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Semistable sheaves on homogeneous spaces and abelian varieties

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Abstract. In this paper we prove that semistable sheaves with zero Chern classes on homogeneous spaces are trivial and semistable sheaves on abelian varieties with zero Chern classes are filtered by line bundles numerically equivalent to zero. The method consists in reducing mod p and then showing that the Frobenius morphism preserves semistability on the above class of varieties. For technical reasons, we have to assume boundedness of semistable sheaves in char p.

Keywords. Semistable sheaf; homogeneous space; abelian variety; Frobenius morphism; purely inseparable descent.

Let X be a smooth projective variety of dimension d defined over an algebraically closed field k and equipped with a very ample line bundle H. For a coherent sheaf F on X we denote by $F(n)$ the sheaf $F \otimes H^n$ and by $\chi_F(n)$ the Hilbert polynomial of F, $\chi_F(n) = \chi[F(n)]$. Let $c_1(F)$ denote the first Chern class of F. If F is torsion-free, then a subsheaf E of F is defined to be a *proper* subsheaf of F if $0 < rkE < rkF$. Define

$$\mu(F) = \frac{c_1(F) \cdot H^{d-1}}{rkF}$$

and

$$p_F(n) = \frac{\chi_F(n)}{rk\,F}$$

A torsion-free sheaf V on X is defined to be *μ-stable* (resp. *μ-semistable*) if for all subsheaves W of V we have (Takemoto [22])

$$\mu(W) < \mu(V) \text{ (resp. } \mu(W) \leqslant \mu(V))$$

V is defined to be χ-stable (resp. χ-semistable) if for all proper subsheaves W of V we have

$$p_W(n) < p_V(n) \text{ (resp. } p_W(n) \leqslant p_V(n))$$

for all $n \gg 0$ (Gieseker [6]; Maruyama [13]).

Fix a *Weil* cohomology $X \to H^*(X)$ (cf. [10]) and let $\mathscr{S} = \{V, r, c_1 \ldots, c_d\}$ denote the set of all χ-semistable torsion-free sheaves on X of rank r and fixed Chern classes $c_i = c_i(V)$, $1 \leqslant i \leqslant d$, where $c_i(V) \in H^*(X)$. A basic problem which arises in the construction of the moduli spaces of these sheaves, is the *boundedness* of $\mathscr{S}$ (cf. [13]). If char $k = 0$ then the boundedness of $\mathscr{S}$ has been proved in [5]. If char $k = p > 0$, then the boundedness of $\mathcal{S}$ has been proved only in some special cases: $\dim X = 2$, $rk\, V$ arbitrary in [14], and $\dim X$ arbitrary and $rk\, V < 3$ in [15].

In this paper we make the following assumption:

* {The set $\mathcal{S}$ is bounded for all values of $\dim X$ and $rk\, V$.}

Using * we prove

Theorem 1: Let X be a complete homogeneous space under a reductive algebraic group and let V be a χ-semistable torsion free sheaf on X. Then if $c_i(V) = 0$ for $1 \leqslant i \leqslant d$ we have V is *trivial i.e.* $V \simeq \mathbf{O}_X^r$, where $r = rk\, V$.

Theorem 2: Let X be an abelian variety and V a χ-semistable torsion-free sheaf on X. If $c_i(V) = 0$ for $1 \leqslant i \leqslant d$, then there is a filtration of V

$$0 = V_0 \subset V_1 \ldots \subset V_r = V$$

where each V_i/V_{i-1}, $1 \leqslant i \leqslant r$, is a *line bundle* on X *algebraically equivalent* to zero. In particular V is *locally free*.

Combining theorems 1 and 2 we get

Theorem 3: Let G be a smooth algebraic group acting transitively on a smooth projective variety X such that for any $x \in X$, the canonical map $G \to X$ given by $g \to gx$ is smooth. Let V be a χ-semistable torsion-free sheaf on X with $c_i(V) = 0$ for $1 \leqslant i \leqslant \dim X$. Then the conclusion of theorem 2 holds for V, *i.e.* V has a filtration by subsheaves whose successive quotients are line bundles algebraically equivalent to zero.

Theorem 1 for rank 2 semistable bundles on $\mathbf{P}^2$ or on $\mathbf{P}^1 \times \mathbf{P}^1$ has been proved by several authors (Takemoto [22]; Maruyama [16]; Schwarzenberger [28]). These results depend on the Riemann-Roch theorem for surfaces. For rank 2 semistable bundles on $\mathbf{P}^n$ in characteristic zero theorem 1 can also be deduced from the results of Ellencwajg-Forster [4] and Maruyama [14].

Theorem 2 for rank 2 bundles on abelian surface has been proved by Umemura [29] and Takemoto [22], and for bundles of arbitrary rank on abelian surfaces by Mukai [25].

Theorem 2 can be generalized as follows: (a) A vector bundle V on an abelian variety is called *weakly-translation invariant* (*semi-homogeneous* in the sense of Mukai [24]) if $T_x V \simeq V \otimes L_x$ for all $x \in X$, where L_x is a line bundle depending on x.

(b) A vector bundle V on X is said to be *special* if there exists an isogeny: $f\colon Y \to X$ such that $f^{\#}(V)$ has a filtration: $f(V) = V_n \supset V_{n-1} \supset \ldots \supset V_0 = 0$ with $V_i/V_{i-1} \in \operatorname{Pic} Y$ for all i and V_i/V_{i-1} algebraically equivalent to V_f/V_{f-1} for all i and f.

(c) A vector bundle V on X is said to be *semistable with projective Chern classes zero* if $c(V)$, the total *Chern class* of V, is equal to $[1 + c_1(V)/n]^n$, where $n = rk\, V$, and V is semistable for some polarization of X.

Mukai [24] proved that (a) is equivalent to (b) and that (b) $\Rightarrow$ (c). That (c) $\Rightarrow$ (b) for abelian surfaces has been proved by Oda [30] and Takemoto [22]. We sketch a proof below that (b) $\Rightarrow$ (c) for abelian varieties of any dimension. So let V be semistable with $c(V) = [1 + c_1(V)/n]^n$. Let $n_X\colon X \to X$ be multiplication by n and put $L = c_1(V)$. Then $n_X^{\#}(L) \simeq L^{n^2} \otimes M$ for some M in $\operatorname{Pic}^0(X)$. Hence there exists $N \in \operatorname{Pic}(X)$ with $N \otimes n_X^{\#}(V)$ semistable and with zero Chern classes. By theorem 2, $n_X^{\#}(V) \otimes N$ has a filtration $n_X^{\#}(V) \otimes N = W_n \supset W_{n-1} \ldots \supset W_0 = 0$ with $W_i/W_{i-1} \in \operatorname{Pic}^0(X)$ for all i. It follows that $n_X^{\#}(V)$ has the desired filtration.

Mukai [24] has proved that a vector bundle V on an abelian variety X is stable and semi-homogeneous if and only if there exists an isogeny $f\colon Y \to X$ and a line bundle L on Y with $K(L) \cap \ker f = 0$ such that $V = f_{\#}(L)$. However, neither f nor L is uniquely determined. It can be proved (Mehta and Nori [31]) that such vector bundles on X are in 1–1 correspondence with pairs (g, L), where $g\colon Z \to X$ is an isogeny, L is a line bundle on Z with $\ker g \subset K(L)$ and e^L/K non-degenerate.

We first prove theorems 1 and 2 in the case where k is the algebraic closure of a finite field. If X is a homogeneous space or an abelian variety then $H^0(X, T_X)$ generates T_X, where T_X is the tangent bundle of X. This implies that for any semistable sheaf V on X, $\pi^{\#}(V)$ is also semistable, where $\pi\colon X \to X$ is the *Frobenius* morphism (cf. Proposition 1.1). We note that this is false unless T_X is generated by $H^0(X, T_X)$ (Gieseker [7]). Now if $X = G/P$ then we invoke [3] and [11, theorem 1.4] to assert that any V as in theorem 1 becomes trivial on some étale cover of X and hence on X itself as $\pi_1^{\mathrm{alg}}(X) = 0$.

If X is an abelian variety, then we show that for any V as in theorem 2, there exists an integer n such that $n_X^{\#}(V)$ is trivial on X, where $n_X\colon X \to X$ is multiplication by n. From this the existence of the desired filtration for V follows at once. The case where the ground field k is arbitrary is handled by the now standard method of reduction mod p. In § 1 we establish some notations and recall some known results and then prove theorem 1. Theorems 2 and 3 are proved in §2. "Stable" and "semistable" will always mean "χ-stable" and "χ-semistable" unless stated explicitly otherwise.

1. Semistable sheaves on homogeneous spaces

Let E be a torsion-free sheaf on X and define rational numbers $h_i(E)$ by

$$\chi_E(n) = \sum_{i=0}^{d} h_i(E) n^i,$$

and for any subsheaf F of E define numbers $\beta_i(F)$, in analogy with [12], by

$$\beta_i(F) = h_i(F)\, rk\, E - h_i(E)\, rk\, F, \quad 0 \leqslant i \leqslant d.$$

If $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence of subsheaves of F, then we have $\beta_i(F_2) = \beta_i(F_1) + \beta_i(F_3)$, $0 \leqslant i \leqslant d$.

The Hirzebruch-Riemann-Roch formula gives

$$\chi_E(n) = \frac{(\deg X\, rk\, E) n^d}{d!} + \frac{[c_1(E)\cdot H^{d-1} + (rk\cdot E/2) c_1(X)\cdot H^{d-1}] n^{d-1}}{d-1!}$$

+ terms of lower degree in n.

Hence we see that E is χ-semistable (resp. χ-stable) if and only if, for any subsheaf F of E, if $\beta_d(F) = \ldots \beta_{d-r+1}(F) = 0$ then $\beta_{d-r}(F) \leqslant 0$ (resp. < 0) for $1 \leqslant r \leqslant d$.

Now let E be a χ-unstable (*i.e.* not semistable) sheaf on X. Hence there exists $F \subset E$ with $\beta_d(F) = \ldots \beta_{d-r+1}(F) = 0$ but $\beta_{d-r}(F) > 0$. We may assume r to be *minimal* with respect to this property. We have two cases:

Case (i) $r = 1$: Then Langton [12] establishes the existence of a *unique* subsheaf B of E with the following properties: (i) E/B is torsion-free. (ii) $\beta_{d-1}(B) = \sup_{F \subset E} \{\beta_{d-1}(F)\}$. (iii) B is the subsheaf of smallest rank with properties 1 and 2. (iv) $\operatorname{Hom}_{\mathbf{O}_X}(B, E/B) = 0$.

Case (*ii*) $r > 1$: (This case has been treated in [17], but we recall the results for the sake of completeness). Let

$$S = \{G \subset E/\beta_d(G) = \ldots \beta_{d-r+1}(G) = 0\}.$$

As $r \geqslant 2$ and for any $G \in S$ we have $1 \leqslant rk\, G \leqslant rk\, E$, it follows from [8, 221–307] that S is a *bounded* family of sheaves on X. In particular we can define $\beta_{d-r} = \sup_{G \in S} \beta_{d-r}(G)$ and $T = \{G \in S/\beta_{d-r}(G) = \beta_{d-r}\}$. Let B be an element of T of smallest rank and define $\hat{B}$ *by*: $B \subset \hat{B} \subset E$ and $E/\hat{B}$ torsion-free. Then $\beta_d(\hat{B}) = 0$ (this is true for any subsheaf of E) and since $B \subset \hat{B}$ we have $h_{d-1}(B) \leqslant h_{d-1}(\hat{B})$, hence $\beta_{d-1}(\hat{B}) \geqslant 0$. If $\beta_{d-1}(\hat{B}) > 0$, then this contradicts the minimality of r. Hence $\beta_{d-1}(\hat{B}) = 0$. Similarly $\beta_{d-2}(\hat{B}) = \ldots \beta_{d-r+1}(\hat{B}) = 0$. So $\hat{B} \in S$. And as $\beta_{d-r}(B) \leqslant \beta_{d-r}(\hat{B})$, by maximality of $\beta_{d-r}(B)$ we get $\beta_{d-r}(B) = \beta_{d-r}(\hat{B})$. So we may assume $B = \hat{B}$ and hence we get that B is a subsheaf of E of *smallest* rank with $B \in S$ and E/B torsion-free and $\beta_{d-r}(B) = \beta_{d-r}$.

Further, we assert that B is the *unique* subsheaf of E of smallest rank with $B \in S$, E/B torsion-free and $\beta_{d-r}(B) = \beta_{d-r}$. For, let $F \subset E$ be another such. Then we claim that both $F \cap B$ and $F + B$ belong to S. Taking $F \cap B$ first, we have $\beta_d(F \cap B) = 0$ and $\beta_{d-1}(F \cap B) \leqslant 0$. If $\beta_{d-1}(F \cap B) < 0$, then in the isomorphism $B/F \cap B \simeq F + B/F$ we get $\beta_{d-1}(B/F \cap B) > 0$ and hence $\beta_{d-1}(F + B/F) > 0$ which in turn implies that $\beta_{d-1}(F + B) > 0$, a contradiction. Hence $\beta_{d-1}(F \cap B) = 0$. Similarly $\beta_{d-2}(F \cap B) = \ldots = \beta_{d-r+1}(F \cap B) = 0$ and also, by the same reasoning, we get $\beta_{d-1}(F + B) = \ldots = \beta_{d-r+1}(F + B) = 0$. Hence both $F \cap B$ and $F + B$ belong to S. Now we claim $\beta_{d-r}(F \cap B) = \beta_{d-r}(B)$. In any case we have $\beta_{d-r}(F \cap B) \leqslant \beta_{d-r}(B)$. Suppose strict inequality holds. Then, $\beta_{d-r}(B/F \cap B) > 0 \Rightarrow \beta_{d-r}(F + B/F) > 0 \Rightarrow \beta_{d-r}(F + B) > \beta_{d-r}(F) = \beta_{d-r}$, a contradiction. Now $rk\, B \cap F \leqslant rk\, B$ and so by the definition of B, $B \cap F = B$ or $F \supset B$. The uniqueness of B is hence established.

Now we can prove that $\mathrm{Hom}\, \mathbf{O}_X(B, E/B) = 0$. If not, let $f: B \to E/B$ be a map with image G/B and kernel K, where $B \subset G \subset E$. From the two exact sequences

$$0 \to B \to G \to G/B \to 0 \tag{1}$$

$$0 \to K \to B \to G/B \to 0 \tag{2}$$

we get

(1) $\beta_i(G) = \beta_i(G/B)$ for $d - r + 1 \leqslant i \leqslant d$

(2) $\beta_i(G/B) + \beta_i(K) = 0$ for $d - r + 1 \leqslant i \leqslant d$.

Hence $\beta_i(G) + \beta_i(K) = 0$ for $d - r + 1 \leqslant i \leqslant d$. But then we must have $\beta_i(G) = \beta_i(K) = 0$ for $d - r + 1 \leqslant i \leqslant d$, because if one of them, say $\beta_i(K) > 0$, then $\beta_d(K) = \beta_{d-1}(K) \ldots = \beta_{i+1}(K) = 0, \beta_i(K) > 0$, a contradiction. Hence both G and K belong to S. From (2) we get $2\beta_{d-r}(B) = \beta_{d-r}(K) + \beta_{d-r}(G)$. Hence at least one of $\beta_{d-r}(B)$ and $\beta_{d-r}(G)$ must be greater than or equal to $\beta_{d-r}(B)$. By our assumptions on B it now follows easily that $G = B$ and $f = 0$. In what follows we shall call B the β subsheaf of E.

Now we have

Proposition 1.1: Let V be a semistable torsion-free sheaf on X with X as in theorem 1 or theorem 2. Assume both X and V are defined over a finite field F_q and let $\pi: X \to X$ be the F_q-linear Frobenius. Then $\pi^{\#}(V)$ is also semistable on X.

Proof: If $\pi^{\#}(V)$ is not semistable on X then let B be the β subsheaf of V. Consider

$\mathrm{Der}_X(X_\pi, X_\pi)$, where $X_\pi = X$ is considered as a scheme over X via π. Every $D \in \mathrm{Der}_X(X_\pi, X_\pi)$ induces a $\mathbf{O}_X$-linear map $\bar{D}: \pi^{\#}(V) \to \pi^{\#}(V)$. We compose $\bar{D}$ with the projection $\pi^{\#}(V) \to \pi^{\#}(V)/B$. This induces a map $B \to \pi^{\#}(V)/B$ which can be seen to be $\mathbf{O}_{X_\pi}$-linear. We have observed that such a map must be zero. Hence $\bar{D}$ takes B to B. Now the assumptions on X and [9, 190–23] imply that B descends to a subsheaf, say $\bar{B}$ of V. We have to show that $\bar{B}$ contradicts the χ-semistability of V and this follows from

Lemma 1.2: Let V be a torsion-free sheaf on a smooth projective variety X, both being defined over a finite field F_q. Let $\pi: X \to X$ be the F_q-linear Frobenius and define β_i for subsheaves of V and $\pi^{\#}(V)$ as before. Let $\bar{B} \subset V$ with $B = \pi^{\#}(\bar{B}) \subset \pi^{\#}(V)$ and assume that $\beta_d(B) = \ldots \beta_{d-r+1}(B) = 0$. Then $\beta_{d-r}(B) = q^r \beta_r(\bar{B})$.

Proof of Lemma 1.2: We use the *H-R-R* formula

$$\chi V(n) = [\mathrm{ch}\, V(n) \tau(X)]_d$$

where $\mathrm{ch}\, V(n)$ is the *Chern character* of $V(n)$, $\tau(X)$ is the *Todd class* of X and $[\quad]_d$ denotes the homogeneous component of degree d. Let $\tau(X)\ \sum_{i=0}^{d} w_i$ and $\mathrm{ch}\, V = \sum_{i=0}^{d} l_i$. Then $\mathrm{ch}\, V(n) = \mathrm{ch}(V)\,\mathrm{ch}(H^n) = \left(\sum_{i=0}^{d} l_i\right)(1 + H^n + \ldots + (H^n)^d/d!)$. It follows that $h_k(V) = \sum_{i+j=d-k} (l_i w_j) H^k/k!$. Similarly, for $\bar{B} \subset V$, if $\mathrm{ch}(\bar{B}) = \sum_{i=0}^{d} t_i$ then $h_k(\bar{B}) = \left(\sum_{i+j=d-k} t_i w_j\right) H^k/k!$. Now $\mathrm{ch}(\pi^{\#}(V)) = \sum_{i=0}^{d} q^i l_i$ and $\mathrm{ch}(B) = \sum_{i=0}^{d} q^i t_i$. So $h_k(\pi^{\#}(V)) = \sum_{i+j=d-k} (q^i l_i w_j) H^k/k!$ and $h_k(B) = \sum_{i+j=d-k} (q^i t_i w_j) H^k/k!$. It now follows, after a short calculation, that if $\beta_d(B) = \ldots = \beta_{d-r+1}(B) = 0$ then $\beta_{d-r}(B) = q^r \beta_{d-r}(\bar{B})$, which completes the proof of Lemma 1.2 and hence that of Proposition 1.1. Q.E.D.

Remark 1.3: Proposition 1.1. in the case of $rk\,2$ sheaves on $\mathbf{P}^n$ has been proved by Barth [2].

We now take up the proof of theorem 1. We first assume that $k = \bar{F}_p$. Let X be as in theorem 1 and let $\mathcal{S} = \{V, r, c_1 \ldots, c_d\}$ be the set of all torsion-free semistable sheaves of $rk\, r$ on X with $c_i(V) = 0, 1 \leqslant i \leqslant d$. By our assumption $(*)$ it follows that there exists a scheme T of finite type over k and a coherent torsion-free sheaf $\mathcal{V}$ on $X \times T$ flat over T such that if $V \in \mathcal{S}$ then there exists a closed point $t \in T$ with $\mathcal{V}_t \simeq V$. There also exists an integer n_0 such that for all $n \geqslant n_0$, we have

(i) $H^0(V(n))$ generates $V(n)$ for all $V \in \mathcal{S}$

(ii) $H^i(V(n)) = 0$ for all $i > 0$, for all $V \in \mathcal{S}$.

Fix an $n \geqslant n_0$. Let P be the Hilbert Polynomial of $V(n)$, $V \in \mathcal{S}$ and E be a trivial vector bundle on X of rank $= \dim H^0(V(n))$, $V \in \mathcal{S}$. Let $Q = Q(E/P)$ be the *Quot* scheme of coherent quotients of E with Hilbert polynomial P. For any $V \in \mathcal{S}$, we may assume the existence of a finite extension $F_q \supset F_p$ with X, Q and V all being defined over F_q, which gives us a F_q-rational point of Q. As Q has only finitely many F_q-rational points, we get the important consequence that there are only finitely many isomorphism classes (over k) of elements of $\mathcal{S}$ which are defined over F_q. Let $\pi: X \to X$ be the F_q-linear Frobenius, we get $\pi^{\#}(V) \in \mathcal{S}$ by proposition 1.1. Hence we infer the existence of an infinite sequence of integers $(n_k)_{k=1}^{\infty}$ with $(\pi^{n_i})(V) \simeq (\pi^{n_j})(V)$. Now we deduce V is locally free:

Lemma 1.4: Let M be a finitely generated module over a regular local ring in char $p > 0$ and assume that $\pi^{\#}(M) \simeq M$. Then M is free.

Proof: Let

$$0 \to F_d \xrightarrow{\phi_d} F_{d-1} \ldots F_1 \xrightarrow{\phi_1} F_0 \to M \to 0$$

be a minimal resolution of M. Then by [20, theorem 1.13], the minimal resolution of $\pi^{\#}(M)$ is

$$0 \to F_d \xrightarrow{\phi_d^q} F_{d-1} \ldots F_1 \xrightarrow{\phi_1^q} F_0 \to \pi^{\#}(M) \to 0$$

If $M \simeq \pi^{\#}(M)$ then these two complexes are also isomorphic and hence it follows by comparing their minors for instance, that M is free.

Q.E.D.

So now each $V \in \mathscr{S}$ is a vector bundle on X and there exist integers m, n with $m > n$ and $(\pi^m)^{\#}(V) \simeq (\pi^n)^{\#}(V)$. Putting $(\pi^n)^{\#}(V) = \bar{V}$ and $m - n = t$, we get $(\pi^t)^{\#}(\bar{V}) \simeq \bar{V}$. Now we quote

Proposition 1.5: (Lange and Stuhler [11, theorem 1.4]) Let V be a vector bundle on a scheme X in char $p > 0$. Then $(\pi^t)^{\#}(V)$ is isomorphic to V for some t if and only if V becomes trivial on an étale cover of X.

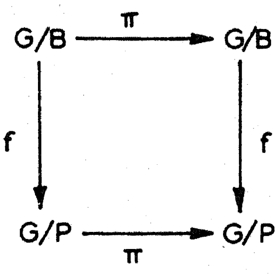
So it follows that $\bar{V}$ is trivial on an étale cover of X.

Proposition 1.6: Let $X = G/P$ be a homogeneous space in char $p > 0$ with G reductive and P parabolic. Let V be a vector bundle on X such that $(\pi^n)^{\#}(V)$ is trivial on X. Then V itself is trivial.

Proof: We first assume P is a Borel subgroup of G. Then by [3] there is a sequence

$$Z = Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} Z_2 \ldots \to Z_m = \mathbf{P}^1$$

with all the f_i locally trivial $\mathbf{P}^1$ fibration and $f: Z \to X$ a birational morphism. Let $g_i: Z \to Z_i$ be the projection and assume that we have proved that $g_i^{\#}(W) \simeq f^{\#}(V)$ for some W on Z_i. It is clear by induction that for any $\mathbf{P}^1 \subset Z_i$, there is a section $s: \mathbf{P}^1 \to Z$ so that $g_i \circ s$ is the inclusion of $\mathbf{P}^1$ in Z_i. Consequently $W \mid \mathbf{P}^1 \simeq s^{\#} f^{\#}(V)$ and this bundle has the property that a high power of its Frobenius is étale-trivialisable, and is therefore easily seen to be trivial by writing it as a direct sum of line bundles. In particular W is trivial when restricted to all the fibres of f_i, showing that $W \simeq f_i^{\#} f_{i\#}(W)$. Putting $f_{i\#}(W) = W_1$, we find that $g_{i+1}(W_1) \simeq f^{\#}(V)$. By induction, $f^{\#}(V)$ is trivial, and because $f_{\#} f(V)^{\#} V$, V itself is trivial. Now assume that P is an arbitrary parabolic subgroup of G and V is a vector bundle on G/P such that $(\pi^n)^{\#}(V)$ is trivial. Let $B \subset P$ be a Borel subgroup of G and let $f: G/B \to G/P$ be the canonical map. We have a commutative diagram:



If $(\pi^n)^{\#}(V)$ is trivial on G/P, then $f^{\#}(\pi^n)^{\#}(V) = (\pi^n)^{\#} f^{\#}(V)$ is trivial on G/B. So $f^{\#}(V)$ is a bundle on G/B with $(\pi^n)^{\#} f^{\#}(V)$ trivial on G/B. By the first part of this proposition, it follows that $f^{\#}(V)$ is trivial on G/B and since V is isomorphic to $f_{\#} f^{\#}(V)$, V is trivial on G/P.

Now suppose $X = G/P$ is defined over an arbitrary algebraically closed field k and V is a semistable torsion-free sheaf on X with $c_i(V) = 0$, $1 \leqslant i \leqslant d$. We may assume (i) There exists a subring A of k which is finitely generated over $\mathbf{Z}$, with quotient field K, such that X, V and G are defined over K. (ii) There is a smooth reductive group-scheme $G_A \to A$ whose general fibre is G. (iii) $P_A \subset G_A$ is a parabolic subgroup scheme with quotient $G_A | P_A$ denoted by X_A. (iv) There is a sheaf V_A on X_A flat over A with $V_A \otimes_A (\Omega)$ semistable and torsion-free on $X_A \otimes_A (\Omega)$ for every geometric point Ω of A and $c_i(V_A \otimes_A \Omega) = 0$, $1 \leqslant i \leqslant d$ and $V_A \otimes_A K = V$.

Now for every maximal ideal $\mathbf{m}$ of A, $F_q = A/\mathbf{m}$ is a finite field and by propositions 1.1–1.6, $V_A \otimes_A \overline{F}_q$ is a trivial vector bundle on $X_A \otimes_A \overline{F}_q$. It easily follows that V_K is trivial on X_K and hence V is trivial on X. Q.E.D.

If we work instead with μ-stability and μ-semistability we have the following improvement of proposition 1.1:

Proposition 1.7: Let X be a projective space or a Grassmannian in char $p > 0$ and let V be a μ-stable sheaf on X. Then $\pi^*(V)$ is also μ-stable.

Proof: In any case, by the proof of proposition 1.1 we infer that $\pi^*(V)$ is μ-semistable. According to [26, theorem 3.23], for any smooth variety X in char $p > 0$ with $\pi^*(V)$ μ-semistable whenever V is μ-semistable, we have that $V_1 \otimes V_2$ is μ-semistable if both V_1 and V_2 are μ-semistable. Now we prove that $\pi^*(V)$ is μ-stable. Let W be a subsheaf of $\pi^*(V)$ with $\mu(W) = \mu(\pi^*(V))$. Now if T_X is the tangent bundle of X, we get a $\mathbf{O}_X$-linear homomorphism $f: T_X \to Hom(W, \pi^*(V)/W)$. If X is a projective space or a Grassmannian, then T_X is μ-semistable with $\mu(T_X) > 0$, as can easily be checked (see below and [12]). But $Hom(W, \pi^*(V)/W)$ is μ-semistable and of degree zero and hence $f = 0$. Hence W descends to a subsheaf $\overline{W}$ say, of V with $\mu(\overline{W}) = \mu(V)$, contradicting the μ-stability of V.

Remark: The tangent bundle of the Grassmanian is a homogeneous bundle, from which it follows that each member of the Harder–Narasimhan flag is also homogeneous. Now the Grassmanian is $Sl(n)/P$ and the action of P on Lie $(Sl(n))$/Lie (P) is irreducible, and therefore there are no proper non-zero homogeneous subbundles of the tangent bundle. Thus the tangent bundle is semi-stable. Note that its μ is positive because its determinant is in fact very ample.

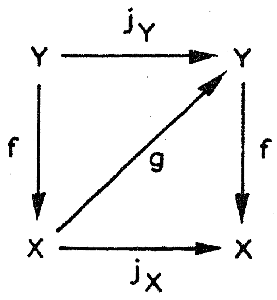
2. Semistable sheaves on abelian varieties

We now turn to the proof of theorem 2. X is an abelian variety over an algebraically closed field k and V is a semistable torsion-free sheaf on X with $c_i(V) = 0, 1 \leqslant i \leqslant d$. We first assume that $k = \overline{F}_p$. Now propositions 1.1, 1.2, 1.4 and 1.5 remain valid for X and V and so we have

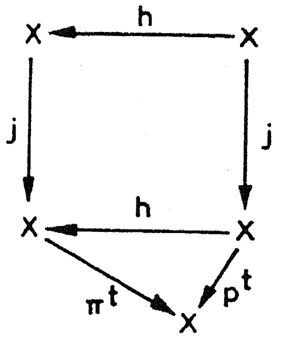
(i) V is a *vector bundle* on X.

(ii) $(\pi^t)^{\#}(\overline{V}) \simeq \overline{V}$, where $\overline{V} = (\pi^n)^{\#}(V)$ and m and n are such that $(\pi^m)^{\#}(V) \simeq (\pi^n)^{\#}(V)$.

Again by [11, theorem 1.4] there exists an étale covering $f\colon Y \to X$ with $f^{\#}(\bar{V})$ trivial on Y. By [18, § 18] Y is an abelian variety, f is a homomorphism and there is a commutative diagram



where j_Y and j_X are multiplication by j on X and Y respectively. Hence $(j_X)^{\#}(\bar{V})$ is trivial on X. Now multiplication by $p^t\colon X \to X$ ($p = \text{char } k$) induces the zero map on the tangent spaces at all the points of X hence there is a commutative diagram



where $h\colon X \to X$ is some homomorphism. We therefore get that $m^{\#}(V)$ is trivial on X, where $m = p^t j$. Let X_m be the *group-scheme* kernel of $m\colon X \to X$. Then $m\colon X \to X$ is a *principal bundle* with structure group X_m. As $m^{\#}(V)$ is trivial on X, V is the bundle on X associated to a *linear representation* of X_m on a finite dimensional vector space say W. As X_m is a commutative group scheme there is a filtration on W

$$0 = W_0 \subset W_1 \ldots \subset W_r = W$$

with each W_i a X_m-subspace of W and dim $W_{i+1}/W_i = 1$ for $0 \leqslant i \leqslant r-1$. It follows that V has a filtration

$$0 = V_0 \subset V_1 \ldots \subset V_r = V$$

with each V_{i+1}/V_i a line bundle on X, say L_{i+1}. Now $m^{\#}(L_{i+1})$ is trivial on X for all i and hence by [18, page 75] each L_{i+1} is algebraically equivalent to zero, *i.e.* each $L_{i+1} \in \mathrm{Pic}^0(X)$.

Now assume k to be arbitrary. As in § 1 we may assume (i) There exists a subfield K of k which is finitely generated over the prime field such that both X and V are defined over K. (ii) There exists a Noetherian subring A of K with K the quotient field of A such that there is an *abelian scheme* X_A over A and a sheaf V_A on X_A flat over A with general fibres X and V respectively and $V_A \times {}_A\Omega$ semistable torsion-free with zero Chern classes for every geometric point Ω of A.

Let $f: X_A \to A$ be the structural map and let $\hat{f}: \hat{X}_A \to A$, where $\hat{X}_A$ is the dual of X_A. Let L be a Poincare bundle on $X_A \otimes_A X_A$. Consider $W = Hom[L, p_1^{\#}(V_A)]$ on $X_A x_A X_A$ where $p_1 : X_A x_A \hat{X}_A \to X_A$ is the first projection. Define $D = (t \in \hat{X}_A | H^0(W_t) \geqslant 1)$. D is a closed subset of X_A. Giving it to the reduced structure we have

Proposition 2.1: The restriction of $\hat{f}$ to D is a surjection: $D \to A$.

Proof: If $A - \hat{f}(D) \neq \phi$ then there exists $g \in A$ and a maximal ideal $\mathbf{m}$ with $g\mathbf{m} \neq 0$. But for any such $\mathbf{m}$, $A/\mathbf{m}$ is a finite field, which contradicts what has been proved earlier. Hence $f: \hat{D} \to A$ is surjective Q.E.D.

So after restricting A suitably we may assume that $D \to A$ and hence $\hat{X}_A \to A$ has a section $\sigma: A \to \hat{X}_A$. We get a line bundle $M = (id_X \times \sigma)^{\#}(L)$ on X and we consider $Hom(M, V)$ on X. We have $H^0[Hom(M_x, V_x)] \geqslant 1$ for every closed point x of A and hence for every point of A. Again by restricting A if necessary we may assume $H^0[Hom(M_x, V_x)] = 1$ for every $x \in A$. Then $f_{\#}[Hom(M, V)]$ is a line bundle on A and we consider the composite

$$M \otimes f^{\#} f_{\#}[Hom(M, V)] \to [M \otimes Hom(M, V)] \to V$$

which we know to be an inclusion of bundles for every closed point of A. Hence M is a line subbundle of V, at least up to a line bundle coming from A. Now $P_M(n) = P_V(n)$ and both M and V are semistable, hence V/M is semistable with zero Chern classes. By induction V/M has a filtration

$$0 = V_1 \subset V_2 \ldots \subset V_r = V/M$$

where each $V_i / V_{i-1} \in \mathrm{Pic}^0(X), 1 \leqslant i \leqslant r$. Hence V has a filtration

$$0 = V_0 \subset V_1 = M \subset V_2 \ldots \subset V_r = V \text{ with}$$

$$V_i / V_{i-1} \in \mathrm{Pic}^0(X),\ 1 \leqslant i \leqslant r.$$ Q.E.D.

Now we can deduce theorem 3 from theorems 1 and 2. We are given a smooth surjective map $f: G \to X$ where G is a smooth algebraic group. Now there exists a *maximal* linear subgroup L of G and an exact sequence.

$$0 \to L \to G \to G/L \to 0$$

with G/L being an abelian variety. We also get a map $G/H \to G/L + H$ with all fibres isomorphic to $L/L \cap H$, and a map $G/L \to G/L + H$ with fibres $L + H/L$. So finally there is a map, also denoted by f, from X to $G/L + H$, which is an abelian variety, which we denote by A. We also have $f^{-1}(a) \simeq L/L \cap H$ for all $a \in A$.

Let V be a semistable torsion free sheaf on X with $c_i(V) = 0, 0 \leqslant i \leqslant d$. Then, by what has been proved earlier, we get

(i) V is a vector bundle on X,

(ii) $(\pi^n)^{\#}(V) \simeq (\pi^n)^{\#}(V)$.

Then $(\pi^m)^{\#}(V)/f^{-1}(a) \simeq (\pi^n)^{\#}(V)/f^{-1}(a)$ for all $a \in A$ and hence $V/f^{-1}(a)$ is trivial for all $a \in A$.

Hence $V = f^{-1}(W)$ with W being a vector bundle on A. Put $\bar{V} = (\pi^t)^{\#}(V)$ and $\bar{W} = (\pi^t)^{\#}(W)$ where $t = m - n$. Then $(\pi^t)^{\#}(\bar{V}) \simeq \bar{V}$ and $(\pi^t)^{\#}(\bar{W})$ is a bundle on

A with $f^{\#}(\pi^t)^{\#}(\bar{W}) = (\pi^t of)^{\#}(\bar{W}) = \overline{(\pi^t)^{\#} f^{\#}(\bar{W})} = (\pi^t)^{\#}(\bar{V}) = \bar{V} = f^{\#}(\bar{W})$. So $(\pi^t)^{\#}(\bar{W}) \simeq \bar{W}$ on A and hence $m^{\#}(W)$ is trivial on A for some integer m. It follows that W has a filtration.

$$0 = W_0 \subset W_1 \ldots \subset W_r = W$$

with $W_i/W_{i-1} \in \mathrm{Pic}^0(A)$, $1 \leqslant i \leqslant r$. Hence V has a filtration

$$0 = V_0 \subset V_1 \ldots \subset V_r = V$$

with V_i/V_{i-1} $\mathrm{Pic}^0(X)$, $1 \leqslant i \leqslant r$. Q.E.D.

Remark 2.2: Theorem 2 generalizes a theorem of Atiyah [1, theorem 5] in the case where $\dim X = 1$.

Remark 2.3: Let V be a semistable sheaf on $\mathbf{P}^n$ in *Characteristic zero.* Then Spindler [21, theorem 3.2] has proved that the instability degree of the restriction of V to a general $\mathbf{P}^1 \subset \mathbf{P}^n$ is bounded above only by the rank of V and not by the Chern classes of V. Theorem 1 shows this result to be false in char p. Take a semistable V on $\mathbf{P}^n$ with $V/\mathbf{P}^1$ not semistable for a general $\mathbf{P}^1 \subset \mathbf{P}^n$. Then $(\pi^t)^{\#}(V)$ is semistable on $\mathbf{P}^n$ for all t but the instability degree of $(\pi^t)^{\#}(V)/\mathbf{P}^1$ goes to infinity.

Remark 2.4: Ellencwajg and Forster [4] have proved the following

Take the set S of semistable sheaves on $\mathbf{P}^n$ in characteristic zero with a fixed rank and c_1 and c_2. Then S is *bounded.* If the same result is true in char p one gets an interesting consequence: take S with $c_1 = c_2 = 0$, Then $(\pi^t)^{\#}(V) \in S$ for all $V \in S$ and $t > 0$ hence every $V \in S$ must become trivial on every $\mathbf{P}^1 \subset \mathbf{P}^n$. This easily implies that each $V \in S$ must be trivial on $\mathbf{P}^n$.

Remark 2.5: Assume V is a vector bundle on an abelian variety A such that V has a filtration

$$0 = V_0 \subset V_1 \ldots \subset V_r = V$$

with $V_i/V_{i-1} \in \mathrm{Pic}^0(A)$, $1 \leqslant i \leqslant r$. We can classify such V as follows

(i) Let $V = \bigoplus_{i=1}^{k} W_i$, with each W_i indecomposable. Then for each i, $1 \leqslant i \leqslant k$, there is a *unique* line bundle $L_i \in \mathrm{Pic}^0(A)$, such that $L_i \otimes W_i$ has a filtration by subbundles with each successive quotient *trivial.* (cf. [19], Chp. 4, [23] Thm. 2.3). (ii) Vector bundles V on A admitting filtrations with trivial quotients as in (i) are in canonical 1–1 correspondence with the isomorphism classes of finite length modules over the local ring at the identity of the dual abelian variety $\hat{A}$. (cf. [19], Chp. 4, [24] Thm. 4.19, [25] Ex. 2.9).

Remark 2.6: Let X be a homogeneous space or an abelian variety. Then it is easy to see that all the results of this paper remain valid for *μ-semistable* torsion-free sheaves on X if one makes the following assumption:

The set V of all μ-semistable sheaves on X is *bounded* for fixed values of $rk\ V$ and $c_i(V)$, $0 \leqslant i \leqslant \dim X$.

S Ramanan has pointed out to us the following application of theorem 2.

Proposition 2.7: Let V be a stable vector bundle on an abelian variety X of dim 3 in char zero. Assume that $NS(X) \simeq \mathbf{Z}$. If Bogomolov's inequality is not strict, then there exists an isogeny $f: Y \to X$ and a line bundle L on Y with $f_*(L) \simeq V$.

Proof: By [26, theorem 3.18] End V is semistable. By our assumption on $NS(X)$, $c_2(\text{End } V) = \alpha H^2$ in $H^4(X; Q)$ for some rational number. α and Bogomolov's inequality says that $c_2(\text{End } V). H \geqslant 0$ (here H denotes the polarization on X). If the inequality is not strict, then $c_2(\text{End } V) = 0$. In any case, $c_i(\text{End } V) = 0$ for odd i.

Hence it follows from theorem 2 that $E \in \text{Pic}^0(X)$ with $E \subset Ad\ V \subset \text{End } V$ inducing $\phi: E \otimes V \to V$. As V is stable, ϕ is an isomorphism and as $E \subset Ad\ V$, E is not the trivial line bundle. By taking determinants we get $E^n \simeq \mathbf{O}_x$, where $n = rk\ V$. Now let $f: Y \to X$ be the isogeny of least degree such that $f^*(E)$ is trivial. It is easy to see that there is a bundle W on Y with $f_*(W) \simeq V$ (See, for example, [32], Lemma 2.5). Repeating the above argument for (Y, W) in place of (X, V), the result follows. Note that the entire argument is valid in char p is one knew Bogomolov's inequalities for an abelian three-fold in char p.

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