Indian J. Pure Appl. Math., **41**(1): 303-312, February 2010

MINIMAX SECOND-ORDER DESIGNS OVER CUBOIDAL REGIONS FOR THE DIFFERENCE BETWEEN TWO ESTIMATED RESPONSES

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Abstract Minimization of the variance of the difference between estimated responses at two points, maximized over all pairs of points in the factor space, is taken as the design criterion. Optimal designs under this criterion are derived, via a combination of algebraic and numerical techniques, for the full second-order regression model over cuboidal regions. Use of a convexity argument and a surrogate objective function significantly reduces the computational burden.

Key words Central composite design, convexity, surrogate objective function.

1. Introduction

Exploration of a response surface, representing the behavior of a quantity of interest in response to the variation in the settings of quantitative explanatory variables or factors, is of crucial importance in many fields of scientific investigation. For instance, in an industrial experiment, the object of interest can be the durability of a product while the explanatory variables are control factors such as the temperature and time settings of heat treatment, amounts of various raw materials, and so on. Similarly, in an agricultural experiment, interest often lies in exploring how the output behaves in response to the doses of various fertilizers and nutrients. An experimental plan for the study of a response surface is called a response surface design.

The criterion for choosing a good response surface design depends on the specific objective of the experiment. For instance, if efficient estimation of the regression coefficients underlying the response surface is of main interest, then a popular design criterion is that of D-optimality (Galil and Kiefer, [3]) which aims at keeping the generalized variance of the estimated regression coefficients small. On the other hand, it has been well recognized, notably by Herzberg [4] and Box and Draper [1], that there are also situations where differences between estimated responses at various points in the factor space are of primary concern. These differences play an important role, for example, in sensitivity analysis with reference to the fitted response surface. In situations of this kind, minimization of the variance of the difference between estimated responses at two points, maximized over all pairs of points in the factor space, is an attractive design criterion (Huda, [5]). The present article aims at investigating such minimax designs when the factor space, i.e., the admissible range of variation of the explanatory variables or factors, is a cuboidal region. Such cuboidal factor spaces arise naturally when the ranges of variation of the individual factors are independent of one another. Then the factor space is the Cartesian product of these individual ranges and, with appropriate scaling for each factor, it becomes cuboidal.

Earlier, optimal second-order designs for regression over hyperspheres under the aforesaid minimaxity criterion were obtained by Huda and Mukerjee [8]. As observed by these authors, the minimax design is then rotatable, i.e., has a spherical variance function (Box, Hunter and Hunter, [2]), and this helps in simplifying its derivation. The corresponding problem for cuboidal regions is, however, much more challenging from both mathematical and computational perspectives. Specifically, unlike what happens with hyperspheres, reduction of the problem by taking recourse to rotatability is no longer possible. Furthermore, as indicated in Section 2, a direct computer search is also infeasible in this context because the computational burden quickly becomes too heavy and hence unmanageable. These reasons hindered the development of a complete solution to the problem of obtaining minimax designs for second-order models over cuboidal regions. Results have so far been obtained only in some restricted cases, either with additional assumptions on the pair of points (e.g., assuming that one of them is the origin) or via consideration of a truncated model; see, for instance, Huda [6, 7]. In what follows, we propose to solve this problem in complete generality, deriving minimax designs for full second-order models over cuboidal regions without any restriction on the pair of points. A convexity argument via a change of variables and the use of an appropriately chosen surrogate objective function help significantly in attaining this goal.

2. Preliminaries

Consider the full second-order model in $k(\geq 2)$ factors

$$E\{y(x)\} = \theta_0 + \sum_{i=1}^{k} \theta_{ii} x_i^2 + \sum_{i=1}^{k} \theta_i x_i + \sum_{1 \leq j < i \leq k} \theta_{ij} x_i x_j,$$

where y is a univariate response and $x = (x_1 \ldots, x_k)' \in [-1,1]^k = \chi$, say. The observations are assumed to be uncorrelated and homoscedastic. Without loss of generality it suffices to consider symmetric, permutation invariant design measures

(Kiefer, [9]). Then the only nonzero elements of the information matrix of a design ξ are

$$\begin{aligned} \alpha_2 &= \int_\chi x_i^2 \xi(dx), \qquad \alpha_4 = \int_\chi x_i^4 \xi(dx) \\ \alpha_{22} &= \int_\chi x_i^2 x_j^2 \xi(dx) (1 \le i \ne j \le k), \end{aligned}$$

with $0 \le \alpha_{22} \le \alpha_4 \le \alpha_2 \le 1$. It is not hard to see that the information matrix is given by

$$M(\xi) = \mathrm{diag}[M_1(\xi), \alpha_2 I_k, \alpha_{22} I_{k^*}],$$

where

$$M_1(\xi) = \begin{bmatrix} 1 & \alpha_2 1_k' \\ \alpha_2 1_k & (\alpha_4 - \alpha_{22}) I_k + \alpha_{22} 1_k 1_k' \end{bmatrix};$$

1_k is the $k \times 1$ vector of 1's, I_k is the identity matrix of order k and $k^* = \frac{1}{2}k(k-1)$. The matrix $M(\xi)$ is positive definite provided $0 < \alpha_{22} < \alpha_4 \le \alpha_2 \le 1$ and $\alpha_4 + (k-1)\alpha_{22} > k\alpha_2^2$. Under these conditions, one can work out an expression for $M^{-1}(\xi)$ and hence show that for any $z = (z_1, \ldots, z_k)'(\in \chi)$ and $t = (t_1, \ldots, t_k)'(\in \chi)$, the variance of the difference between the estimated responses at z and t is proportional to $V(\xi; z, t)$, where

$$\begin{aligned} V(\xi; z, t) &= \frac{1}{\alpha_4 - \alpha_{22}} \sum_{i=1}^k (z_i^2 - t_i^2)^2 + \frac{1}{\alpha_2} \sum_{i=1}^k (z_i - t_i)^2 \\ &+ \frac{1}{\alpha_{22}} \sum_{1 \le j < i \le k} (z_i z_j - t_i t_j)^2 \\ &+ \frac{1}{k} \left\{ \frac{1}{\alpha_4 + (k-1)\alpha_{22} - k\alpha_2^2} - \frac{1}{\alpha_4 - \alpha_{22}} \right\} \left\{ \sum_{i=1}^k (z_i^2 - t_i^2) \right\}^2. \end{aligned} \tag{1}$$

We aim at finding ξ so as to minimize $\max_{z,t \in \chi} V(\xi; z, t)$. The expression for $V(\xi; z, t)$ remains unaltered if (z_i, t_i) is replaced by $(-z_i, -t_i)$ for any i. Hence defining

$$\triangle = \{(z, t) : z = (z_1, \ldots, z_k)' \in \chi, t = (t_1, \ldots, t_k)' \in \chi \text{ and } z_i \ge t_i, 1 \le i \le k\},$$

the maximum of $V(\xi; z, t)$ over $z, t \in \chi$ equals that over $(z, t) \in \triangle$ and it suffices to find ξ so as to minimize the latter. Furthermore, since $\alpha_4 \le \alpha_2$ and, for $(z, t) \in \triangle$, the right-hand side of (1) is nonincreasing in α_4, hereafter we consider ξ such that $\alpha_4 = \alpha_2$. Then, with

$$\tau(\alpha_2) = \max\{0, \alpha_2(k\alpha_2 - 1)/(k-1)\},$$

the aforesaid nonsingularity conditions reduce to

$$\tau(\alpha_2) < \alpha_{22} < \alpha_2, \qquad 0 < \alpha_2 < 1. \tag{2}$$

A naïve direct computational approach to the present minimaxity problem would consist of the following steps:

I. For each fixed ξ, i.e., each fixed α_2 and α_{22} satisfying (2), maximize $V(\xi; z, t)$ over $(z, t) \in \triangle$.

II. Minimize the maximum in step I with respect to α_2 and α_{22}, subject to (2).

The maximization in step I is over $2k$ variables and has to be executed for *every* α_2 and α_{22} satisfying (2). As a result, the direct computational approach takes more than two hours even in the simple case $k = 2$, and quickly becomes unmanageable with increase in k. This underscores the need for reduction of the problem via theoretical arguments. In the next two sections, we work towards achieving this.

3. Reduction of the Objective Function

We first show how a change of variables, coupled with a convexity argument, entails a reduction of the present optimization problem. For any $(z, t) \in \triangle$, let $u = (u_1, \ldots, u_k)' = \frac{1}{2}(z - t)$ and $w = (w_1, \ldots, w_k)' = \frac{1}{2}(z + t)$. Then $u \in [0, 1]^k (= \chi^+$, say) and given u, conditionally $w \in \Omega(u)$, where $\Omega(u)$ is the Cartesian product of the intervals $[-(1 - u_i), 1 - u_i], 1 \leq i \leq k$. Also, with $\alpha_4 = \alpha_2$, it follows from (1) that $V(\xi; z, t) = 4\tilde{V}(\xi; u, w)$, where

$$\begin{aligned}\tilde{V}(\xi; u, w) &= \frac{4}{\alpha_2 - \alpha_{22}} \sum_{i=1}^{k} u_i^2 w_i^2 + \frac{1}{\alpha_2} \sum_{i=1}^{k} u_i^2 + \frac{1}{\alpha_{22}} \sum_{1 \leq j < i \leq k} (u_i w_j + u_j w_i)^2 \\ &+ \frac{4}{k} \left\{ \frac{1}{\alpha_2 + (k-1)\alpha_{22} - k\alpha_2^2} - \frac{1}{\alpha_2 - \alpha_{22}} \right\} \left(\sum_{i=1}^{k} u_i w_i \right)^2. \qquad (3)\end{aligned}$$

Thus our objective is equivalent to choosing ξ so as to minimize the maximum of $\tilde{V}(\xi; u, w)$ over $w \in \Omega(u)$ and $u \in \chi^+$.

For any fixed $u \in \chi^+$, from (3) now observe that $\tilde{V}(\xi; u, w)$ is convex in w, so that it is maximized, with respect to w, at some extreme point of the convex set $\Omega(u)$. Consider a typical extreme point $e = (e_1, \ldots, e_k)'$, where $e_i = 1 - u_i$ for $i \in S, e_i = -(1 - u_i)$ for $i \in \overline{S}$, and S is a possibly empty set in $\{1, \ldots, k\}$ with $\overline{S} = \{1, \ldots, k\} - S$. From (3), after some algebra,

$$\begin{aligned}\tilde{V}(\xi; u, e) &= A_1(\alpha) \sum_{i=1}^{k} u_i^2 (1 - u_i)^2 + A_2(\alpha) \left\{ \sum_{i \in S} u_i (1 - u_i) - \sum_{i \in \bar{S}} u_i (1 - u_i) \right\}^2 \\ &+ \frac{1}{\alpha_{22}} \left(\sum_{i=1}^{k} u_i^2 \right) \left\{ \sum_{i=1}^{k} (1 - u_i)^2 \right\} + \frac{1}{\alpha_2} \sum_{i=1}^{k} u_i^2, \qquad (4)\end{aligned}$$

where

$$\begin{aligned}A_1(\alpha) &= \frac{4}{\alpha_2 - \alpha_{22}} - \frac{2}{\alpha_{22}}, \\ A_2(\alpha) &= \frac{4}{k} \left\{ \frac{1}{\alpha_2 + (k-1)\alpha_{22} - k\alpha_2^2} - \frac{1}{\alpha_2 - \alpha_{22}} \right\} + \frac{1}{\alpha_{22}}. \qquad (5)\end{aligned}$$

By (4), $\widetilde{V}(\xi; u, e)$ remains unaltered if the roles of S and $\overline{S}$ are interchanged, and its maximum, over $u \in \chi^+$, depends on S only through the cardinality of S. Hence, our objective function, namely, the maximum of $\tilde{V}(\xi; u, w)$ over $w \in \Omega(u)$ and $u \in \chi^+$ reduces to, say,

$$\phi(\alpha_2, \alpha_{22}) = \max_{u \in \chi^+} h(\alpha_2, \alpha_{22}; u), \tag{6}$$

where $h(\alpha_2, \alpha_{22}; u) = \max\{\widetilde{V}(\xi; u, e^{(v)}) : m \leq v \leq k\}, m$ is the greatest integer in $\frac{1}{2}(k+1)$, and

$$e^{(v)} = (1-u_1, \ldots, 1-u_v, -(1-u_{v+1}), \ldots, -(1-u_k))'.$$

Since for $u \in \chi^+$

$$\left|\sum_{i=1}^{v} u_i(1-u_i) - \sum_{i=v+1}^{k} u_i(1-u_i)\right| \leq \sum_{i=1}^{k} u_i(1-u_i),$$

from (4) we also get

$$\begin{aligned} h(\alpha_2, \alpha_{22}; u) &= A_1(\alpha) \sum_{i=1}^{k} u_i^2(1-u_i)^2 + A_2(\alpha)\psi(u) \\ &+ \frac{1}{\alpha_{22}} \left(\sum_{i=1}^{k} u_i^2\right) \left\{\sum_{i=1}^{k} (1-u_i)^2\right\} + \frac{1}{\alpha_2} \sum_{i=1}^{k} u_i^2, \end{aligned} \tag{7}$$

with

$$\begin{aligned} \psi(u) &= \left\{\sum_{i=1}^{k} u_i(1-u_i)\right\}^2 \quad \text{if} \quad A_2(\alpha) \geq 0, \\ &= \min_{m \leq v \leq k} \left\{\sum_{i=1}^{v} u_i(1-u_i) - \sum_{i=v+1}^{k} u_i(1-u_i)\right\}^2 \text{if} A_2(\alpha) < 0. \end{aligned} \tag{8}$$

In view of (8), the sign of $A_2(\alpha)$ is crucial in the subsequent development. To that effect, for $0 < \alpha_2 < 1$, let

$$\begin{aligned} p(\alpha_2) &= \frac{1}{2}(k+3)^{-1}\alpha_2[k-2+(k+4)\alpha_2 \\ &+ \{k^2+16-2(k^2+4k+8)\alpha_2+(k+4)^2\alpha_2^2\}^{1/2}], \\ q(\alpha_2) &= \frac{1}{2}(k+3)^{-1}\alpha_2[k-2+(k+4)\alpha_2 \\ &- \{k^2+16-2(k^2+4k+8)\alpha_2+(k+4)^2\alpha_2^2\}^{1/2}]. \end{aligned}$$

One can check that both $p(\alpha_2)$ and $q(\alpha_2)$ are real and that

$$q(\alpha_2) \leq \tau(\alpha_2) < p(\alpha_2) < \alpha_2, \tag{9}$$

where $\tau(\alpha_2)$ is as defined in the context of (2). Then the following lemma holds.

Lemma 1. *The quantity $A_2(\alpha)$ is positive, zero or negative according as α_{22} is less than, equal to or greater than $p(\alpha_2)$, respectively.*

Proof. From (5), with additional algebra, the sign of $A_2(\alpha)$ turns out to be the same as that of

$$\alpha_2^2(1-k\alpha_2)+\{k-2+(k+4)\alpha_2\}\alpha_2\,\alpha_{22}-(k+3)\alpha_{22}^2.$$

For fixed α_2 , the above is a concave quadratic function of α_{22}, with zeros at $p(\alpha_2)$ and $q(\alpha_2)$. The lemma is now evident from (2) and (9). □

4. Final Results via a Surrogate Objective Function

Notwithstanding the reduction achieved in the last section, the objective function $\phi(\alpha_2,\alpha_{22})$ in (6) still remains in the form of a maximum over the k elements of u. Therefore, direct minimization of $\phi(\alpha_2,\alpha_{22})$, subject to (2), would call for maximization over k variables for *every* (α_2,α_{22}), and the computational burden is formidable unless k is rather small. Consideration of a surrogate objective function entails significant further simplification. Let $J(\subset\chi^+)$ consist of those u with all elements equal, and $T=\{0,1\}^k$. Then $J\cup T\subset\chi^+$, so that

$$\phi(\alpha_2,\alpha_{22})\geq\phi^*(\alpha_2,\alpha_{22}), \tag{10}$$

where, for any (α_2,α_{22}) satisfying (2),

$$\phi^*(\alpha_2,\alpha_{22})=\max_{u\in J\cup T}h(\alpha_2,\alpha_{22};u), \tag{11}$$

with $h(\alpha_2,\alpha_{22};u)$ as in (7).

We now study $\phi^*(\alpha_2,\alpha_{22})$ in some detail as a surrogate objective function. If $\alpha_{22}\leq p(\alpha_2)$ then by Lemma 1, $A_2(\alpha)\geq 0$, and hence for any fixed $u(\in J\cup T)$ and $\alpha_2(0<\alpha_2<1)$, it can be seen from (5), (7) and (8) that $h(\alpha_2,\alpha_{22};u)$ is nonincreasing in α_{22}. Thus in order to minimize $\phi^*(\alpha_2,\alpha_{22})$ it is enough to consider the region

$$p(\alpha_2)\leq\alpha_{22}<\alpha_2,\qquad 0<\alpha_2<1. \tag{12}$$

For (α_2,α_{22}) satisfying (12), from (7) and (8), one can show that

$$\max_{u\in T}h(\alpha_2,\alpha_{22};u)=\alpha_{22}^{-1}m(k-m)+\alpha_2^{-1}m=c_1(\alpha), \tag{13}$$

say, m being as before the greatest integer in $\frac{1}{2}(k+1)$. Furthermore, for any such (α_2,α_{22}) and any $u\in J$, writing $\overline{u}$ for the common value of the elements of u, by (5), (7) and (8),

$$h(\alpha_2,\alpha_{22};u)=k[\{\overline{u}(1-\overline{u})\}^2g(\alpha)+\alpha_2^{-1}\overline{u}^2]=c_2(\alpha;\overline{u}), \tag{14}$$

say, where

$$g(\alpha)=4(\alpha_2-\alpha_{22})^{-1}+(k-2)\alpha_{22}^{-1}+\delta k^{-1}A_2(\alpha),$$

with $\delta = 0$ or 1 for even or odd k respectively. It can be seen that $g(\alpha) > 0$. Suppose $3 \leq k \leq 10$ (the case $k = 2$ is discussed later). Then we also have $0 < u_0 < 1$, where

$$u_0 \equiv u_0(\alpha) = \frac{3}{4} - \frac{1}{4}[1 - 8\{\alpha_2 g(\alpha)\}^{-1}]^{1/2}.$$

Consideration of the sign of the first derivative with respect to $\overline{u}$ now shows that $c_2(\alpha; \overline{u})$ in (14) is maximum, over $\overline{u}$, at $\overline{u} = u_0$ or 1. Since by (13) and (14), $c_1(\alpha) > c_2(\alpha; 1)$ from (11) it follows that

$$\phi^*(\alpha_2, \alpha_{22}) = \max\{c_2(\alpha; u_0), c_1(\alpha)\}. \tag{15}$$

By (13)-(15), the surrogate objective function is much easier to compute than the original one and its numerical minimization, subject to (12), is quite straightforward. For $3 \leq k \leq 10$, Table 1 shows the minimizer $(\alpha_2^0, \alpha_{22}^0)$ of $\phi^*(\alpha_2, \alpha_{22})$ as well as the value of $\phi^*(\alpha_2^0, \alpha_{22}^0)$. Interestingly, every $(\alpha_2^0, \alpha_{22}^0)$ in this table satisfies $\alpha_{22}^0 = p(\alpha_2^0)$

Table 1: Minimizer and the minimum value of the surrogate objective function.

k	3	4	5	6	7	8	9	10
α_2^0	0.766	0.794	0.827	0.844	0.863	0.874	0.887	0.895
α_{22}^0	0.600	0.642	0.693	0.720	0.751	0.769	0.790	0.804
$\phi^*(\alpha_2^0, \alpha_{22}^0)$	5.94	8.75	12.29	16.05	20.62	25.37	30.95	36.69

We now return to the original objective function $\phi(\alpha_2, \alpha_{22})$ given by (6). For each $(\alpha_2^0, \alpha_{22}^0)$ in Table 1, it is seen that

$$\phi(\alpha_2^0, \alpha_{22}^0) = \phi^*(\alpha_2^0, \alpha_{22}^0). \tag{16}$$

In view of (10), this springs the pleasant surprise that for each k ($3 \leq k \leq 10$), the tabulated $(\alpha_2^0, \alpha_{22}^0)$ really represents the minimax design which minimizes $\phi(\alpha_2, \alpha_{22})$ and that $\phi^*(\alpha_2^0, \alpha_{22}^0)$ is actually the minimum possible value of $\phi(\alpha_2, \alpha_{22})$. In fact, our computations suggest that this technique should work for even larger values of k. Note that in order to verify (16) one needs to compute $\phi(\alpha_2, \alpha_{22})$ only at $(\alpha_2^0, \alpha_{22}^0)$ which is far simpler than direct minimization of $\phi(\alpha_2, \alpha_{22})$.

The above technique, however, fails for k=2, where $\phi(\alpha_2, \alpha_{22})$ turns out to be greater than $\phi^*(\alpha_2, \alpha_{22})$ at the minimizer of the latter. Fortunately, in this case, evaluation of $\phi(\alpha_2, \alpha_{22})$ is relatively simple, and a direct search shows that $\phi(\alpha_2, \alpha_{22})$ is minimum, subject to (2), at $\alpha_2 = 0.702, \alpha_{22} = 0.514$, the corresponding minimum value being 3.49. Here also $\alpha_{22} = p(\alpha_2)$ for the optimal solution.

5. Concluding Remarks

Having obtained the minimax designs, it makes sense to compare them with the optimal ones under other criteria. Specifically, for $2 \leq k \leq 10$, Table 2 shows the D-efficiency of our minimax design as well as the minimax efficiency of the D-optimal design. As usual, these are defined respectively as

$$D\text{-eff}\,(\xi_{\text{minimax}}) = \left[\frac{\det\{M(\xi_{\text{minimax}})\}}{\det\{M(\xi_D)\}}\right]^{1/s}$$

and

$$\text{minimax-eff}(\xi_D) = \frac{\max_{z,t\in\chi} V(\xi_{\text{minimax}}; z, t)}{\max_{z,t\in\chi} V(\xi_D; z, t)},$$

where ξ_{minimax} and ξ_D are the minimax and D-optimal designs and $s = \frac{1}{2}(k+1)(k+2)$ is the number of regression coefficients in the model. Note that the D-optimal designs for second-order models over cuboidal regions are already available in the literature (Galil and Kiefer, [3]) and that, like the minimax designs, these also satisfy $\alpha_4 = \alpha_2$. Thus, in view of the findings in Section 3, the minimax efficiency of a D-optimal design can be calculated simply as the ratio of $\phi(\alpha_2, \alpha_{22})$ for the minimax design to that for the D-optimal design.

Table 2: D-efficiency of minimax design and minimax efficiency of D-optimal design

k	2	3	4	5	6	7	8	9	10
D-eff (ξ_{minimax})	0.993	0.995	0.993	0.995	0.994	0.995	0.995	0.996	0.995
minimax-eff (ξ_D)	0.900	0.910	0.876	0.886	0.866	0.872	0.858	0.862	0.852

From Table 2, the minimax efficiencies of the D-optimal designs are not unsatisfactory. However, they come nowhere close to the D-efficiencies of our minimax designs, which remain consistently above 0.99, for $2 \leq k \leq 10$.

Since the minimax designs have $\alpha_4 = \alpha_2$, they are supported at the points of the 3^k factorial having coordinates 0 and ± 1. Partition these 3^k points into k+1 sets with the ith set G_i consisting of the $\binom{k}{i} 2^i$ points with i nonzero coordinates. To obtain the actual minimax design for any k, it suffices to distribute a mass $\rho_i (\geq 0)$ over each point of G_i $(0 \leq i \leq k)$, such that the equations

$$\sum_{i=0}^{k} \binom{k}{i} 2^i \rho_i = 1, \quad \sum_{i=1}^{k} \binom{k-1}{i-1} 2^i \rho_i = \alpha_2, \quad \sum_{i=2}^{k} \binom{k-2}{i-2} 2^i \rho_i = \alpha_{22}, \tag{17}$$

hold, where the pair (α_2, α_{22}) corresponds to the minimax design. For $2 \leq k \leq 10$, with the minimax (α_2, α_{22}) as shown in the last section, a particular nonnegative solution to these equations emerges as

$$\begin{aligned}\rho_0 &= 1-2\alpha_2+\alpha_{22}, \qquad \rho_{k-1}=(\alpha_2-\alpha_{22})/2^{k-1},\\ \rho_k &= \{(k-1)\alpha_{22}-(k-2)\alpha_2\}/2^k, \text{ and}\\ \rho_i &= 0 \text{ for all other } i.\end{aligned}$$

The results in this article concern optimal continuous designs under the criterion of minimaxity. These results serve as useful benchmarks and provide guidelines for the efficient construction of exact N-observation designs. In the spirit of the last paragraph, for this purpose we consider exact designs that assign n_i observations to each point of $G_i, 0 \le i \le k$. The n_i should be so chosen that, with $\rho_i = n_i/N$, the first equation in (17) is satisfied exactly and the next two equations are met as far as practicable. Examples show that this technique can yield exact designs of reasonable size and high minimax efficiency. Thus, with k = 2, the exact designs $n_0 = n_1 = n_2 = 1(N = 9)$ and $n_0 = n_1 = 2, n_2 = 3(N = 22)$ have minimax efficiencies 0.929 and 0.976 respectively. Similarly, with k = 3, the exact designs $n_0 = n_2 = 0, n_1 = n_3 = 1(N = 14)$ and $n_0 = 2, n_1 = 0, n_2 = n_3 = 1(N = 22)$ have respective minimax efficiencies 0.911 and 0.926. Note that both designs for k = 2 as well as the first design for k = 3 are central composite designs while the second design for k = 3 is a Kôno design. Their minimax efficiencies are impressive especially in the light of the fact that these efficiencies are relative to optimal continuous designs that are not attainable with a finite number of observations.

Acknowledgement

Thanks are due to a referee for very constructive suggestions. The work of RM was supported by a grant from the Indian Institute of Management Calcutta.

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