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QUANTUM CONSTRAINT ALGEBRA FOR AN INTERACTING SUPERSTRING

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ABSTRACT

We perform the Hamiltonian BRST quantization of the type II superstrings interacting with a gravitational background. The quantum constraint algebra is derived to lowest (non-trivial) order in the space-time curvature, exhibiting the complete structure of c and q-number anomalies.

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1. - INTRODUCTION AND OUTLINE

The basic concern of the modern first quantized approach to string theory is Polyakov's path integral [1] describing a string moving in a set of external fields, e.g., in the simplest case in a gravitational field (or metric) $g_{\mu\nu}(x)$.

The importance of this point of view goes back to the work of Ademollo et al. [2] who noticed that string propagation in a background generates the dual model S-matrix.

In the case of a gravitational field this S-matrix describes interacting gravitons. Thus Polyakov's path integral represents a new - and possibly consistent - approach to quantum gravity itself [3] based on the quantization of a two-dimensional field theory.

Classically, an arbitrary gravitational field can be introduced without spoiling the two-dimensional local invariances of the theory, the resulting constraints or the BRST charge nilpotency. This is particularly transparent in the Hamiltonian BRST approach [4]. Indeed, while the metric $g_{\mu\nu}(X)$ enters in the expression of the constraints in term of canonical co-ordinates and momenta, it disappears completely from their algebra, which takes the free form (i.e., $g_{\mu\nu} = \eta_{\mu\nu}$) even for arbitrary $g_{\mu\nu}(X)$ [5].

By contrast, it is well known that this is not true at the quantum level. Already in the free case the algebra gets modified by the appearance of the so-called central charge [6] (giving the famous D = 26 or D = 10 for the dimensionality of space-time). In the presence of external fields, further anomalies appear, which can be eliminated only by imposing suitable differential equations on the backgrounds themselves. The above conditions turn out to be the (string modified) classical equations of motions for the external fields [e.g., Einstein-type equations for $g_{\mu\nu}(X)$] and thus contain all the dynamical information. These anomalies can be studied in several ways, the most popular being the one based on the renormalization of the associated σ-model and on the condition that the corresponding β-functions vanish [7,8,9].

Not much has been done instead on the Hamiltonian approach for the interacting case, which is certainly the most appropriate one for studying BRST invariance, unitarity and the modifications of the constraint algebra. Indeed, while the Hamiltonian approach is conceptually rather direct, it is usually unpractical for computations in contrast to the more conventional Lagrangian framework.

On the basis of experience gained on analogous problems for the relativistic particle [10], one can guess that great simplifications should occur in the

Hamiltonian approach when the theory is supersymmetric. This is precisely what we shall exploit in this paper for the case of (type II closed) superstrings in a gravitational background.

The outline of the paper is follows. In Section 2 we shall recall the action of the superstring model to be dealt with and determine the classical form of the constraints by working in the orthonormal gauge. An Appendix deals with the analogous problem for the free superstring in an arbitrary gauge. In Section 3, we show, in a simple way allowed by supersymmetry that the classical constraint algebra is not affected by $g_{\mu\nu}(X)$. In Sections 4 and 5 we deal with the quantum case. After a brief summary of the free case, we discuss our model to lowest (non-trivial) order in the normal co-ordinate expansion of the background gravitational field. Extensive use of supersymmetry and of the techniques developed in Ref. [11] allow us to fix many quantization ambiguities and to obtain in a simple way the known constraints on $g_{\mu\nu}(X)$ and the - so far unknown - quantum constraint algebra with the complete c and q-number anomaly structure.

2. - CLASSICAL CONSTRAINTS AND CANONICAL VARIABLES

We consider throughout the case of type II closed superstrings propagating in a gravitational background. As it will become clear, our method can be straightforwardly extended to other kinds of type II superstrings in arbitrary massless backgrounds.

A question we have to face at the outset is the construction of the corresponding action. The best attitude would be to work with an action formulated in an arbitrary gauge, which would have the form of a two-dimensional supergravity with matter couplings (the bosonic and fermionic superstring variables) and auxiliary two-dimensional metric and gravitino fields. Although this is possible in principle, by suitably generalizing the strategy of Ref. [12] to the present case, it is much more simple to work in the orthonormal gauge (ON gauge), where the zweibein is constant (normalized to one) and the gravitino field vanishes. Correspondingly, the action takes the simple form [13]

$$I = \int dt\, d\sigma \left[\frac{1}{2} g_{\mu\nu}(X)\, \partial_a X^\mu \partial^a X^\nu + \frac{i}{2} \chi^\alpha \gamma^0 \gamma^a D_a \chi_\alpha + \frac{1}{12} R_{\alpha\beta\gamma\delta}(X) (\chi^\alpha \gamma^0 \chi^\gamma)(\chi^\beta \gamma^0 \chi^\delta) \right] \tag{2.1}$$

where the spinor covariant derivative is defined by

$$D_a \chi^\alpha \equiv \partial_a \chi^\alpha - \omega_{\mu,\alpha\beta}(X)\,\chi^\beta \partial_a X^\mu . \tag{2.2}$$

Some notational comments are useful. Latin indices are (two-valued) world-sheet labels, raised and lowered by the flat metric $\eta_{oo} = -\eta_{11} = 1$ (ON gauge) - they are throughout never explicitly exhibited for spinors, whose Majorana nature ($\chi^* = \chi$) has already been used in Eq. (2.1). Our convention for the γ matrices is

$$\gamma^0 = \sigma_2 \ , \quad \gamma^1 = i\,\sigma_1 \ , \quad \gamma_5 \equiv \gamma^0\gamma^1 = \sigma_3 \ .$$

Greek indices are D-valued target space labels, and in particular $\mu,\nu,\ldots$ and $\alpha,\beta,\ldots$ are world and tangent space indices, respectively. Finally, $g_{\mu\nu}(X)$ is the background gravitational metric, expressed in terms of the target space vielbein by $g_{\mu\nu}(X) = e_{\mu\alpha}(X)e^\alpha_\nu(X)$, while the metric connection, the spin connection and the curvature tensor are given by the usual expressions

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}\, g^{\mu\sigma}\left(\partial_\lambda g_{\sigma\nu} + \partial_\nu g_{\sigma\lambda} - \partial_\sigma g_{\lambda\nu}\right), \tag{2.3a}$$

$$\omega_{\mu,\alpha\beta} = -e_{\alpha\lambda}\,\partial_\mu e^\lambda{}_\beta - e_{\alpha\nu}\,\Gamma^\nu{}_{\mu\lambda}\, e^\lambda{}_\beta , \tag{2.3b}$$

$$R_{\alpha\beta\gamma\delta} = e_\alpha{}^\mu e_\beta{}^\nu \left(\partial_\mu \omega_{\nu,\gamma\delta} - \partial_\nu \omega_{\mu,\gamma\delta} - \omega_{\mu,\gamma\epsilon}\,\omega_{\nu,\epsilon\delta} + \omega_{\nu,\gamma\epsilon}\,\omega_{\mu,\epsilon\delta}\right). \tag{2.3c}$$

The action (2.1) is invariant under general co-ordinate transformations and local Lorentz rotations in the target space, as well as under global world-sheet supersymmetry transformations

$$\delta X^\mu = \bar{\epsilon}\, e^\mu{}_\alpha\, \chi^\alpha , \tag{2.4a}$$

$$\delta\chi^\alpha = -i\,\partial\!\!\!/ X^\mu e_\mu{}^\alpha \epsilon - \Gamma^\mu{}_{\nu\rho}\,\bar{\epsilon}\,\chi^\beta\chi^\gamma\, e^\alpha{}_\mu\, e^\nu{}_\beta\, e^\rho{}_\gamma , \tag{2.4b}$$

where ε is a constant two-dimensional Majorana spinor (indeed manifest local world-sheet supersymmetry is spoiled by the ON gauge choice).

We are now in position to derive the constraints of our dynamical system. Suppose we had started with an action formulated in an arbitrary gauge. Then the desired constraints would have arisen as the equations of motion for the zweibein and the gravitino field (they are indeed auxiliary variables). Now, the variation of the action with respect to these fields yields the energy-momentum tensor and the supercurrent, and so we conclude that the constraints we are looking for consist in the vanishing of these two objects. These constraints can be obtained quite simply in the ON gauge, by setting to zero the energy-momentum tensor and the supercurrent as computed by means of the Noether procedure applied to Eq. (2.1).

We remark that this strategy is already known to work for the bosonic string [14] and is explicitly checked for the free superstring (see Appendix), for which a gauge-independent formulation exists [13]. Moreover, again on the basis of the bosonic string experience, we expect that the constraints derived by the above procedure should become gauge-independent, once they are expressed in terms of canonical variables.

Explicitly, the Noether construction yields the following expressions for the supercurrent and energy-momentum tensor - to be set to zero as constraints

$$J^a = e_{\mu\alpha} \partial X^\mu \gamma^a \chi^\alpha = \left(e^\mu{}_\alpha \gamma^0 \Pi_\mu + e_{\mu\alpha} \gamma^1 X'^\mu \right) \gamma^a \chi^\alpha , \tag{2.5}$$

$$\Lambda \equiv T_{01} = T_{10} = P_\mu X'^\mu + \frac{i}{2} \chi^\alpha \chi'_\alpha , \tag{2.6a}$$

$$\begin{aligned} B \equiv T_{00} = T_{11} = {} & \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu + \frac{1}{2} g_{\mu\nu} X'^\mu X'^\nu + \\ & + \frac{i}{2} \chi^\alpha \gamma_5 \chi'_\alpha - \frac{i}{2} \omega_{\mu,\alpha\beta} \chi^\alpha \gamma_5 \chi^\beta X'^\mu - \\ & - \frac{1}{12} R_{\alpha\beta\gamma\delta} (\chi^\alpha \gamma^0 \chi^\gamma)(\chi^\beta \gamma^0 \chi^\delta) . \end{aligned} \tag{2.6b}$$

Observe that - with an eye to the subsequent Hamiltonian treatment - we have expressed the constraints (2.5), (2.6) in terms of the "co-ordinates" X^μ, χ^α and the correspondingly conjugate "momenta", with

$$P_\mu = g_{\mu\nu} \dot{X}^\nu - \frac{i}{2} \omega_{\mu,\alpha\beta} \chi^\alpha \chi^\beta , \tag{2.7}$$

while the canonical momentum associated with χ^α coincides (up to a constant) with χ^α itself (second-class constraint). Moreover, for later convenience we have introduced the quantity

$$\Pi_\mu \equiv P_\mu + \frac{i}{2} \omega_{\mu,\alpha\beta} \chi^\alpha \chi^\beta = g_{\mu\nu} \dot{X}^\nu \tag{2.8}$$

As usual, the constraint $\Lambda = 0$ does not involve $g_{\mu\nu}(X)$ - its kinematical nature stems from the σ-reparametrization invariance of the theory. As already mentioned, there is a second-class constraint in the fermionic sector. A similar situation already occurred for the N = 1 supersymmetric particle [15] and for the compactified bosonic string upon fermionization of the compact co-ordinates [14], and it can be straightforwardly handled. Observe further that this second-class constraint would have also involved X^μ had we taken $\chi^\mu \equiv e^\mu_\alpha(X)\chi^\alpha$ as fermionic co-ordinates, a fact that would greatly complicate the constraint algebra. This explains our choice of χ^α as co-ordinates. Below, we shall discuss yet another choice of fermionic co- ordinates which is completely free of second-class constraints.

As we shall see in Section 3, the classical constraint algebra takes a particularly eloquent form by introducing the chiral fermionic projections

$$\psi_\pm^\alpha \equiv \frac{1}{2}\left(1 \mp \gamma_5\right) \chi^\alpha \tag{2.9}$$

in terms of which the Lagrangian in Eq. (2.1) reads

$$\begin{aligned} L = {} & \frac{1}{2} g_{\mu\nu}(X)\partial_a X^\mu \partial^a X^\nu + \frac{i}{2}\left(\psi_+^\alpha D_0 \psi_{+\alpha} - \psi_+^\alpha D_1 \psi_{+\alpha}\right) + \\ & + \frac{i}{2}\left(\psi_-^\alpha D_0 \psi_{-\alpha} + \psi_-^\alpha D_1 \psi_{-\alpha}\right) - \\ & - \frac{1}{12} R_{\alpha\beta\gamma\delta}(X)\left[\psi_+^\alpha, \psi_-^\gamma\right]\left[\psi_+^\beta, \psi_-^\delta\right] . \end{aligned} \tag{2.10}$$

Defining further the auxiliary bosonic quantities

$$S_\alpha^\pm \equiv \frac{1}{2}\left(\pi_\mu e^\mu{}_\alpha \pm X'^\mu e_{\mu\alpha}\right) \tag{2.11}$$

and the chiral supersymmetry charges

$$Q^\pm \equiv \frac{1}{2}\left(1 \mp \gamma_5\right) J^0 \tag{2.12}$$

the constraints (2.5) and (2.6) can be rewritten as follows

$$Q^\pm = 2\, S^\pm{}_\alpha \psi_\pm^\alpha \tag{2.13}$$

$$\begin{aligned} T^\pm \equiv {} & \frac{1}{2}\left(T_{00} \pm T_{01}\right) = S_\alpha^\pm S^{\pm\alpha} \mp \frac{i}{2}\left(D_\sigma \psi_\pm^\alpha\right)\psi_{\pm\alpha} + \\ & + \frac{1}{24} R_{\alpha\beta\gamma\delta}\left[\psi_+^\alpha, \psi_-^\gamma\right]\left[\psi_+^\beta, \psi_-^\delta\right] . \end{aligned} \tag{2.14}$$

Now, Lagrangian (2.10) suggests other convenient sets of fermionic canonical variables, which are free of second-class constraints. Explicitly, they are given by

$$\psi^{\alpha} \equiv \frac{1}{\sqrt{2}}\left(\psi_+^{\alpha} + i\,\psi_-^{\alpha}\right) \qquad (2.15a)$$

$$\hat{\psi}^{\alpha} \equiv \frac{1}{\sqrt{2}}\left(\psi_+^{\alpha} - i\,\psi_-^{\alpha}\right) \qquad (2.15b)$$

from which we can also construct

$$\psi^{\mu} \equiv e^{\mu}{}_{\alpha}\,\psi^{\alpha} \quad , \quad \psi_{\mu} \equiv e_{\mu\alpha}\,\psi^{\alpha} \qquad (2.16a)$$

$$\hat{\psi}_{\mu} \equiv e_{\mu\alpha}\,\hat{\psi}^{\alpha} \quad , \quad \hat{\psi}^{\mu} \equiv e^{\mu}{}_{\alpha}\,\hat{\psi}^{\alpha} . \qquad (2.16b)$$

Correspondingly, the fermionic kinetic part of Lagrangian (2.10) can be written in three equivalent forms (up to a total τ derivative)

$$\frac{i}{2}\left(\psi_+^{\alpha} D_0 \psi_{+\alpha} + \psi_-^{\alpha} D_0 \psi_{-\alpha}\right) \sim \begin{cases} i\,\hat{\psi}^{\alpha}\dot{\psi}_{\alpha} + i\,\omega_{\mu,\alpha\beta}\,\dot{X}^{\mu}\hat{\psi}^{\alpha}\psi^{\beta} & (2.17a) \\ i\,\hat{\psi}_{\mu}\dot{\psi}^{\mu} + i\,\Gamma^{\lambda}{}_{\mu\nu}\dot{X}^{\mu}\hat{\psi}_{\lambda}\psi^{\nu} & (2.17b) \\ i\,\hat{\psi}^{\mu}\dot{\psi}_{\mu} - i\,\Gamma^{\lambda}_{\mu\nu}\,\dot{X}^{\mu}\hat{\psi}^{\nu}\psi_{\lambda} & (2.17c) \end{cases}$$

showing that ψ and $\hat{\psi}$ are canonically conjugate variables. This entails in turn different expressions for the bosonic momenta. We are thus led - exactly like in Ref. [11] - to three different sets of classical canonical variables

$$\Omega \equiv \left(X^{\mu}, P_{\mu}\,;\,\psi^{\alpha}, \hat{\psi}^{\alpha}\right) \qquad (2.18a)$$

$$\Omega_1 \equiv \left(X^{\mu}, P^{(1)}_{\mu}\,;\,\psi^{\mu}, \hat{\psi}_{\mu}\right) \qquad (2.18b)$$

$$\Omega_2 \equiv \left(X^{\mu}, P^{(2)}_{\mu}\,;\,\psi_{\mu}, \hat{\psi}^{\mu}\right) \qquad (2.18c)$$

obeying the canonical Poisson brackets*)

$$\left\{X^{\mu}(\sigma), P_{\nu}(\sigma')\right\}_{PB} = i\,\delta^{\mu}_{\nu}\,\delta(\sigma-\sigma') , \qquad (2.19a)$$

$$\left\{\psi^{\alpha}(\sigma), \hat{\psi}_{\beta}(\sigma')\right\}_{PB} = \delta^{\alpha}{}_{\beta}\,\delta(\sigma-\sigma') ; \qquad (2.19b)$$

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*) Note that our Poisson brackets are defined somewhat unconventionally, so that they turn into (anti)commutators at the quantum level (without i factors).

$$\{X^\mu(\sigma), P^{(1)}_\nu(\sigma')\}_{PB} = i\,\delta^\mu{}_\nu\,\delta(\sigma-\sigma') \quad , \tag{2.20a}$$

$$\{\psi^\mu(\sigma), \hat{\psi}_\nu(\sigma')\}_{PB} = \delta^\mu{}_\nu\,\delta(\sigma-\sigma') \quad ; \tag{2.20b}$$

$$\{X^\mu(\sigma), P^{(2)}_\nu(\sigma')\}_{PB} = i\,\delta^\mu{}_\nu\,\delta(\sigma-\sigma') \quad , \tag{2.21a}$$

$$\{\psi_\mu(\sigma), \hat{\psi}^\nu(\sigma')\}_{PB} = \delta^\nu{}_\mu\,\delta(\sigma-\sigma') \quad . \tag{2.21b}$$

The newly defined momenta P_μ, $P^{(1)}_\mu$, $P^{(2)}_\mu$ are related to the velocity $\Pi_\mu = g_{\mu\nu}\dot{X}^\nu$ [see Eq. (2.8)] as follows

$$\Pi_\mu = \begin{cases} P_\mu + i\,\omega_{\mu,\alpha\beta}\,\hat{\psi}^\alpha\psi^\beta & (2.22a) \\ P^{(1)}_\mu - i\,\Gamma^\lambda{}_{\mu\nu}\,\hat{\psi}_\lambda\,\psi^\nu & (2.22b) \\ P^{(2)}_\mu - i\,\Gamma^\lambda{}_{\mu\nu}\,\psi_\lambda\,\hat{\psi}^\nu & (2.22c) \end{cases}$$

so their interrelations are given by

$$P^{(2)}_\mu - P^{(1)}_\mu = -i\,\partial_\mu g_{\lambda\rho}\,\hat{\psi}^\lambda\psi^\rho \tag{2.23a}$$

$$P^{(1)}_\mu - P_\mu = i\,e^\nu{}_\alpha\,\partial_\mu e_{\nu\beta}\,\hat{\psi}^\alpha\psi^\beta \tag{2.23b}$$

$$P^{(2)}_\mu - P_\mu = -i\,e^\nu{}_\alpha\,\partial_\mu e_{\nu\beta}\,\hat{\psi}^\alpha\psi^\beta \ . \tag{2.23c}$$

3. - CLASSICAL CONSTRAINT ALGEBRA

The advantage of dealing with the newly defined sets of canonical variables Ω_1, Ω_2 at once, becomes manifest by introducing the following (linear combination of the) supersymmetry charges

$$Q \equiv \frac{1}{\sqrt{2}}\left(Q^+ + i\,Q^-\right) \tag{3.1a}$$

$$\hat{Q} \equiv \frac{1}{\sqrt{2}}\left(Q^+ - i\,Q^-\right) \tag{3.1b}$$

As it can be very easily discovered, their canonical expressions read

$$Q = P^{(1)}_\mu \psi^\mu + X'^\mu \hat{\psi}_\mu \tag{3.2a}$$

$$\hat{Q} = P^{(2)}_\mu \hat{\psi}^\mu + X'^\mu \psi_\mu \tag{3.2b}$$

which turn out to be a direct extension to the superstring of the canonical expressions found for the supersymmetry charges in N = 2 non-relativistic quantum mechanics [11]. Quite remarkably, Q has a free form in the variables Ω_1, and the same happens for $\hat{Q}$ in the variables Ω_2. Moreover, the variables which appear multiplied in the above expressions have vanishing Poisson brackets - this fact will play a crucial rôle when the quantization procedure will be carried out.

On the basis of our previous experience [see Ref. (11)], we expect that all relevant quantities can be expressed in terms of Q, $\hat{Q}$ and their algebra. Since Q and $\hat{Q}$ have the free form in the variables Ω_1 and Ω_2, respectively, the metric $g_{\mu\nu}(X)$ only enters through the relations (2.23). In particular, one can express more symmetrically both Q and Q in the variables Ω. <u>It is precisely in delaying this last step as much as possible, that the quantization procedure can be greatly simplified</u>.

<u>3.1 - The basic Poisson brackets</u>

We now proceed to evaluate the Poisson brackets involving Q and $\hat{Q}$, starting with $\{Q,Q\}_{PB}$ and $\{\hat{Q},\hat{Q}\}_{PB}$. Obviously, the first one is computed (as in the free theory) in the canonical set Ω_1, leading to

$$\{Q(\sigma), Q(\sigma')\}_{PB} = \left(\Lambda(\sigma) + \Lambda(\sigma')\right) \delta(\sigma-\sigma') \ . \tag{3.3}$$

Likewise, the second Poisson bracket is evaluated (as in the free theory) in the set Ω_2, giving

$$\{\hat{Q}(\sigma), \hat{Q}(\sigma')\}_{PB} = \left(\hat{\Lambda}(\sigma) + \hat{\Lambda}(\sigma')\right) \delta(\sigma-\sigma') \ . \tag{3.4}$$

Explicitly, we find

$$\Lambda = P^{(1)}_\mu X'^\mu - \frac{i}{2} \hat{\psi}'_\mu \psi^\mu - \frac{i}{2} \psi'^\mu \hat{\psi}_\mu \ , \tag{3.5a}$$

$$\hat{\Lambda} = P^{(2)}_\mu X'^\mu - \frac{i}{2} \hat{\psi}'^\mu \psi_\mu - \frac{i}{2} \psi'_\mu \hat{\psi}^\mu \ . \tag{3.5b}$$

It can be easily checked [through Eqs. (2.23)] that

$$\hat{\Lambda} = \Lambda \ . \tag{3.6}$$

Moreover, we have

$$\Lambda = P_\mu X'^\mu - \frac{i}{2}\hat{\psi}'_\alpha \psi^\alpha - \frac{i}{2}\psi'^\alpha \hat{\psi}_\alpha =$$
$$= \Pi_\mu X'^\mu - \frac{i}{2}\left(D_\sigma \hat{\psi}^\alpha\right)\psi_\alpha - \frac{i}{2}\left(D_\sigma \psi^\alpha\right)\hat{\psi}_\alpha , \tag{3.7}$$

in agreement with Eq. (2.6a).

The Poisson brackets of Q and $\hat{Q}$ involve $g_{\mu\nu}(X)$ explicitly, since there is no canonical basis where both Q and $\hat{Q}$ take the free form. A direct calculation yields

$$\left\{Q(\sigma), \hat{Q}(\sigma')\right\}_{PB} = \left(B(\sigma) + B(\sigma')\right)\delta(\sigma-\sigma') \tag{3.8}$$

where

$$B = \frac{1}{2}\Big[g_{\mu\nu}X'^\mu X'^\nu - i\left(D_\sigma \hat{\psi}^\alpha\right)\hat{\psi}_\alpha - i\left(D_\sigma \psi^\alpha\right)\psi_\alpha +$$
$$+ \Pi_\mu g^{\mu\nu}\Pi_\nu - \frac{2}{3}R_{\alpha\beta\gamma\delta}\psi^\alpha\hat{\psi}^\beta\hat{\psi}^\gamma\psi^\delta\Big] \tag{3.9}$$

which can be checked to coincide with Eq. (2.6b).

We report for further convenience the form taken by the above Poisson brackets in the ± fermionic variables

$$\left\{Q^+(\sigma), Q^+(\sigma')\right\}_{PB} = 2\left(T^+(\sigma) + T^+(\sigma')\right)\delta(\sigma-\sigma') , \tag{3.10a}$$

$$\left\{Q^-(\sigma), Q^-(\sigma')\right\}_{PB} = 2\left(T^-(\sigma) + T^-(\sigma')\right)\delta(\sigma-\sigma') , \tag{3.10b}$$

$$\left\{Q^+(\sigma), Q^-(\sigma')\right\}_{PB} = 0 , \tag{3.10c}$$

with $T^\pm = \frac{1}{2}(B\pm\Lambda)$ given by Eq. (2.14).

3.2 - The Remaining Algebra

As it is clear from Eq. (3.9), the explicit expression of $B(\sigma)$ is much more complicated than the one of $\Lambda(\sigma)$, and so a direct computation of the Poisson brackets containing $B(\sigma)$ would be cumbersome. Thus, we proceed first to evaluate the Poisson brackets involving Λ, Q and $\hat{Q}$. Remarkably, all the remaining Poisson brackets will next be obtained by making use of appropriate (graded) Jacobi identities.

Indeed, taking advantage from the fact that Λ and Q have simple canonical expressions in the variables Ω_1, and the same happens for $\hat{\Lambda}$ and $\hat{Q}$ in the variables Ω_2, we easily obtain

$$\{\Lambda(\sigma), Q(\sigma')\}_{PB} = (-i)\left(Q(\sigma) + \tfrac{1}{2} Q(\sigma')\right) \delta'(\sigma-\sigma') \tag{3.11a}$$

$$\{\Lambda(\sigma), \hat{Q}(\sigma')\}_{PB} = (-i)\left(\hat{Q}(\sigma) + \tfrac{1}{2} \hat{Q}(\sigma')\right) \delta'(\sigma-\sigma') \tag{3.11b}$$

$$\{\Lambda(\sigma), \Lambda(\sigma')\}_{PB} = (-i)\left(\Lambda(\sigma) + \Lambda(\sigma')\right) \delta'(\sigma-\sigma') \tag{3.11c}$$

which imply at once

$$\{\Lambda(\sigma), Q^+(\sigma')\}_{PB} = (-i)\left(Q^+(\sigma) + \tfrac{1}{2} Q^+(\sigma')\right) \delta'(\sigma-\sigma') \tag{3.12a}$$

$$\{\Lambda(\sigma), Q^-(\sigma')\}_{PB} = (-i)\left(Q^-(\sigma) + \tfrac{1}{2} Q^-(\sigma')\right) \delta'(\sigma-\sigma') \tag{3.12b}$$

Now, use of the Jacobi identities involving $[\{Q^+,Q^+\},Q^-]$ and $[\{Q^-,Q^-\},Q^+]$ entails

$$\{T^+(\sigma), Q^-(\sigma')\}_{PB} = 0 \tag{3.13a}$$

$$\{T^-(\sigma), Q^+(\sigma')\}_{PB} = 0 \,. \tag{3.13b}$$

Recalling now that

$$T^+ = T^- + \Lambda \,, \quad T^- = T^+ - \Lambda \tag{3.14}$$

and using Eqs. (3.12) and (3.13) we immediately get

$$\{T^+(\sigma), Q^+(\sigma')\}_{PB} = (-i)\left(2Q^+(\sigma) + Q^+(\sigma')\right) \delta'(\sigma-\sigma') \tag{3.15a}$$

$$\{T^-(\sigma), Q^-(\sigma')\}_{PB} = i\left(2Q^-(\sigma) + Q^-(\sigma')\right) \delta'(\sigma-\sigma') \tag{3.15b}$$

Again, use of the Jacobi identity for $[T^+\{Q^-,Q^-\}]$ yields

$$\{T^+(\sigma), T^-(\sigma')\}_{PB} = 0 \tag{3.16}$$

and finally, the Jacobi identities involving $[T^+,\{Q^+,Q^+\}]$, $[T^-,\{Q^-,Q^-\}]$ and Eqs. (3.15) lead to

$$\{T^+(\sigma), T^+(\sigma')\}_{PB} = (-2i)\left(T^+(\sigma) + T^+(\sigma')\right)\delta'(\sigma-\sigma') \tag{3.17a}$$

$$\{T^-(\sigma), T^-(\sigma')\}_{PB} = 2i\left(T^-(\sigma) + T^-(\sigma')\right)\delta'(\sigma-\sigma') \tag{3.17b}$$

The full classical constraint algebra is summarized in Table 1. Some concluding remarks are in order. As in the case of the spinning particle [10] and the bosonic string [14], the classical constraint algebra is precisely the same as the free theory. Moreover, the + and - sectors do not couple. This fact is a direct consequence of the particularly simple canonical expressions of Q and Q [Eqs. (3.2)] and of the Jacobi identities.

4. - QUANTIZATION

4.1 - The Free Case

When going from a system with finitely many degrees of freedom (the particle) to one with infinitely many (the string), one encounters a well-known complication: the product of two local operators $A(\sigma_1)$, $B(\sigma_2)$ is often singular as $\sigma_1 \to \sigma_2$. Consequently, unless some appropriate limiting procedure is used, the local product $A(\sigma)B(\sigma)$ is ill defined.

In order to illustrate this problem - and the way to handle it - we discuss in some detail the well-known case of free string theory. This will also be our starting point for dealing with the interacting case.

The four fundamental operators of the free superstring are:

$$P_\pm^\mu(\sigma) \equiv \frac{1}{2}\left(P^\mu(\sigma) \pm X'^\mu(\sigma)\right) \tag{4.1}$$

$$\psi_\pm^\alpha(\sigma) \tag{4.2}$$

They yield the well-known non-vanishing correlation functions:

$$\langle P_\pm^\mu(\sigma)\, X^\nu(\sigma')\rangle = \pm\, \eta^{\mu\nu}\, \frac{1}{4\pi}\, (\Delta \pm i\epsilon)^{-1} \tag{4.3a}$$

$$\langle X^\mu(\sigma)\, P_\pm^\nu(\sigma')\rangle = \mp\, \eta^{\mu\nu}\, \frac{1}{4\pi}\, (\Delta \pm i\epsilon)^{-1} \tag{4.3b}$$

$$\langle P_\pm^\mu(\sigma)\, P_\pm^\nu(\sigma')\rangle = \eta^{\mu\nu}\, \frac{1}{4\pi}\, (\Delta \pm i\epsilon)^{-1} \tag{4.3c}$$

$$\langle \psi_\pm^\alpha(\sigma)\, \psi^\beta{}_\pm(\sigma')\rangle = \pm\, \frac{i}{2\pi}\, \eta^{\alpha\beta}\, (\Delta \pm i\epsilon)^{-1} \tag{4.3d}$$

where $\Delta = \sigma - \sigma'$. Recalling

$$(\Delta + i\epsilon)^{-1} - (\Delta - i\epsilon)^{-1} = -\, 2\pi i\, \delta(\Delta) \tag{4.4}$$

eqs. (4.3) imply canonical equal-time (anti)commutators.

The products appearing in Eqs. (4.3) are divergent for $\sigma \to \sigma'$. In order to define finite local operators one normal orders (creation precedes destruction operators) which, for bilinear operators, simply amounts to subtracting the corresponding vacuum expectation values (4.3).

Let us apply the procedure to the fermionic constraint operators (2.13):

$$Q_0^\pm(\sigma) = 2\, \psi_\pm^\alpha(\sigma)\, P_{\pm\alpha}(\sigma) \tag{4.5}$$

and to their anticommutators. Since ψ and P describe independent dynamical variables, no particular care is needed for defining $Q_0^\pm$. As for their algebra, one has immediately:

$$\{ Q_0^\pm(\sigma),\, Q_0^\mp(\sigma')\} = 0 \tag{4.6}$$

since, again, independent operators appear in Q_0^+, Q_0^- .

By contrast, for $\{Q_0^\pm(\sigma), Q_0^\pm(\sigma')\}$ we have to be careful and write, for instance:

$$\begin{aligned} Q_0^+(\sigma)\, Q_0^+(\sigma') = {} & :Q_0^+(\sigma)\, Q_0^+(\sigma'): + 4 :P_+^\mu(\sigma) P_{+\mu}(\sigma'): \frac{i}{2\pi}(\Delta + i\epsilon)^{-1} + \\ & + 4 :\psi_+^\alpha(\sigma)\psi_{+\alpha}(\sigma'): \frac{1}{4\pi}(\Delta + i\epsilon)^{-2} + 4\, D\, \frac{i}{8\pi^2}\, (\Delta + i\epsilon)^{-3}, \end{aligned} \tag{4.7}$$

where D is the number of space-time dimensions. This yields immediately, through Eq. (4.4) and its derivatives, the well-known result

$$\left\{ Q_0^+(\sigma), Q_0^+(\sigma') \right\} = 4\, T_0^+(\sigma)\, \delta(\sigma-\sigma') + \frac{D}{2\pi}\, \delta''(\sigma-\sigma') \tag{4.8}$$

and similarly

$$\left\{ Q_0^-(\sigma), Q_0^-(\sigma') \right\} = 4\, T_0^-(\sigma)\, \delta(\sigma-\sigma') + \frac{D}{2\pi}\, \delta''(\sigma-\sigma') \tag{4.9}$$

where

$$\begin{aligned} T_0^+(\sigma) &= \tfrac{1}{2}\left(B_0(\sigma) + \Lambda_0(\sigma)\right) = \,:P_+^{\mu}(\sigma) P_{+\mu}(\sigma): - \tfrac{i}{2} :\psi_+'(\sigma)\psi_+(\sigma): , \\ T_0^-(\sigma) &= \tfrac{1}{2}\left(B_0(\sigma) - \Lambda_0(\sigma)\right) = \,:P_-^{\mu}(\sigma) P_{-\mu}(\sigma): + \tfrac{i}{2} :\psi_-'(\sigma)\psi_-(\sigma): . \end{aligned} \tag{4.10}$$

Equations (4.8)-(4.10) differ from the classical ones [Eqs. (2.14) for the free case] in two respects:

i) $T_0^{\pm}$ are normal ordered;

ii) The new term $D/2\pi\ \delta''(\sigma - \sigma')$ appears. This kind of Schwinger term is the famous anomaly (or central extension of the Virasoro algebra) first discussed in string theory by J.H. Weis [6].

4.2 - The Fundamental Transformation

In the interacting theory, these anomalies will originate very much in the same way as in the free case, i.e., from the need to work with finite, local, composite operators. In the presence of a non-trivial metric or vielbein $e_\mu^\alpha(x)$, the operator X^μ appears, in general, in a non-polynomial way (cf. non-linear σ-model) a further complication adding up to the ill-defined local products.

As it is customary, we shall resort to a perturbative approach based on expanding X^μ and all background fields in normal Riemannian co-ordinates. This expansion, justified for slowly varying backgrounds (in α' units) is described in detail in Ref. [16]. At the lowest non-trivial order one has

$$X^\mu(\sigma) = X_B^{\mu} + \xi^\mu(\sigma) \tag{4.11a}$$

$$e_{\mu\alpha}(X(\sigma)) = \eta_{\mu\alpha} - \frac{1}{6} R_{\mu\lambda\alpha\kappa} \; : \xi^{\lambda}(\sigma)\, \xi^{\kappa}(\sigma) : + \cdots \tag{4.11b}$$

where $R_{\mu\lambda\alpha\kappa}$, now a c-number, stands for the Riemann tensor at the background point X_B. In order to give a meaning to Eqs. (4.11) we had, of course, to normal order the product $\xi\xi$.

Equations (4.11) suggest to introduce the operator:

$$U = 1 + K = 1 + \int d\sigma\, K(\sigma) \,, \tag{4.12a}$$

$$K(\sigma) = -\frac{i}{6} R_{\alpha\lambda\beta\kappa} \; : \xi^{\lambda}(\sigma)\, \xi^{\kappa}(\sigma)\, \psi_{+}^{\alpha}(\sigma)\, \psi_{-}^{\beta}(\sigma) : \tag{4.12b}$$

The importance of U lies in the fact that it allows to express in a simple and neat way the relations among the quantum version of the three sets of variables Ω, Ω_1, Ω_2 of Section 2 to lowest order in R. One finds:

$$\Omega_1 = U \Omega U^{-1} \simeq \Omega + [K, \Omega] \,, \tag{4.13a}$$

$$\Omega_2 = U^{-1} \Omega U \simeq \Omega - [K, \Omega] \,. \tag{4.13b}$$

Equations (4.13) show that, to this order, all the dynamical information due to the presence of a gravitational field is encoded in the single operator K. Indeed we know from the previous discussion [Eqs. (3.2)] that Q is bilinear [and $g_{\mu\nu}(X)$ independent] in the variables Ω_1, the same being true of Q in the set Ω_2. If Ω_1 and Ω_2 would coincide (e.g., for K = 0), the algebra would be totally independent of $g_{\mu\nu}(X)$ and hence identical to the free one.

For a non-trivial K, using the expressions:

$$Q(\sigma) = P_{\mu}^{(1)}(\sigma)\, \psi^{\mu}(\sigma) + X'^{\mu}(\sigma)\, \hat{\psi}_{\mu}(\sigma) \,, \tag{4.14a}$$

$$\hat{Q}(\sigma) = P_{\mu}^{(2)}(\sigma)\, \hat{\psi}^{\mu}(\sigma) + X'^{\mu}(\sigma)\, \psi_{\mu}(\sigma) \tag{4.14b}$$

and (4.13), we find

$$Q(\sigma) = UQ_0(\sigma)U^{-1} \simeq Q_0(\sigma) + [K, Q_0(\sigma)] \quad , \tag{4.15a}$$

$$\hat{Q}(\sigma) = U^{-1}\hat{Q}_0(\sigma)U \simeq \hat{Q}_0(\sigma) - [K, \hat{Q}_0(\sigma)] \quad , \tag{4.15b}$$

where Q_0, $\hat{Q}_0$ are the free fermionic constraints expressed in the variables Ω. From Eq. (4.15) the more useful quantities $Q^{\pm}$ can be immediately expressed as:

$$Q^+(\sigma) = Q_0^+(\sigma) + i\,[K, Q_0^-(\sigma)] \quad , \tag{4.16a}$$

$$Q^-(\sigma) = Q_0^-(\sigma) - i\,[K, Q_0^+(\sigma)] \; . \tag{4.16b}$$

Equations (4.16) can be now used to obtain convenient expressions for the relevant anticommutators to first order in K. These are:

$$\{Q^+(\sigma), Q^+(\sigma')\} = \{Q_0^+(\sigma), Q_0^+(\sigma')\} + i\,\{[K, Q_0^-(\sigma)], Q_0^+(\sigma')\} + \\ + i\,\{Q_0^+(\sigma), [K, Q_0^-(\sigma')]\} \quad , \tag{4.17}$$

$$\{Q^-(\sigma), Q^-(\sigma')\} = \{Q_0^-(\sigma), Q_0^-(\sigma')\} - i\,\{[K, Q_0^+(\sigma)], Q_0^-(\sigma')\} - \\ - i\,\{Q_0^+(\sigma), [K, Q_0^+(\sigma')]\} \quad , \tag{4.18}$$

$$\{Q^+(\sigma), Q^-(\sigma')\} = i\,\{[K, Q_0^-(\sigma)], Q_0^-(\sigma')\} - \\ - i\,\{Q_0^+(\sigma), [K, Q_0^+(\sigma')]\} \; . \tag{4.19}$$

The strategy of our approach is now clear. At zero order [Section (4.1)] only two bilinear operators $Q_0^{\pm}$ occur and, consequently, we have to deal with two-point functions. At first order, dynamics enters through the quadrilinear operator $K = \int d\sigma k(\sigma)$. We thus have to deal with three-point functions (integrated over one point). This pattern should of course extend to higher orders in the normal coordinate expansion.

5. - THE QUANTUM CONSTRAINT ALGEBRA: FERMIONIC CONSTRAINTS

We now insert into Eqs. (4.16) the expressions (4.5), (4.12) for $Q_0^{\pm}$ and $K = \int d\sigma\, k(\sigma)$. The first step consists of the evaluation of the commutator $[k(\sigma), Q^{\pm}(\sigma')]$ yielding easily:

$$\left[K(\sigma), Q_0^{+}(\sigma')\right] = :C^{+}(\sigma):\, \delta(\sigma-\sigma') + \tag{5.1}$$
$$+ \frac{1}{12\pi} R_{\alpha\beta}\, \psi_-^{\alpha}(\sigma)\, \xi^{\beta}(\sigma)\, \delta'(\sigma-\sigma')$$

$$\left[K(\sigma), Q_0^{-}(\sigma')\right] = :C^{-}(\sigma):\, \delta(\sigma-\sigma') + \tag{5.2}$$
$$+ \frac{1}{12\pi} R_{\alpha\beta}\, \psi_+^{\alpha}(\sigma)\, \xi^{\beta}(\sigma)\, \delta'(\sigma-\sigma')$$

where

$$C^{+}(\sigma) = -\frac{1}{3} \hat{R}_{\mu\lambda\alpha\kappa} \left(P_+^{\mu} \xi^{\lambda} \xi^{\kappa} \psi_-^{\alpha} + i\, \xi^{\kappa} \psi_-^{\mu} \psi_+^{\alpha} \psi_+^{\lambda} \right) \tag{5.3a}$$

$$C^{-}(\sigma) = \frac{1}{3} \hat{R}_{\mu\lambda\alpha\kappa} \left(P_-^{\mu} \xi^{\lambda} \xi^{\kappa} \psi_+^{\alpha} + i\, \xi^{\kappa} \psi_+^{\mu} \psi_-^{\alpha} \psi_-^{\lambda} \right) \tag{5.3b}$$

with

$$\hat{R}_{\mu\lambda\alpha\kappa} \equiv \frac{1}{2} \left(R_{\mu\lambda\alpha\kappa} + R_{\mu\kappa\alpha\lambda} \right) . \tag{5.4}$$

Inserting Eqs. (5.3) into Eqs. (5.1) and (5.2) and integrating over σ_3, Eqs. (4.16) take the form:

$$Q^{+}(\sigma) = Q_0^{+}(\sigma) + :C^{-}(\sigma): - \frac{1}{12\pi} R_{\alpha\beta} \frac{\partial}{\partial\sigma} \left(\psi_+^{\alpha}(\sigma)\, \xi^{\beta}(\sigma) \right) \tag{5.5a}$$

$$Q^-(\sigma) = Q_0^-(\sigma) - :C^+(\sigma): + \frac{1}{12\pi} R_{\alpha\beta} \frac{\partial}{\partial\sigma} \left(\psi_-^\alpha(\sigma)\xi^\beta(\sigma)\right) . \qquad (5.5b)$$

Let us identify the various terms appearing in Eqs. (5.5). The terms $Q_0^\pm \pm C^\mp$ coincide, apart from ordering complications, with the corresponding first order approximations to the classical expressions (2.12).

On the other hand, the last terms in Eqs. (5.5) are essentially quantum corrections arising from the correct treatment of composite operators.

In order to evaluate expressions (4.17), (4.18), we now compute the relevant triple commutators

$$\begin{aligned}
&\left\{\left[K(\sigma_3), Q_0^-(\sigma_1)\right], Q_0^+(\sigma_2)\right\} + \left\{Q_0^+(\sigma_1), \left[K(\sigma_3), Q_0^-(\sigma_2)\right]\right\} = \\
&= -\left\{\left[K(\sigma_3), Q_0^+(\sigma_1)\right], Q_0^-(\sigma_2)\right\} - \left\{Q_0^-(\sigma_1), \left[K(\sigma_3), Q_0^+(\sigma_2)\right]\right\} = \\
&= -i\ :E(\sigma_1):\ \delta(\sigma_1-\sigma_3)\,\delta(\sigma_2-\sigma_3) + \qquad (5.6)\\
&+ \frac{i}{6\pi} R_{\alpha\beta}\, \xi'^\alpha(\sigma_3)\,\xi^\beta(\sigma_3) \left(\delta'(\sigma_1-\sigma_3)\,\delta(\sigma_2-\sigma_3) + \delta'(\sigma_2-\sigma_3)\,\delta(\sigma_1-\sigma_3)\right) - \\
&- \frac{i}{3\pi^2}\, \delta'(\sigma_3-\sigma_1)\,\delta'(\sigma_3-\sigma_2)\, R
\end{aligned}$$

where

$$\begin{aligned}
E(\sigma) \equiv \frac{4}{3} \hat{R}_{\mu\nu\rho\sigma} \Big(& P_+^\rho \xi^\nu \xi^\sigma P_-^\mu - \qquad (5.7)\\
& - i P_-^\mu \xi^\sigma \psi_+^\nu \psi_+^\rho - i P_+^\mu \xi^\sigma \psi_-^\nu \psi_-^\rho + \frac{1}{2} \psi_+^\mu \psi_-^\rho \psi_+^\sigma \psi_-^\nu \Big) .
\end{aligned}$$

Moreover, for the evaluation of (4.19) we compute:

$$\{Q^+(\sigma_1), [K(\sigma_3), Q^+(\sigma_2)]\} - \{Q^-(\sigma_2), [K(\sigma_3), Q^-(\sigma_1)]\} = \tag{5.8}$$
$$= i\, V(\sigma_3)\left(\delta'(\sigma_3-\sigma_2)\,\delta(\sigma_3-\sigma_1) + \delta'(\sigma_3-\sigma_1)\,\delta(\sigma_3-\sigma_2)\right)$$

where

$$V(\sigma) \equiv -\frac{i}{6\pi} R_{\alpha\beta}\, \psi_-^{\alpha}(\sigma)\,\psi_+^{\beta}(\sigma) . \tag{5.9}$$

Integrating now over σ_3 and inserting Eqs. (5.6), (5.8) into Eqs. (4.17), (4.18) and (4.19), we finally obtain

$$\{Q^\pm(\sigma), Q^\pm(\sigma')\} = \delta(\sigma-\sigma')\Big(4 : T_0^{\pm}(\sigma) : + : E(\sigma) : -$$
$$- \frac{1}{6\pi} R_{\alpha\beta} \frac{\partial}{\partial\sigma}\left(\xi'^{\alpha}(\sigma)\,\xi^{\beta}(\sigma)\right)\Big) + \frac{1}{2\pi}\,\delta''(\sigma-\sigma')\left(D - \frac{R}{6\pi}\right), \tag{5.10}$$

$$\{Q^+(\sigma), Q^-(\sigma')\} = \delta(\sigma-\sigma')\, V'(\sigma) . \tag{5.11}$$

Equations (5.10) and (5.11) are our basic result since the rest of the algebra will follow from these anticommutators and appropriate use of Jacobi identities.

Let us discuss first Eq. (5.10). The coefficient of δ is:

$$4\, T^{\pm}(\sigma) \equiv 4 : T_0^{\pm}(\sigma) : + : E(\sigma) : - \frac{1}{12\pi} R_{\alpha\beta} \frac{\partial^2}{\partial\sigma^2} : \left(\xi^{\alpha}(\sigma)\,\xi^{\beta}(\sigma)\right) : \tag{5.12}$$

Here the first two terms coincide, except for ordering complications, with the first order expansion of the classical expressions for $4T^{\pm}$, while the last term of Eq. (5.12) represents a quantum modification of $T^{\pm}$ which does <u>not</u> affect the algebra. Finally, $(D-R/6\pi)$, is the first-order modification of the free c-number anomaly (4.8), (4.9).

By contrast, Eq. (5.11) represents a <u>new complete departure</u> from the classical constraint algebra in two respects:

i) it couples Q^+ and Q^-, which are completely decoupled at the classical level;

ii) it introduces a q-number-type Schwinger term (anomaly).

Unless $R_{\mu\nu} = 0$, this would make it very problematic to close in any simple way the constraint algebra and to construct a nilpotent quantum BRST operator. Because of the importance of this new anomaly we wish to present a more direct derivation of Eq. (5.11) based on the identity

$$i\left\{Q^+(\sigma), Q^-(\sigma')\right\} + i\left\{Q^+(\sigma'), Q^-(\sigma)\right\} = \left\{Q(\sigma), Q(\sigma')\right\} - \left\{\hat{Q}(\sigma), \hat{Q}(\sigma')\right\} \tag{5.13}$$

and on the fact that Q and $\hat{Q}$ can be written in terms of Q_0, $\hat{Q}_0$ and K through Eqs. (4.15). It is a simple algebra to show that

$$\left\{Q(\sigma), Q(\sigma')\right\} = 2\Lambda(\sigma)\,\delta(\sigma-\sigma') \tag{5.14}$$

$$\left\{\hat{Q}(\sigma), \hat{Q}(\sigma')\right\} = 2\hat{\Lambda}(\sigma)\,\delta(\sigma-\sigma') \tag{5.15}$$

where

$$\Lambda(\sigma) = \Lambda_0(\sigma) + \left[K, \Lambda_0(\sigma)\right] \tag{5.16a}$$

$$\hat{\Lambda}(\sigma) = \Lambda_0(\sigma) - \left[K, \Lambda_0(\sigma)\right] \tag{5.16b}$$

and

$$\Lambda_0(\sigma) = P_+^{\mu}(\sigma)P_{+\mu}(\sigma) - P_-^{\mu}(\sigma)P_{-\mu}(\sigma) - \frac{i}{2}\left(\psi_+'^{\alpha}(\sigma)\psi_{+\alpha}(\sigma) + \psi_-'^{\alpha}(\sigma)\psi_{-\alpha}(\sigma)\right) \tag{5.17}$$

It remains to compute the commutator in Eqs. (5.16). One gets

$$\left[K(\sigma), \Lambda_0(\sigma')\right] = -i\,V(\sigma)\,\delta'(\sigma-\sigma') + \frac{\partial}{\partial\sigma}(\cdots) \tag{5.18}$$

with $V(\sigma)$ given by Eq. (5.9). After integration over σ, this immediately yields:

$$\Lambda(\sigma) - \hat{\Lambda}(\sigma) = 2i\,V'(\sigma) \tag{5.19}$$

which, after use of Eq. (5.13), agrees with Eq. (5.11).

6. - THE REST OF THE QUANTUM CONSTRAINT ALGEBRA

Let us recall one of the central results of the previous section, i.e., the anomalous anticommutator:

$$\{Q^+(\sigma), Q^-(\sigma')\} = V'(\sigma)\,\delta(\sigma-\sigma') \tag{6.1}$$

with V given by Eq. (5.9).

In order to extend the algebra to the remaining (bosonic and mixed) commutators, it is convenient to introduce three more operators through:

$$[Q^+(\sigma), V(\sigma')] = W_1(\sigma)\,\delta(\sigma-\sigma') \tag{6.2}$$

$$[Q^-(\sigma), V(\sigma')] = W_2(\sigma)\,\delta(\sigma-\sigma') \tag{6.3}$$

$$[Q^+(\sigma), W_2(\sigma')] = [Q^-(\sigma), W_1(\sigma')] = Z(\sigma)\,\delta(\sigma-\sigma') \tag{6.4}$$

One finds easily

$$W_1(\sigma) = -\frac{i}{6\pi} R_{\alpha\beta}\,\psi_-^{\alpha}(\sigma)\,P_+^{\beta}(\sigma) \tag{6.5}$$

$$W_2(\sigma) = -\frac{i}{6\pi} R_{\alpha\beta}\,\psi_+^{\alpha}(\sigma)\,P_-^{\beta}(\sigma) \tag{6.6}$$

$$Z(\sigma) = -\frac{i}{6\pi} R_{\alpha\beta}\,P_+^{\alpha}(\sigma)\,P_-^{\beta}(\sigma)\,. \tag{6.7}$$

Let us first proceed to compute $[T^+(\sigma),Q^-(\sigma')]$ and $[T^-(\sigma),Q^+(\sigma')]$ by using Jacobi identities for $[\{Q^+,Q^+\},Q^-]$ and $[\{Q^-,Q^-\},Q^+]$. One easily obtains

$$[T^+(\sigma), Q^-(\sigma')] = \left(W_1(\sigma') - 2W_1(\sigma)\right)\delta'(\sigma-\sigma') \tag{6.8}$$

$$[T^-(\sigma), Q^+(\sigma')] = \left(W_2(\sigma') - 2W_1(\sigma)\right)\delta'(\sigma-\sigma')\,. \tag{6.9}$$

In order to compute $[T^{\pm}(\sigma),Q^{\pm}(\sigma')]$, we recall from Eqs. (5.12) and (4.10):

$$T^+(\sigma) - T^-(\sigma) = T_0^+(\sigma) - T_0^-(\sigma) = \frac{1}{2}\left(\Lambda(\sigma) + \hat{\Lambda}(\sigma)\right) = \Lambda_0(\sigma) \tag{6.10}$$

and that the pairs Q, Λ and $\hat{Q}$, $\hat{\Lambda}$ have the free form in the variables Ω_1 and Ω_2, respectively. Thus

$$[\Lambda(\sigma), Q(\sigma')] = (-i)\left(Q(\sigma) + \frac{1}{2}Q(\sigma')\right)\delta'(\sigma-\sigma') \tag{6.11}$$

$$\left[\hat{\Lambda}(\sigma),\, \hat{Q}(\sigma')\right] = (-i)\left(\hat{Q}(\sigma) + \tfrac{1}{2}\,\hat{Q}(\sigma')\right)\delta'(\sigma-\sigma') \tag{6.12}$$

Using Eqs. (6.10), (6.11) and (6.12), a slightly cumbersome but easy calculation yields

$$\left[T^+(\sigma), Q^+(\sigma')\right] = -\left(2\left(W_2(\sigma) + iQ^+(\sigma)\right) + \left(W_2(\sigma') + i\,Q^+(\sigma')\right)\right)\delta'(\sigma-\sigma') \tag{6.13a}$$

$$\left[T^-(\sigma), Q^-(\sigma')\right] = -\left(2\left(W_1(\sigma) - iQ^-(\sigma)\right) + \left(W_1(\sigma') - i\,Q^-(\sigma')\right)\right)\delta'(\sigma-\sigma') \tag{6.13b}$$

This exhausts the mixed commutators. Proceeding to the purely bosonic algebra, we use the Jacobi identities again, this time for $[T^\pm,\{Q^\pm,Q^\pm\}]$ and for $[T^\pm,\{Q^\mp,Q^\mp\}]$, and we obtain

$$\begin{aligned}\left[T^+(\sigma), T^+(\sigma')\right] &= -2\left(\left(Z(\sigma) + iT^+(\sigma)\right) + \left(Z(\sigma') + i\,T^+(\sigma')\right)\right)\delta'(\sigma-\sigma') - \\ &- \frac{i}{2\pi}\left(D - \frac{R}{6\pi}\right)\delta'''(\sigma-\sigma'),\end{aligned} \tag{6.14a}$$

$$\begin{aligned}\left[T^-(\sigma), T^-(\sigma')\right] &= -2\left(\left(Z(\sigma) - iT^-(\sigma)\right) + \left(Z(\sigma') - i\,T^-(\sigma')\right)\right)\delta'(\sigma-\sigma') + \\ &+ \frac{i}{2\pi}\left(D - \frac{R}{6\pi}\right)\delta'''(\sigma-\sigma')\end{aligned} \tag{6.14b}$$

and, finally,

$$\left[T^+(\sigma), T^-(\sigma')\right] = 2\,Z'(\sigma)\,\delta(\sigma-\sigma') \,. \tag{6.15}$$

This exhausts the whole quantum constraint algebra, whose complete structure is summarized in Table 2. It is immediate to check that, to this order, the necessary and sufficient condition for preserving the free algebra is $R_{\mu\nu} = 0$. In this case, taking into account the ghost contribution to the BRST operator [4], nilpotency is obtained at $D = 10$.

On the other hand, if $R_{\mu\nu} \neq 0$, the square of Q_{BRST} contains (even for $D = 10$,

$R = 0$) a term proportional to $R_{\mu\nu}$ and to some of bilinears in left and right moving ghost fields.

7. - CONCLUSIONS

One of the main conclusions to be drawn from this work is the crucial difference between quantization of the supersymmetric point particle and of the superstring.

While at the classical level, even in the presence of external fields, both cases can be dealt with on similar grounds and without difficulty, at the quantum level the case of the string brings up (besides the known problem of operator ordering already present for the point) the difficulty of regularizing the product of local fields at the same point. In order to obtain finite, acceptable results, it is necessary to make reference to a well-defined Hilbert space and to normal ordering the product of operators with respect to the ground state.

This new problem is already present for the free string and leads to the well-known c-number Schwinger term (the anomaly) which enforces a critical value for the dimensionality of flat space-time.

In going to the interacting case, a great simplification is brought in by world-sheet supersymmetry, which allows to focus one's attention on the fermionic constraints and their algebra, which then generates, through Jacobi identities, the remaining commutators. In this paper we have tried to take full advantage of the simplifications allowed by supersymmetry.

Noticing that each of the two independent fermionic constraints takes the free form in a convenient set of canonical variables, we have been able to connect the anomalies to the transformation from one canonical set to the other: we have then worked out the results to leading non-trivial order in the space-time curvature tensor $R_{\mu\nu\rho\sigma}$. To this order, the correct closure of the quantum constraint algebra (alternatively the nilpotency of the BRST operator) can be shown to demand the vacuum Einstein equation $R_{\mu\nu} = 0$, in agreement with earlier claims.

Besides showing explicitly how a non-vanishing $R_{\mu\nu}$ modifies the algebra through c as well as q-number Schwinger terms, we may speculate that our method can be extended to higher-orders in the normal co-ordinate expansion (or σ-model loops) with considerable simplifications with respect to the equivalent σ-model calculations. Moreover, the present approach can be straightforwardly generalized to other kinds of type II superstrings propagating in arbitrary massless backgrounds.

We hope to report on new results in this direction in a future publication.

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APPENDIX

SUPERSTRING IN FLAT SPACE-TIME AND IN AN ARBITRARY GAUGE

In this appendix we present the Hamiltonian formalism for the free superstring in an arbitrary gauge. The action is [13]

$$I = \int d\tau\, d\sigma \sqrt{e}\, \Big[\tfrac{1}{2} g^{ab} \partial_a X^\alpha \partial_b X_\alpha + \tfrac{i}{2} \chi^\alpha \gamma^0 \gamma^a \partial_a \chi_\alpha - \qquad \text{(A.1)}$$
$$- i \bar{\lambda}_a \gamma^b \gamma^a \chi^\alpha \partial_b X_\alpha - \tfrac{1}{4} \chi^\alpha \gamma^0 \chi_\alpha \bar{\lambda}_a \gamma^b \gamma^a \lambda_b \Big]$$

The notations are as follows. X^α, χ^α are bosonic and (NRS) fermionic co-ordinates, respectively, where $\alpha = 0,\ldots, D-1$; $e^i_{\ a}$ is the zweibein ($e = \det\|e^i_{\ a}\|$), η_{ij} is the local flat metric on the world-sheet and $g_{ab} = e^i_{\ a} e^j_{\ b} \eta_{ij}$ is the world-sheet metric - Latin indices take the values 0, 1. λ_a is the (spinor-vector) gravitino field on the world-sheet, obeying the Majorana condition $\lambda^*_a = \lambda_a$. Finally $\gamma^a = e^a_{\ i} \gamma^i$ where γ^i are the two-dimensional Dirac matrices defined in the local co-ordinate frame.

The action (A.1) is invariant under local reparametrizations, Weyl rescaling and local supersymmetry transformations. Our aim here is to derive the constraints associated with the local symmetries of Eq. (A.1), by working in the Hamiltonian formalism.

Owing to the fact that the kinetic energy terms for $e^a_{\ i}$ and λ_a are missing in Eq. (A.1), the equations of motion for these fields are actually constraint relations. Hence

$$\frac{\delta I}{\delta e^a_{\ i}} \equiv T_a^{\ i} = - e^i_{\ a} \Big(\partial_b X^\alpha \partial_c X_\alpha g^{bc} + i \chi^\alpha \gamma^0 \gamma^a \partial_a \chi_\alpha -$$
$$- 2i \bar{\lambda}_b \gamma^c \gamma^b \chi^\alpha \partial_c X_\alpha - \tfrac{1}{2} \chi^\alpha \gamma^0 \chi_\alpha \bar{\lambda}_a \gamma^b \gamma^a \lambda_b \Big) + \qquad \text{(A.2)}$$
$$+ 2 \partial_a X^\alpha e^b_{\ j} \partial_b X_\alpha \eta^{ij} + i \chi^\alpha \gamma^0 \gamma^i \partial_a \chi_\alpha -$$
$$- 2i \bar{\lambda}_b \gamma^i \gamma^b \chi^\alpha \partial_a X_\alpha = 0$$

$$\frac{\delta I}{\delta \lambda_a} \equiv J^a = \partial_b X^\alpha \gamma^b \gamma^a \chi_\alpha - \frac{i}{2} \chi^\alpha \gamma^0 \chi_\alpha \gamma^b \gamma^a \chi_b = 0 \qquad (A.3)$$

Since the energy-momentum tensor is symmetric and traceless, the two constraints (A.2) are expressed in terms of the Hamiltonian phase-space variables as follows

$$T^\pm = \frac{1}{4}\left(P \pm X'\right)^2 \pm \Pi^\alpha_\pm \partial_\sigma \psi_{\pm\alpha} = 0 \qquad (A.4)$$

where

$$P^\alpha \equiv e\, \partial_a X^\alpha g^{0a} - i e \bar{\lambda}_a \gamma^b e^0{}_b \gamma^a \chi^\alpha \qquad (A.5)$$

$$\Pi^\alpha_\pm \equiv \frac{i}{2} e\, \bar{\psi}^\alpha_\pm \gamma^a e^0{}_a \qquad (A.6)$$

and

$$\psi^\alpha_\pm = \frac{1}{2}\left(1 \mp \gamma_5\right) \chi^\alpha \qquad (A.7)$$

Likewise, the two constraints (A.3) - vanishing of the supercurrent - take the Hamiltonian form

$$Q^\pm = \frac{1}{2}\left(P^\alpha \pm X'^\alpha\right) \psi_{\pm\alpha} = 0 \qquad (A.8)$$

The space-space realization of the classical constraint algebra for the free superstring then takes the same form as in Table 1.

TABLE 1

Classical constraint algebra

$$\{Q^+(\sigma),Q^+(\sigma')\}_{PB} = 2(T_+(\sigma)+T_+(\sigma'))\delta(\sigma-\sigma')$$

$$\{Q^-(\sigma),Q^-(\sigma')\}_{PB} = 2(T_-(\sigma)+T_-(\sigma'))\delta(\sigma-\sigma')$$

$$\{Q^+(\sigma),Q^-(\sigma')\}_{PB} = 0$$

$$\{T^+(\sigma),Q^+(\sigma')\}_{PB} = -i(2Q^+(\sigma)+Q^+(\sigma'))\delta'(\sigma-\sigma')$$

$$\{T^-(\sigma),Q^-(\sigma')\}_{PB} = i(2Q^-(\sigma)+Q^-(\sigma'))\delta'(\sigma-\sigma')$$

$$\{T^+(\sigma),Q^-(\sigma')\}_{PB} = 0$$

$$\{T^-(\sigma),Q^+(\sigma')\}_{PB} = 0$$

$$\{T^+(\sigma),T^+(\sigma')\}_{PB} = -2i(T+(\sigma)+T_+(\sigma'))\delta'(\sigma-\sigma')$$

$$\{T^-(\sigma),T^-(\sigma')\}_{PB} = +2i(T_-(\sigma)+T_-(\sigma'))\delta'(\sigma-\sigma')$$

$$\{T^+(\sigma),T^-(\sigma')\}_{PB} = 0$$

TABLE 2

Quantum constraint algebra

$$\{Q^+(\sigma),Q^+(\sigma')\} = 2\big(T^+(\sigma)+T^+(\sigma')\big)\delta(\sigma-\sigma') + \frac{1}{2\pi}\,(D - \frac{R}{6\pi})\,\delta''(\sigma-\sigma')$$

$$\{Q^-(\sigma),Q^-(\sigma')\} = 2\big(T^-(\sigma)+T^-(\sigma')\big)\delta(\sigma-\sigma') + \frac{1}{2\pi}\,(D - \frac{R}{6\pi})\,\delta''(\sigma-\sigma')$$

$$\{Q^+(\sigma),Q^-(\sigma')\} = V'(\sigma)\delta(\sigma-\sigma')$$

$$[T^+(\sigma),Q^+(\sigma')] = -\big(2(W_2(\sigma)+iQ^+(\sigma)) + (W_2(\sigma')+iQ^+(\sigma'))\big)\,\delta'(\sigma-\sigma')$$

$$[T^-(\sigma),Q^-(\sigma')] =-\big(2(W_1(\sigma)+iQ^-(\sigma)) + (W_1(\sigma')-iQ^-(\sigma'))\big)\,\delta'(\sigma-\sigma')$$

$$[T^+(\sigma),Q^-(\sigma')] = \big(W_1(\sigma')-2W_1(\sigma)\big)\delta'(\sigma-\sigma')$$

$$[T^-(\sigma),Q^+(\sigma')] = \big(W_2(\sigma')-2W_2(\sigma)\big)\delta'(\sigma-\sigma')$$

$$[T^+(\sigma),T^+(\sigma')] = -2\big((Z(\sigma)+iT^+(\sigma))+((Z(\sigma')+iT^+(\sigma'))\big)\delta'(\sigma-\sigma') - $$

$$- \frac{i}{2\pi}\,(D - \frac{R}{6\pi})\,\delta'''(\sigma-\sigma')$$

$$[T^-(\sigma),T^-(\sigma')] = -2\big((Z(\sigma)-iT^-(\sigma))+((Z(\sigma')-iT^-(\sigma'))\big)\delta'(\sigma-\sigma') + $$

$$+ \frac{i}{2\pi}\,(D - \frac{R}{6\pi})\,\delta'''(\sigma-\sigma')$$

$$[T^+(\sigma),T^-(\sigma')] = 2Z'(\sigma)\delta(\sigma-\sigma')$$

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