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# HIDDEN SYMMETRIES OF TWO DIMENSIONAL STRING EFFECTIVE ACTION

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## Abstract

The ten dimensional heterotic string effective action with graviton, dilaton and antisymmetric tensor fields is dimensionally reduced to two spacetime dimensions. The resulting theory, with some constraints on backgrounds, admits infinite sequence of conserved nonlocal currents. It is shown that generators of the infinitesimal transformations associated with these currents satisfy Kac-Moody algebra.

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The purpose of this investigation is to unravel hidden symmetries of dimensionally reduced string effective action in two spacetime dimensions. Recently, we have shown [1] the existence of an infinite set of nonlocal conserved currents (NCC) for the reduced action with some constraints. The starting point is to consider the heterotic string effective action in ten dimensions with massless backgrounds such as graviton, dilaton and antisymmetric tensor fields. Then, one toroidally compactifies $d$ of its *internal* coordinates and requires that the backgrounds are independent of these $d$ coordinates. It has been demonstrated that the dimensionally reduced effective action is invariant under global noncompact $O(d,d)$ symmetry transformations [2,3]. Thus in $1+1$ dimensions the group is $O(8,8)$, and its algebra is denoted by $\mathcal{G}$. The infinite sequence of currents were derived for this action with some restrictions on the backgrounds. It is well known that Kac-Moody algebra is intimately connected with integrable systems, theories that admit NCC and string theory [4].

We exhibit the infinite parameter Lie algebra responsible for the NCC to be the affine Kac-Moody algebra. First, it is shown, following the work of Dolan and Roos [5], that there is an infinitesimal symmetry transformation, associated with each of these currents, which leave the Lagrangian invariant up to a total derivative term [6]. Then, the existence of the Kac-Moody algebra is proved, for the problem at hand, by suitably adopting the remarkable result of Dolan [7], derived for loop space and two dimensional chiral models. We identify the infinite parameter Lie algebra, crucial for the NCC, to be the affine Kac-Moody subalgebra $C[\xi]\otimes\mathcal{G}$ following ref.7. Here $C[\xi]\otimes\mathcal{G}$ is an infinite dimensional Lie algebra defined over a ring of polynomials in the complex variable $\xi$. A simple representation of the generators of the algebra $C[\xi]\otimes\mathcal{G}$ is, $\mathcal{M}_\alpha^{(n)} = T_\alpha \xi^n$, where $\{T_\alpha\}$ are the generators of the finite parameter algebra $\mathcal{G}$, and $n = 1, 2, ...\infty$. The generators of $C[\xi]\otimes\mathcal{G}$ satisfy $[\mathcal{M}_\alpha^{(n)}, \mathcal{M}_\beta^{(m)}] = f_{\alpha\beta\gamma}\mathcal{M}_\gamma^{(m+n)}$, when the algebra of the generators of $\mathcal{G}$ is $[T_\alpha, T_\beta] = f_{\alpha\beta\gamma}T_\gamma$ and $f_{\alpha\beta\gamma}$ are the structure constants antisymmetric in their indices and satisfy the Jacobi identity.

In what follows, we recapitulate the results of ref.2. The effective action in $\hat{D} = D+d$ dimensions ($\hat{D} = 10$ for the present case ) is,

$$\hat{S} = \int d^{\hat{D}}x \, \sqrt{-\hat{g}}e^{-\hat{\phi}}[\hat{R}(\hat{g}) + \hat{g}^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi} - \frac{1}{12}\, \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\, \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}}]. \tag{1}$$

Note that $\hat{S}$ is the bosonic part of the heterotic string effective action in critical dimension. $\hat{H}$ is the field strength of antisymmetric tensor and $\hat{\phi}$ is the dilaton. Here all the field backgrounds to have been set to zero. We consider the theory in a spacetime $M \times K$, where $M$ is $D$ dimensional spacetime and the coordinates on $M$ are denoted by $x^\mu$. The internal space, $K$, is $d$ dimensional and $\{y^\alpha\}$, $\alpha = 1, 2, ..d$, are the coordinates. When the backgrounds are independent of $y^\alpha$ and the internal space is taken to be torus, the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ can be decomposed as

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + A^{(1)\gamma}_{\mu}A^{(1)}_{\nu\gamma} & A^{(1)}_{\mu\beta} \\ A^{(1)}_{\nu\alpha} & G_{\alpha\beta} \end{pmatrix}, \tag{2}$$

where $G_{\alpha\beta}$ is the internal metric and $g_{\mu\nu}$, the $D$-dimensional space-time metric, depend on the coordinates $x^\mu$. The dimensionally reduced action is,

$$\begin{aligned} S_D = \int d^Dx\sqrt{-g}e^{-\phi}\Big\{&R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} \\ &+ \frac{1}{8}\mathrm{tr}(\partial_\mu M^{-1}\partial^\mu M) - \frac{1}{4}\mathcal{F}^i_{\mu\nu}(M^{-1})_{ij}\mathcal{F}^{\mu\nu j}\Big\}. \end{aligned} \tag{3}$$

Here $\phi = \hat{\phi} - \frac{1}{2}\log\det G$ is the shifted dilaton.

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2}\mathcal{A}^i_\mu\eta_{ij}\mathcal{F}^j_{\nu\rho} + (\text{cyc. perms.}), \tag{4}$$

$\mathcal{F}^i_{\mu\nu}$ is the $2d$-component vector of field strengths

$$\mathcal{F}^i_{\mu\nu} = \begin{pmatrix} F^{(1)\alpha}_{\mu\nu} \\ F^{(2)}_{\mu\nu\alpha} \end{pmatrix} = \partial_\mu\mathcal{A}^i_\nu - \partial_\nu\mathcal{A}^i_\mu\ , \tag{5}$$

$A^{(2)}_{\mu\alpha} = \hat{B}_{\mu\alpha} + B_{\alpha\beta}A^{(1)\beta}_{\mu}$ (recall $B_{\alpha\beta} = \hat{B}_{\alpha\beta}$), and the $2d \times 2d$ matrices $M$ and $\eta$ are defined as

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \qquad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

The action (3) is invariant under a global $O(d,d)$ transformation,

$$M \to \Omega^T M \Omega, \qquad \Omega\eta\Omega^T = \eta, \qquad \mathcal{A}^i_\mu \to \Omega^i{}_j \mathcal{A}^j_\mu, \qquad \text{where} \qquad \Omega \in O(d,d). \tag{7}$$

and the shifted dilaton, $\phi$, remains invariant under the $O(d,d)$ transformations. Note that $M \in O(d,d)$ also and $M^T\eta M = \eta$. Thus if we solve for a set of backgrounds, $M$,$\mathcal{F}$ and $\phi$, satisfying the equations of motion they correspond to a vacuum configuration of the string theory.

Let us consider the reduced action, eq.(3), in $1+1$ dimension. Note that $H_{\mu\nu\rho}H^{\mu\nu\rho}$ term does not contribute to the action in two spacetime dimensions. Moreover, we assume that the dilaton, $\phi$, entering the action (3) is constant. We recall that a four dimensional action admits solitonic string solution [8,9] when the backgrounds are such that $\phi =$ constant, $H_{\mu\nu\rho} = 0$, $\mathcal{F}^i_{\mu\nu} = 0$ and the metric as well as the moduli depend only on two coordinates. Such a theory is an effective two dimensional theory. Recently, Bakas [10] has considered a four dimensional effective action with $\delta c = 0$, where $\delta c$ is the central charge deficit. One can interprete that the action arises from compactification of a string effective action in critical dimensions through dimensional reduction where $M$ and $\mathcal{F}^i_{\mu\nu}$ are set to zero ( see eq.(3)). Furthermore, the axion ( arising from duality transformation on $H_{\mu\nu\rho}$) and the dilaton can combined to define a complex field which transforms nontrivially under one $SL(2,R)$. Then the existence of two commuting Killing symmetries ( that all backgrounds depend ony on two coordinates), is exploited to derive a form of the metric such that the action is invariant under another $SL(2,R)$

and the resulting theory is described by a two dimensional action. Thus, this dimensionally reduced theory has a symmetry which can be infinitesimally be identified with the $O(2,2)$ current algebra [10]. In contrast, in the present investigation, $M$, expressed in terms of moduli $G$ and $B$, is spacetime dependent and other backgrounds fulfill the restrictions of constant $\phi$ and vanishing $\mathcal{F}^i_{\mu\nu}$. The relevant action is

$$S_2 = \int d^2x \sqrt{-g} \Big\{ R + \frac{1}{8} \mathrm{tr}(\partial_\mu M^{-1} \partial^\mu M) \Big\}. \tag{8}$$

Notice that, for constant $\phi$, $\partial_\mu \phi \partial^\mu \phi$ term is absent. Since we are considering two dimensional spacetime, we can choose the spacetime metric $g_{\mu\nu} = e^{\alpha(x,t)} \eta_{\mu\nu}$. Here $\eta_{\mu\nu}$ is the flat diagonal spacetime metric = diag $(-1,1)$ ( not to be confused with the $O(d,d)$ metric ). The Einstein term of the action in two dimensions is a topological term and it does not contribute to the equations of motion. Thus the equations of motion associated with the matrix M is of primary importance to us. It is more convenient to go over to an $O(8,8)$ metric, $\sigma$, which is diagonal and is related to $\eta$ by the following transformation: $\sigma = \rho^T \eta \rho$, where

$$\rho = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9}$$

and matrix elements 1 stand for $d \times d$ unit matrix. Then, $M \rightarrow \mathcal{U} = \rho^T M \rho$ and the $\mathcal{U}$ satisfies the property: $\mathcal{U}^T = \mathcal{U}$ and $\sigma \mathcal{U} \sigma = \mathcal{U}^{-1}$. The action eq.(8) takes the form

$$S_2 = \int d^2x \Big\{ R + \frac{1}{8} \mathrm{tr}(\partial_\mu \mathcal{U}^{-1} \partial^\mu \mathcal{U}) \Big\}. \tag{10}$$

The equations of motion for the $\mathcal{U}$ are

$$\partial^\mu \mathcal{A}_\mu = 0, \qquad \mathcal{A}_\mu = \mathcal{U}^{-1} \partial_\mu \mathcal{U} \tag{11}$$

and we observe that $\mathcal{A}_\mu$ is a pure gauge. Therefore, $[\mathcal{D}_\mu, \mathcal{D}_\nu] = 0$, with $\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu$. It is worthwhile mentioning that $\mathcal{A}_\mu$ coincides with the vector field introduced in ref.1 to construct the infinite set of NCC. The equations of motion (11) and the curvaturelessness proprties of $\mathcal{A}_\mu$ were utilized to construct these currents by employing the known techniques [11] for our problem.

The infinitesimal transformations, on the $O(d,d)$ valued function $\mathcal{U}$, associated with the infinite set on NCC are given by

$$\delta^{(n)}\mathcal{U} = -\mathcal{U}\Lambda^{(n)} \tag{12}$$

The set of $\{\Lambda^{(n)}\}$ are recursively defined as

$$\Lambda^{(n+1)}(t,x) = \int_{-\infty}^{x} dy \mathcal{D}_0 \Lambda^{(n)}(t,y) = \int_{-\infty}^{x} dy \Big\{ \partial_0 \Lambda^{(n)}(t,y) + [\mathcal{A}_0(t,y), \Lambda^{(n)}] \Big\}, \tag{13}$$

with $\Lambda^0 \equiv T$, $T$ being a generic form of an infinitesimal transformation of the group $\mathcal{G}$ and $T$ can be expanded as a linear combination of the set $\{T_\alpha\}$. Furthermore,

$$\Lambda^{(1)} = [X_1, T] = \int_{-\infty}^{x} dy [\mathcal{A}_0(t,y), T] \tag{14}$$

$$\Lambda^{(2)} = [X_2, T] + \frac{1}{2}[X_1, [X_1, T]] \tag{15}$$

where $X_1 = \int_{-\infty}^{x} dy \mathcal{A}(t,y)$ and $X_2$ satisfies the equation $\partial_1 X_2 = \partial_0 X_1 - \frac{1}{2}[\partial_1 X_1, X_1]$.

In what follows, we present the essential steps to construct the generators of the Kac-Moody algebra and demonstrate the existence of the algebra for the theory described by eq.(9). Here we adopt an elegant and economic technique due to Devchand and Fairlie [12] to derive the algebra. Let us introduce the generating function for the $\Lambda$'s as

$$S(\xi) = \sum_{0}^{\infty} \Lambda^{(r)} \xi^r \tag{16}$$

using the recursion relation, eq.(12), and the properties of $\mathcal{A}_\mu$, we can show

$$(\partial_1 - \xi \partial_0) S = \xi [\mathcal{A}_0, S] \tag{17}$$

and $S$ can be expressed as

$$S(\xi) = Q(\xi)TQ^{-1}(\xi). \tag{18}$$

Now $Q$ satisfies the equation

$$Q(\partial_1 - \xi\partial_0)Q^{-1} = -\xi\mathcal{A}_0 \tag{19}$$

and $Q$ is defined as limit: $Q = lim_{N\to\infty}Q_N$; with

$$Q_N = e^{X_N\xi^N}..e^{X_2\xi^2}e^{X_1\xi}. \tag{20}$$

We can check by explicit calculations that coefficients of $\xi$ and $\xi^2$ in (18) and (19) give us equns. (14) and (15).

Moreover, it can be shown, following ref.12 that, under an infinitesimal transformation, $\delta\mathcal{U} = -\mathcal{U}S$, the variation of the Lagrangian density (9) is

$$\delta\mathcal{L} = \frac{1}{4}\partial_\mu\epsilon^{\mu\nu}tr[\xi\mathcal{A}_\nu + (\xi + \frac{1}{\xi})Q^{-1}\partial_\nu QT] \tag{21}$$

In order to derive the algebra, first we define the generators of the transformations and then evaluate commutators of two transformations. Now, we label each transformation with an index. For definiteness, we choose two transformations to be $\delta_\alpha\mathcal{U} = -\mathcal{U}S_\alpha$ and $\delta_\beta\mathcal{U} = -\mathcal{U}S_\beta$; $\Lambda^0$ appearing in the expansions, eq.(16), for $S_\alpha$ and $S_\beta$ are taken to be $T_\alpha$ and $T_\beta$ respectively and these gerators satisfy $[T_\alpha, T_\beta] = f_{\alpha\beta\gamma}T_\gamma$. Of course, we could have chosen any two arbitrary generator $T_a$ and $T_b \in \mathcal{G}$; in that case each of these generators will be expanded in terms of the basis $\{T_\gamma\}$ and the arguments we are going to present below will go through in that general setting too with some extra calculations. However, we have made this choice here to facilitate simplicity in computations and bring out the essence of the arguments. Let us define following Dolan [7]

$$\mathcal{M}_\alpha(\xi) = \int d^2y\mathcal{U}S_\alpha\frac{\delta}{\delta\mathcal{U}(y)} \tag{22}$$

Then the commutator of two transformations are

$$[\mathcal{M}_\alpha(\xi), \mathcal{M}_\beta(\zeta)] = \int d^2y \mathcal{U}[S_\alpha(\xi), S_\beta(\zeta)] \frac{\delta}{\delta \mathcal{U}(y)} - \int d^2y \mathcal{U}[\delta_\alpha S_\beta(\zeta) - \delta_\beta S_\alpha(\xi)] \frac{\delta}{\delta \mathcal{U}(y)}. \quad (23)$$

The variation, $\delta_\alpha S_\beta(\zeta)$, can be expressed as

$$\delta_\alpha S_\beta(\zeta) = -\frac{\zeta}{\zeta - \xi} \Big\{ [S_\alpha(\xi), S_\beta(\zeta)] - f_{\alpha\beta\gamma} S_\gamma(\zeta) \Big\} \quad (24)$$

after some computations [7,12], and a similar equation holds for $\delta_\beta S_\alpha(\xi)$ with appropriate argument and indices. Using the above relations in eq.(22), we arrive at

$$[\mathcal{M}_\alpha(\xi), \mathcal{M}_\beta(\zeta)] = f_{\alpha\beta\gamma} \int d^2y \frac{\mathcal{U}[\xi S_\gamma(\xi) - \zeta S_\gamma(\zeta)]}{\xi - \zeta} \frac{\delta}{\mathcal{U}(y)} \quad (25)$$

This elegant form of equation was derived in [12]. The Kac-Moody algebra is derived as follows: Note that $\mathcal{M}_\alpha(\xi)$ can be expanded in a power series in $\xi$ as

$$\mathcal{M}_\alpha(\xi) = \sum_0^\infty \mathcal{M}_\alpha^{(l)} \xi^l \quad (26)$$

inserting the expansion eq.(27) in the commutator (28) and comparing the coefficients of $\xi^m \zeta^n$ on both the sides we arrive at the desired Kac-Moody algebra

$$[\mathcal{M}_\alpha^m, \mathcal{M}_\beta^n] = f_{\alpha\beta\gamma} \mathcal{M}_\gamma^{(m+n)} \quad (27)$$

A few remarks are in order here: The NCC constructed in ref.[1] can be expressed in terms of $\mathcal{U} \in O(8,8)$ and is related to the the M-matrix: $\mathcal{U} = \rho^T M \rho$. An arbitrary element of $O(d,d)$ can be expressed in terms of $2d^2 - d$ independent parameters. But we know that $\mathcal{U}$, alternatively $M$, is determined in terms of the moduli $G$ and $B$ and thus has only $d^2$ parameters. In fact, it was shown by Maharana and Schwarz [2] that the moduli appearing in the effective action, parametrize the coset $\frac{O(d,d)}{O(d) \otimes O(d)}$ and thus the matrix valued function $\mathcal{U}$ can be expanded on a basis which belong to the coset $\frac{O(8,8)}{O(8) \otimes )(8)}$. Indeed, the NCC were derived in [1] by going over to the coset reformulation [2] of the effective action (9) and then construct the curvatureless vector field $\mathcal{A}_\mu$. Notice that if we had not set to zero the $U(1)^{16}$ gauge field action in $\hat{S}$ the resulting coset will be $\frac{O(8,24)}{O(8)\otimes O(24)}$ all our arguments will still be valid. Recently, it has been recognized that the string effective actions in lower dimensions exhibit a rich symmetry content. The dimensionally reduced effective theory ( coming from 10-dimensional heterotic string action with the inclusion of 16 Abelian gauge fields ) in 4-dimensions possesses two symmetries [3]: $O(6,22;Z)$ T-duality and $SL(2,Z)$ S-duality [13]. For $D=3$, the theory has a bigger invariance group, $O(8,24;Z)$, and it has been shown that $SL(2,Z)$ and $O(7,23;Z)$ T-duality are a part of this group [14]. Now, we see that in two spacetime dimensions there is an infinite dimensional symmetry algebra.

To summarise, we have demonstrated the existence symmetrty transformations of associated with each of the infinite sequence of conserved currents in the two dimensional effective theory. The generators of the infinitesimal transformations, associated with these currents, satisfy Kac-Moody algebra which is very intimately related with the T-duality group.

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