefore in this case $t=2$.

Suppose $n \geq 3$. If $t=1$ then we are done. Let $t=2$. Let

$$\bar{L} : SO(n+1) \longrightarrow \mathrm{Aut}(H^0(Q, \mathcal{E}nd(\mathbb{V}_{\widetilde{\sigma}})))$$

be the induced map. We are given that $\mathbb{V}_{\widetilde{\sigma}} = \mathbb{U} \oplus \mathbb{U}$, where $\mathbb{U}$ is a stable bundle on Q. But $\mathrm{End}_Q(\mathbb{U})$ consists of scalars, and so

$$\mathrm{End}_Q(\mathbb{V}_{\widetilde{\sigma}}) \simeq M(2,k) \text{ is the algebra of } 2\times 2 \text{ matrices.}$$

Hence $\mathrm{Aut}(H^0(Q, \mathcal{E}nd(\mathbb{V}_{\widetilde{\sigma}}))) \simeq SO(3)$. So, we have the map

$$\bar{L} : SO(n+1) \longrightarrow SO(3).$$

But $SO(n+1)$ is an almost simple group, which implies, that

$$\text{either } \dim \mathrm{Im}\ \bar{L} = 0 \text{ or } \dim SO(n+1) = \dim \mathrm{Im}\ \bar{L} \leq \dim SO(3).$$

Hence, for $n \geq 3$, $\dim \mathrm{Im}\ \bar{L} = 0$, which means $\bar{L}$ is trivial. Therefore, by Lemma 3.7, the bundle $\mathbb{U}$ is homogeneous.

However, by Remark 3.4 and Lemma 3.2, the only G-homogeneous subbundle of $\mathbb{V}_{\widetilde{\sigma}}$, of the same slope as $\mathbb{V}_{\widetilde{\sigma}}$, is $\mathbb{V}_{\widetilde{\sigma}}$ itself. Hence we conclude that $\mathbb{V}_{\widetilde{\sigma}} = \mathbb{U}$ is stable, if $n \geq 3$. This proves the proposition. □

Corollary 3.8. *If $Q \subset \mathbf{P}^n_k$ is a smooth quadric such that k is an algebraically closed field of char $\neq 2$ then*

(1) $\Omega_{\mathbf{P}^n_k}|_Q$ *is strongly semistable if $n=2$ and*
(2) $\Omega_{\mathbf{P}^n_k}|_Q$ *is strongly stable if $n \geq 3$.*

Proof. If $n=2$ then the corollory follows from Proposition 3.6. Suppose $n \geq 3$. Then, by Proposition 3.6, the bundle $\Omega_{\mathbf{P}^n_k}|_Q$ is stable. Moreover, by Remark 2.2, the tangent bundle $\mathcal{T}_Q$ of Q is semistable and $\mu(\mathcal{T}_Q) > 0$. Hence, by Theorem 2.1 of [MR1], the bundle $\Omega_{\mathbf{P}^n_k}|_Q$ is strongly stable. This proves the corollory. □

4. Stablity of $\mathcal{T}_{\mathbf{P}^n_k}|_{\text{smooth cubic}}$

We recall the Bott vanishing theorem for $(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t))$, where k an arbitrary field of arbitrary characteristic.

$$\begin{array}{lcl} H^0(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t)) & \neq & 0, \text{ if } 0 \leq q \leq n, \text{ and } t > q \\ H^n(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t)) & \neq & 0 \text{ if } 0 \leq q \leq n, \text{ and } t < q-n \\ H^p(\mathbf{P}^n_k, \Omega^p_{\mathbf{P}^n_k}) & = & k, \text{ if } 0 \leq p \leq n \\ H^p(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t)) & = & 0 \text{ otherwise.} \end{array}$$

Now throughout this section we fix a smooth hypersurface X of degree $d \geq 3$ in $Y = \mathbf{P}^n$, $(d, \mathrm{char}\ k) = 1$. We have the following short exact sequences

$$0 \longrightarrow \Omega_Y^q(t) \longrightarrow \Omega_Y^q(t+d) \longrightarrow \Omega_Y^q(t+d)\mid_X \longrightarrow 0 \tag{4.1}$$

$$0 \longrightarrow \Omega_X^q(t) \longrightarrow \Omega_Y^{q+1}(t+d)\mid_X \longrightarrow \Omega_X^{q+1}(t+d) \longrightarrow 0 \tag{4.2}$$

(1) If $p+q < \dim\ X$ and $p, q \geq 0$ then from Bott vanishing and the short exact sequences (4.1) and (4.2), it follows that $H^p(X, \Omega_X^q(t)) = 0$ for $t < 0$.

(2) If $p+q < \dim\ X$ then
$$H^p(X, \Omega_X^q) \simeq H^p(Y, \Omega_Y^q).$$

(3) Consider the following commutative diagram of natural maps
$$\begin{array}{ccc} H^p(Y, \Omega_Y^q) & \longrightarrow & H^{p+1}(Y, \Omega_Y^{q+1}) \\ \downarrow & & \downarrow \\ H^p(X, \Omega_X^q) & \longrightarrow & H^{p+1}(X, \Omega_X^{q+1}), \end{array}$$
where the horizontal maps are given by the cup product with $c_1(\mathcal{O}_Y(d)) = d \cdot c_1(\mathcal{O}_Y(1))$ and $c_1(\mathcal{O}_X(d))$ respectively. Since $(\mathrm{char\ k}, \mathrm{d}) = 1$, the map $H^p(Y, \Omega_Y^q) \longrightarrow H^{p+1}(Y, \Omega_Y^{q+1})$ is an isomorphism for every p, q with $p, q \geq 0$ and $p+1 \leq \dim\ Y$. In particular, the induced composite map
$$\eta_{p,q} : H^p(X, \Omega_X^q) \longrightarrow H^{p+1}(Y, \Omega_Y^{q+1}) \tag{4.3}$$
is an isomorphism if $p, q \geq 0$ and $p+q < \dim\ X$.

We prove the following Lemma 4.1 and Corollory 4.2 along the same line of arguements, as given for the case $k = \mathbb{C}$, in [PW].

Lemma 4.1. *Let $X \subseteq \mathbf{P}_k^n$ be a hypersurface of* deg $d \geq 3$*. Let $n \geq 2$ and* $(\mathrm{char\ k}, \mathrm{d}) = 1$*. If $p, q \geq 0$ and $p+q < \dim\ X$ and $t \leq q(n+1-d)/(n-1)$ then*

(1) $H^p(X, \Omega_X^q(t)) = 0$*, if $t \neq 0$ and*

(2) $H^p(X, \Omega_X^q) \simeq H^p(Y, \Omega_Y^q)$.

Proof. As discussed above, (a) for $t < 0$, the statement (1) holds, *i.e.*, for $t < 0$, we have $H^p(X, \Omega_X^q(t)) = 0$, and (b) the statement (2) always holds.

Suppose $t = d$. In particular $q \geq 2$. Now (4.2) gives the long exact sequence

$$H^p(\Omega_X^{q-1}) \xrightarrow{f_{p,q-1}} H^p(\Omega_Y^q(d)\mid_X) \longrightarrow H^p(\Omega_X^q(d)) \longrightarrow H^{p+1}(\Omega_X^{q-1}) \xrightarrow{f_{p+1,q-1}} H^{p+1}(\Omega_Y^q(d)\mid_X).$$

Hence to prove that $H^p(X, \Omega_X^q(d)) = 0$, it is enough to prove the following
<u>Claim</u>: The map $f_{p,q}$ is an isomorphism, if $p, q \geq 0$ and $p+q < \dim\ X$.

<u>Proof of the claim</u>. Note that we have the following commutative diagram

$$\begin{array}{ccc} H^p(X, \Omega_X^q) & \xrightarrow{f_{p,q}} & H^p(Y, \Omega_Y^{q+1}(d)\mid_X) \\ & \searrow^{\eta_{p,q}} & \downarrow^{g_{p,q+1}} \\ & & H^{p+1}(Y, \Omega_Y^{q+1}), \end{array}$$

where, by (4.3) the map $\eta_{p,q}$ is an isomorphism. Hence the map $g_{p,q+1}$ is surjective, in this case. Moreover, by (4.1) we also have the exact sequence

$$H^p(Y, \Omega_Y^{q+1}(d)) \longrightarrow H^p(X, \Omega_Y^{q+1}(d) \mid_X) \xrightarrow{g_{p,q+1}} H^{p+1}(Y, \Omega_Y^{q+1}),$$

where $H^p(Y, \Omega_Y^{q+1}(d)) = 0$, by Bott vanishing. Therefore the map $g_{p,q+1}$ is an isomorphism. This implies that $f_{p,q}$ is an isomorphism. This proves the claim. Hence $H^p(X, \Omega_X^q(d)) = 0$ if $p, q \geq 0$ and $p + q < \dim\ X$

By induction on t, we can assume that for $m < t$ and $m \neq 0$, we have

$$H^i(X, \Omega_X^j(m)) = 0, \text{ where } i, j \geq 0,\ i + j < \dim\ X \text{ and } m \leq \frac{j(n+1-d)}{n-1},$$

Now, to prove the proposition, it remains to show that,

$$t \leq \frac{q(n+1-d)}{(n-1)},\ t \notin \{0, d\},\ p, q \geq 0,\ p + q < \dim\ X \implies H^p(X, \Omega_X^q(t)) = 0.$$

Note that the hypothesis that

$$t \leq \frac{q(n+1-d)}{n-1} \implies t \leq q.$$

Consider the following long exact sequence (obtained from (4.2))

$$H^p(X, \Omega_Y^q(t) \mid_X) \longrightarrow H^p(X, \Omega_X^q(t)) \longrightarrow H^{p+1}(X, \Omega_X^{q-1}(t-d))$$

If $q - 1 < 0$ then the last term is 0. If $q - 1 \geq 0$ then as

$$t \leq \frac{q(n+1-d)}{n-1} \implies t - d \leq \frac{(q-1)(n+1-d)}{n-1},$$

by induction hypothesis on t, the last term of the sequence is 0. Consider the exact sequence (obtained from (4.1))

$$H^p(Y, \Omega_Y^q(t)) \longrightarrow H^p(X, \Omega_Y^q(t) \mid_X) \longrightarrow H^{p+1}(Y, \Omega_Y^q(t-d))$$

then, by Bott vanishing, the first and the last term of the sequence are 0. This implies that $H^p(X, \Omega_Y^q(t) \mid_X) = 0$. Hence $H^p(X, \Omega_X^q(t)) = 0$. This completes the proof of the proposition. $\square$

Corollary 4.2. *Let $X \subset \mathbf{P}_k^n$ be a smooth hypersurface of degree $d \geq 3$. Let $n \geq 4$ and g.c.d.*$(\mathrm{char\ k}, \mathrm{d}) = 1$. *Then Ω_X is stable.*

Proof. Suppose Ω_X is not stable then there exists a subbundle $W \subset \Omega_X$ of rank $q \leq n-2$, such that $\mu(W) \geq \mu(\Omega_X)$. Then $\wedge^q W \hookrightarrow \wedge^q \Omega_X$. Since $\wedge^q W \in \mathrm{Pic}\,(X)$, we have $\wedge^q W = \mathcal{O}_{\mathbf{P}_K^n}(-t) \mid_X$, as $n \geq 4$ implies that the map $\mathrm{Pic}\,(\mathbf{P}_K^n) \to \mathrm{Pic}\,(X)$ is an isomorphism. This implies that $H^0(X, \Omega_X(t)) \neq 0$. Hence to prove that the bundle Ω_X is stable, it is enough to prove that

$$H^0(X, \Omega_X^q) = 0, \text{ for } t \leq \frac{q(n+1-d)}{n-1},$$

which immediately follows by Lemma 4.1. Hence Ω_X is stable. $\square$

Lemma 4.3. *Let $X \subset \mathbf{P}_k^3$ be a smooth hypersurface of degree $d = 3$. Then $\mu_{\min}(\mathcal{T}_X) \geq 0$.*

Proof. Let $H \subset \mathbf{P}^3_k$ be a general hyperplane such that $C = X \cap H$ is a nonsingular complete intersection on $\mathbf{P}^3_k$. In particular C is an elliptic curve. This gives the canonical short exact sequence

$$0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_X \mid_C \longrightarrow \mathcal{N}_{C/X} \longrightarrow 0,$$

which is equivalent to

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{f_1} \mathcal{T}_X \mid_C \xrightarrow{f_2} \mathcal{O}_C(1) \longrightarrow 0.$$

If T_X is semistable then $\mu_{\min}(\mathcal{T}_X) = \mu(\mathcal{T}_X) = 1/2 > 0$. We can assume that T_X is not semistable. Let $\mathcal{L} \subset \mathcal{T}_X$ be the Harder-Narasimhan filtration of $\mathcal{T}_X$, which gives a short exact sequence of coherent sheaves (where $\mathcal{L}$ is a line bundle on X),

$$0 \longrightarrow \mathcal{L} \xrightarrow{g_1} \mathcal{T}_X \xrightarrow{g_2} \mathcal{M} \longrightarrow 0.$$

By definition, $\mu_{\min}(\mathcal{T}_X) = \deg \mathcal{M}$, therefore it is enough to prove that $\deg \mathcal{M} > 0$, which is same as to prove that $\deg \mathcal{M} \mid_C = \mathcal{M} \cdot H > 0$. Consider the composite map

$$\mathcal{O}_C \xrightarrow{f_1} \mathcal{T}_X \mid_C \xrightarrow{g_2|_C} \mathcal{M} \mid_C .$$

<u>Case</u> 1. If $g_2 \mid_C \circ f_1 = 0$ then the induced map $\mathcal{O}_C(1) \longrightarrow \mathcal{M} \mid_C$ is surjective. This implies that $\deg \mathcal{M} \mid_C > 0$. <u>Case</u> 2. If $g_2 \mid_C \circ f_1 \neq 0$ then there exists a nonzero map $\mathcal{O}_C \longrightarrow \mathcal{M} \mid_C$, which implies that $\deg \mathcal{M} \mid_C \geq 0$. This proves the lemma. □

Lemma 4.4. *Let $X \subset \mathbf{P}^n_k$ be a smooth hypersurface of degree $d \geq 3$. Let $n \geq 4$ and g.c.d.*(char k, d) $= 1$. *Then $\Omega_{\mathbf{P}^n_k} \mid_X$ is stable.*

Proof. As argued in Corollory 4.2, it is enough to prove that

$$H^0(X, \Omega^q_{\mathbf{P}^n_k}(t) \mid_X) = 0, \text{ for } t \leq q(n+1)/n \text{ and } 1 \leq q \leq n-1.$$

Now, consider

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n_k}(-d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_k} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

which gives

$$0 \longrightarrow \Omega^q_{\mathbf{P}^n_k}(t-d) \longrightarrow \Omega^q_{\mathbf{P}^n_k}(t) \longrightarrow \Omega^q_{\mathbf{P}^n_k}(t) \mid_X \longrightarrow 0.$$

Since $t \leq q(n+1)/n \implies t \leq q$, by Bott vanishing we have

$$H^0(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t)) = 0, \text{ for } t \leq q(n+1)/n,$$

and

$$H^1(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t-d)) = 0, \text{ if } t \neq d \text{ or } q \neq 1.$$

Therefore the exact sequence

$$H^0(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t)) \longrightarrow H^0(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t) \mid_X) \longrightarrow H^1(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t-d))$$

implies that for $t \leq q(n+1)/n$

$$H^0(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t) \mid_X) = 0, \text{ if } t \neq d \text{ or } q \neq 1.$$

However the case, when $t = d$ and $q = 1$ and $t \leq q(n+1)/n$ does not arise, as these conditions imply that $d = t \leq 1 + (1/n) < 2$. Hence we conclude that $H^0(\mathbf{P}^n_k, \Omega^q_{\mathbf{P}^n_k}(t) \mid_X) = 0$ if $t \leq q(n+1)/n$. This proves the lemma. □

Lemma 4.5. *Let $X \subset \mathbf{P}^n_k$ be a smooth cubic hypersurface such that $n = 2$ or $n = 3$. Then $\Omega_{\mathbf{P}^n_k} \mid_X$ is strongly semistable.*

Proof. Suppose $n = 2$, then X is an elliptic curve. Hence $\Omega_{\mathbf{P}^2_k} \mid_X$ is an indecomposable rank 2 vector bundle on X (see the proof of Theorem 3.16 of [NT]) and is of negative degree. Hence strong semistabilty follows from the facts that a vector bundle of negative degree has no sections and a semistable bundle is strongly semistable on an elliptic curve.

Suppose $n = 3$. Let $Q \subset \mathbf{P}^3_k$ be a general smooth quadric such that $C = Q \cap X$ is a smooth complete intersection nonsingular curve in $\mathbf{P}^3_k$. Then C is curve of genus $= 4$ such that $\mathcal{O}_{\mathbf{P}^3_k}(1) \mid_C = \omega_C$ and the restriction of the short exact sequence

$$0 \longrightarrow \Omega_{\mathbf{P}^3_k}(1) \longrightarrow H^0(\mathbf{P}^3_k, \mathcal{O}_{\mathbf{P}^3_k}(1)) \otimes \mathcal{O}_{\mathbf{P}^3_k} \longrightarrow \mathcal{O}_{\mathbf{P}^3_k}(1) \longrightarrow 0,$$

to C, is

$$0 \longrightarrow \Omega_{\mathbf{P}^3_k}(1) \mid_C \longrightarrow H^0(C, \omega_C) \otimes \mathcal{O}_C \longrightarrow \omega_C \longrightarrow 0.$$

Note that C is a non-hyperelliptic curve, hence by Corollory 3.5 of [PR] (the proof given there for $k = \mathbb{C}$ works for any algebraically closed field k of arbitrary characteristic), the bundle $\Omega_{\mathbf{P}^3_k}(1) \mid_C$ is stable. By Lemma 4.3, we have $\mu_{\min}(\mathcal{T}_X) \geq 0$. Therefore Theorem 2.1 of [MR1] implies that $\Omega_{\mathbf{P}^3_k}(1) \mid_C$ is strongly semistable, for general curve $C \subset X$, of degree 3. Hence $\Omega_{\mathbf{P}^3_k}(1) \mid_X$ is strongly semistable. Hence the lemma. □

Corollary 4.6. *If $X \subset \mathbf{P}^n_k$ is a smooth cubic such that k is an algebraically closed field of characteristic $\neq 3$, then*

(1) $\Omega_{\mathbf{P}^n_k} \mid_X$ *is strongly semistable, if $n = 2$ or 3 and*
(2) $\Omega_{\mathbf{P}^n_k} \mid_X$ *is strongly stable, if $n \geq 4$*

Proof. The cases $n = 2$ and $n = 3$ follow from Lemma 4.5. Hence it is enough to prove the corollory for $n \geq 4$. Now, by Corollory 4.2, the tangent bundle $\mathcal{T}_X = \Omega_X^\vee$ of X is semistable and is of positive slope. By Lemma 4.4, the bundle $\Omega_{\mathbf{P}^n_k} \mid_X$ is stable. Hence, again, by Theorem 2.1 of [MR1], we deduce that $\Omega_{\mathbf{P}^n_k} \mid_X$ is strongly stable. Hence the corollory. □

5. Main results

Notation 5.1. We recall the notion of 'generic' and 'general' as given in Section 1 of [MR2]. Let k be an algebraically closed field of arbitrary characteristic. Let $S_d = \mathrm{Proj}(H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}))$. Then we have

$$\begin{array}{ccc} \mathbf{P}^n_k \times S_d \supseteq Z_d & \xrightarrow{q_d} & S_d \\ \downarrow{\scriptstyle p_d} & & \\ \mathbf{P}^n_k & & , \end{array}$$

wherw $Z_d = \{(x, s) \in \mathbf{P}^n_k \times S_d \mid s(x) = 0\}$ and p_d, q_d are projections. The fiber of q_d over $s \in S_d$ is the embedding in $\mathbf{P}^n_k$ via p_d as the hypersurface of $\mathbf{P}^n_k$ defined by the ideal generated by s. Let K_d be the function field of S_d. Let Y_d be the generic fiber of q_d given by the fiber product

$$\begin{array}{ccc} Z_d & \longrightarrow & S_d \\ \uparrow^{q_d} & & \uparrow \\ Y_d & \longrightarrow & \text{Spec } K_d, \end{array}$$

where Y_d is an absolutely irreducible, nonsingular hypersurface, and there is a nonempty open subset of S_d over which the geometric fibres of q_d are irreducible.

We call Y_d *the generic hypersurface* of degree d. Whenever a property holds for $q_d^{-1}(s)$ for s in a nonempty Zariski open subset of S_d, then we say it holds for *a general s*.

Remark 5.2. For a torsion free sheaf V on a smooth projective variety (which is $\mathbf{P}^n_k$ in our case), the restriction of V to the generic hypersurface Y_d is semistable (geometrically stable) if and only if the restriction of V to a general hypersurface of degree d is semistable (geometrically stable): because, for any coherent torsion free sheaf F of X, the sheaf p_d^*F forms a flat family over a nonempty open subset of S_d (see Proposition 1.5 of [MR2]), and the property of coherent sheaves being semistable (geometrically stable) is open in flat families.

Remark 5.3. If

(1) $X =$ smooth quadric, if char $k \neq 2$, or
(2) $X =$ smooth cubic, if char $k \neq 3$

then, by Corollary 3.8 and Corollary 4.6, the bundle $\Omega_{\mathbf{P}^n_k}|_X$ is strongly semistable. Moreover, by Remark 2.2, Corollary 4.2 and Lemma 4.3, we have $\mu_{\min}(\mathcal{T}_X) \geq 0$. In particular, by Theorem 2.1 of [MR1] and Theorem 3.23 of [RR], any semistable bundle on X remains semistable after applying the functors like Frobenius pull backs, tensor powers, symmetric powers, and exterior powers on X.

Proof of Theorem 1.2. By Remark 5.3, it is enough to prove that $\mathbb{W}_\tau$ is semistable on X. By Proposition 2.4 of [J], given an irreducible representation

$$\tau : GL(n) \longrightarrow GL(W),$$

there exists $\lambda \in \chi(T)$ (for a fixed torus T of $GL(n)$) such that

$$W = L(\lambda),$$

where following the notation of [J], the $GL(n)$-module $L(\lambda) =$ socle of $H^0(\lambda)$. Moreover, by corollary 2.5 of [J], the module dual to $L(\lambda)$ is

$$L(\lambda)^\vee = L(-w_0\lambda).$$

Let $\epsilon_i \in \chi(T)$ be given by $\epsilon_i(t_1, t_2, \ldots, t_n) = t_i$ and let $\omega_i = \epsilon_1 + \cdots + \epsilon_i$. Then any $\nu \in \chi(T)$ can be written as

$$\nu = \sum_i a_i\omega_i = \sum_i \nu_i\epsilon_i,$$

where $\nu_i \in \mathbb{Z}$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$.

Let $\mathbb{H}^0(L_\nu)$ be the vector bundle on $G/P = \mathbf{P}^n_k$ corresponding to the $GL(n)$-representation $H^0(L_\nu)$.

Claim. The bundle $\mathbb{H}^0(L_\nu)\mid_X$ is semistable on $X \subset \mathbf{P}^n_k$ and

$$\mu(\mathbb{H}^0(L_\nu)\mid_X) = (\sum_i \nu_i)(\mu(\mathbb{V}_\sigma\mid_X)),$$

Proof of the claim: Let us denote

$$S(a_1,\ldots,a_n,V) = S^{a_1}(V)\otimes S^{a_2}(\wedge^2 V)\otimes\cdots\otimes S^{a_n}(\wedge^n V),$$

for a vector space V, and let us denote

$$S(a_1,\ldots,a_n,\mathbb{V}) = S^{a_1}(\mathbb{V})\otimes S^{a_2}(\wedge^2 \mathbb{V})\otimes\cdots\otimes S^{a_n}(\wedge^n \mathbb{V}),$$

for a vector bundle $\mathbb{V}$. By definition of $H^0(L_\nu)$, we have a surjection of $GL(n)$-modules

$$S(a_1,\ldots,a_n,V) \longrightarrow H^0(L_\nu), \tag{5.1}$$

where $\sigma : GL(n) \longrightarrow GL(n) = GL(V)$ is the standard representation. Hence we have the surjection of G-homogeneous bundles on $\mathbf{P}^n_k$

$$S(a_1,\ldots,a_n,\mathbb{V}_\sigma) \longrightarrow \mathbb{H}^0(L_\nu), \tag{5.2}$$

where we recall that $\mathbb{V}_\sigma = \mathcal{T}_{\mathbf{P}^n_k}(-1) = (\Omega_{\mathbf{P}^n_k}(1))^\vee$ is the vector bundle associated to the representation σ. Therefore we have the surjection of bundles on X

$$S(a_1,\ldots,a_n,\mathbb{V}_\sigma\mid_X) \longrightarrow \mathbb{H}^0(L_\nu)\mid_X . \tag{5.3}$$

By Theorem 1.1 (and Cor. 1.3), exposé XXV, Schémas en groupes III, [SGA-3], $GL(n)/B$ (B is a Borel group of $GL(n)$) can be lifted to characteristic zero. Therefore the degree and rank of these vector bundles are independent of the characteristic of the field. Now over a field of characteristic 0, sequence (5.1) split, which implies that sequence (5.2) splits as bundles on $\mathbf{P}^n_k$, defined over field of characteristic 0. Now since $S(a_1,\ldots,a_n,\mathbb{V}_\sigma)$ is semistable vector bundle, we have

$$\begin{array}{rcl}\mu(\mathbb{H}^0(L_\nu)) & = & \mu(S(a_1,\ldots,a_n,\mathbb{V}_\sigma)) \\ & = & (a_1+2a_2+\cdots+na_n)\mu(\mathbb{V}_\sigma) \\ & = & (\sum_i \nu_i)\mu(\mathbb{V}_\sigma),\end{array}$$

where the last inequality follows as $\nu_i = a_i+\cdots a_n$. Hence

$$\mu(\mathbb{H}^0(L_\nu)\mid_X) = (\sum_i \nu_i)(\mu(\mathbb{V}_\sigma\mid_X)). \tag{5.4}$$

By Remark 5.3, the bundle $S(a_1,\ldots,a_n,\mathbb{V}_\sigma\mid_X)$ is semistable. Therefore, by (5.3) and (5.4), the bundle $\mathbb{H}^0(L_\nu)\mid_X$ is semistable. Hence the claim.

Now, coming back to $W = L(\lambda)$, let

$$\lambda = \sum_i a_i\omega_i = \sum_i \lambda_i\epsilon_i.$$

Then, as $w_0(\epsilon_i) = \epsilon_{n+1-i}$, we have

$$-w_0\lambda = a_{n-1}\omega_1+\cdots+a_1\omega_{n-1}+(-a_1+\cdots-a_n)\omega_n = -\sum_i(\lambda_{n+1-i})\epsilon_i.$$

This implies that $\mu(\mathbb{H}^0(L_{-w_0\lambda})) = -\mu(\mathbb{H}^0(L_\lambda))$, therefore

$$\mu(\mathbb{H}^0(L_{-w_0\lambda})\mid_X) = -\mu(\mathbb{H}^0(L_\lambda)\mid_X). \tag{5.5}$$

Moreover there exists the surjective map of vector bundles on X

(5.6)
$$S(a_1,\ldots,a_n,\mathbb{V}_\sigma\mid_X)\otimes S(a_{n-1},\ldots,a_1,-(a_1+\cdots+a_n),\mathbb{V}_\sigma\mid_X)\longrightarrow(\mathbb{H}^0(L_\lambda)\otimes\mathbb{H}^0(L_{-w_0\lambda}))\mid_X,$$

where the L.H.S. is a semistable vector bundle of slope $= 0$. Moreover, by (5.5), the slope of R.H.S. is also $= 0$. Hence $\mathbb{H}^0(L_\lambda)\mid_Q\otimes\mathbb{H}^0(L_{-w_0\lambda})\mid_X$ is semistable of slope 0. Now, consider the injective map

$$\mathbb{W}_\tau\otimes\mathbb{W}_\tau^\vee\longrightarrow\mathbb{H}^0(L_\lambda)\otimes\mathbb{H}^0(L_{-w_0\lambda}),$$

which give the injective map

$$\mathbb{W}_\tau\mid_X\otimes\mathbb{W}_\tau^\vee\mid_X\longrightarrow\mathbb{H}^0(L_\lambda)\mid_X\otimes\mathbb{H}^0(L_{-w_0\lambda})\mid_X \tag{5.7}$$

is injective, where the slope of L.H.S is $= 0$, which is same as the slope of R.H.S.. Hence $\mathbb{W}_\tau\mid_X\otimes\mathbb{W}_\tau^\vee\mid_X$ is semistable. This implies that $\mathbb{W}_\tau\mid_X$ is semistable, which proves the theorem. $\square$

Corollary 5.4. *Let $\mathbb{W}_\tau$ be the homogeneous bundle on $\mathbf{P}^n_k$ associated to an irreducible representation $\tau: GL(n)\longrightarrow GL(W)$. Let k be an algebraically closed field of characteristic $\neq 2,3$. Then*

1. *for $s\geq 0$, the s^{th} Frobenius power $F^{s*}\mathbb{W}_\tau\mid_H$ is semistable, for general hypersurface H of degree $d\geq 2$ in $\mathbf{P}^n_k$. In particular*
2. *$\mathbb{W}_\tau\mid_{H_0}$ is strongly semistable, where $H_0\subset\mathbf{P}^n_{K_d}$ is the k-generic hypersurface of degree $d\geq 2$.*

Moreover, if $\mathbb{W}_\tau$ is the tangent bundle on $\mathbf{P}^n_k$ and $n\geq 4$ then we can replace the word 'semistable' by 'stable' everywhere in the above statement.

Proof. By Theorem 1.2, the bundle $\mathbb{W}_\tau\mid_X$ is strongly semistable, where X is a smooth quadric or a smooth cubic in $\mathbf{P}^n_k$. In other words, for $s\geq 0$ and for the s^{th} iterated Frobenius pull back, $F^{s*}\mathbb{W}_\tau$ of $\mathbb{W}_\tau$, the bundle $F^{s*}\mathbb{W}_\tau\mid_X$ is semistable, where X is a smooth quadric or a smooth cubic. Hence, by the proof of the restriction theorem of [MR2], it follows that $F^{s*}\mathbb{W}_\tau\mid_H$ is semistable when restricted to a general hypersurface $H\subset\mathbf{P}^n_k$ of degree ≥ 2 (see also the modified proof of the above mentioned restriction theorem given in [HL]). This proves part (1) of the corollory.

Moreover this implies that, for any $s\geq 0$ and for generic hypersurface H_0 of degree ≥ 2, the bundle $F^{s*}\mathbb{W}_\tau\mid_{H_0}$ is semistable (see Remark 5.2). In particular, the bundle $\mathbb{W}_\tau\mid_{H_0}$ is strongly semistable. This proves the part (2) of the corollory.

Note that, for $n\geq 4$, by Corollories 3.8 and 4.6, the bundle $\mathcal{T}_{\mathbf{P}^n_k}\mid_X$ is strongly stable and hence geometrically strongly stable (as the underlying field k is algebraically closed). Now the similar arguments, as above, applied to the tangent bundle $\mathcal{T}_{\mathbf{P}^n_k}$, prove the rest of the corollory. $\square$

Remark 5.5. By Proposition 3.6, the bundle $\mathcal{T}_{\mathbf{P}^n_k}\ |_Q$ is stable for a smooth quadric $Q \subset \mathbf{P}^n_k$, for $n \geq 3$. One may ask the following: If $\tau : GL(n) \longrightarrow GL(W)$ is an irreducible representation, then is the associated bundle $\mathbb{W}_\tau$ stable on Q? More generally if $\tau : GL(n) \longrightarrow H$ is any irreducible representation, with H semisimple, then is the induced H bundle semistable on Q?

References

[B] Biswas, I., *On the stability of homogeneous vector bundles*, J. Math. Sci. Univ. Tokyo 11 (2004), no. 2, 133–140.

[NT] Fakhruddin, N. and Trivedi, V., *Hilbert-Kunz functions and multiplicities for full flag varieties and elliptic curves*, J. Pure and Applied Algebra, **181** (2003) 23-52.

[F] Flenner, H., *Restrictions of semistable bundles on projective varieties*, Comment. Helv. **59** (1984), 635-650.

[J] Jantzen, J., *Representations of Algebraic Groups*, Vol. **131**, Pure and Applied Mathematics, Academic Press, INC.

[HL] Huybrechts, D., and Lehn, M., *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31. Friedr. Vieweg and Sohn, Braunschweig, 1997. xiv+269 pp. ISBN: 3-528-06907-4.

[MR1] Mehta, V.B. and Ramanathan, A., *Homogeneous bundles in characteristic p*, Algebraic geometry - open problems (Ravello, 1982), 315–320, Lecture Notes in Math., **997**, Springer, Berlin, (1983).

[MR2] Mehta, V. B. and Ramanathan, A., *Semistable sheaves on projective varieties and their restriction to curves*, Math. Ann. **258** (1981/82), no. 3, 213–224.

[PW] Peternell, T. and Wiśniewski, J.A., *On semistability of tangent bundles of Fano manifolds with* $b_2 = 1$, J. Algebraic Geometry, **4** (1995), 363-384.

[PR] Paranjape, K. and Ramanan, S., *On the canonical ring of a curve* Algebraic geoemtry and commutative algebra, Vol. II, 503-516, Kinokuniya, Tokyo (1988)

[R] Ramanan,S., *Holomorphic vector bundles on homogeneous spaces*, Topology, Vol. **5** (1966).

[RR] Ramanan, S. and Ramanathan, A., *Some remarks on the instability flag*, Tohuku Math. J.(2) **36** (1984), no.2, 269-291.

[SGA] *Schémas en groupes, Séminaire de géometrie algébrique de I.H.E.S.*, Lecture notes in Mathematics, **153**, Springer-Verlag.

[U] Umemura, H., *On a theorem of Ramanan*, Nagoya Math. J. **69** (1978), 131-138.

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