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Duality and $O(d,d)$ Symmetries in String Theory*

Jnanadeva Maharana†

California Institute of Technology, Pasadena, CA 91125

Abstract

The evolution of a closed bosonic string is envisaged in the time-dependent background of its massless modes. A duality transformation is implemented on the spatial component of string coordinates to obtain a dual string. It is shown that the evolution equations are manifestly $O(d,d)$ invariant. The tree level string effective actions for the original and the dual string theory are shown to be equivalent.

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† Permanent Address: Institute of Physics, Bhubaneswar - 752005, India.
email maharana@theory3.caltech.edu. until June 30, 1992.

It is now recognized that spacetime duality[1] plays an important role in understanding several aspects of string theory. One of the consequences of R-duality is that the dynamics of a string on a circle of radius R is equivalent to that of another string on a circle of radius $\frac{1}{R}$ (in suitable units). The idea of duality has also been used in a wider context[2,3] and it is conjectured that this symmetry might be maintained to all orders in string perturbation theory.[4] Furthermore, duality has been applied to study string cosmology[5−7] and used to obtain new black hole solutions.[7]

Recently the concept of scale factor duality[6] (SFD) has been introduced as a symmetry group of classical string equations of motion, derived from a low energy string effective action. One of the salient features of SFD is that it does not require compactification of the target space. Moreover, this transformation relates different time-dependent background configurations of string theory. There is an intimate connection between Narain's[8] construction of inequivalent static compactification and the $O(d,d)$ transformations on background fields, which rotate time-dependent backgrounds (solutions of equations of motion) into other ones which are not necessarily equivalent. Subsequently, several cosmological[9] and black hole solutions[10] have been obtained through the implementation of $O(d,d)$ transformations and their generalizations. It has been observed that the roles of the canonical momentum P and X' are interchanged under duality (this amounts to interchange of winding number and momentum zero modes) when constant background fields are suitably transformed along with $P \leftrightarrow X'$. The Hamiltonian remains invariant under duality.

The purpose of this note is to investigate the evolutions of a closed bosonic string in a background in which background of its massless excitations (graviton, $g_{\mu\nu}$, antisymmetric tensor, $B_{\mu\nu}$, and dilaton ϕ) are time-dependent. It is shown that the evolution equations of the string in a time-dependent background have a hidden $O(d,d)$ symmetry; d is the number of space dimensions. The space-time dimensions is $D = d + 1$. We introduce a duality transformation on string coordinates to define a dual Lagrangian. Then a larger manifold is constructed to include string coordinates and their dual coordinates. The equations of motion are derived in a manifestly $O(d,d)$ invariant manner. It is worthwhile to mention that although the equations of motion are manifestly $O(d,d)$ invariant the Lagrangian is not. We may recall that a similar situation also arises in the context of the discussion of noncompact hidden symmetries in supergravity theories,[11] where the equations of motion are manifestly invariant under these hidden symmetries whereas the action is not.

It is argued that the vanishing β-function equations of the original string theory are the same as those of the dual string theory, since the tree level string effective action for both the theories are the same when the background are time-dependent (but not space-dependent)

The two-dimensional sigma model Lagrangian that we consider is

$$\begin{aligned}\bar{L} = &\frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} + \frac{1}{2}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu} \ , \\ &+ L_D \ ,\end{aligned} \tag{1}$$

where γ_{ab} is the world sheet metric; ϵ^{ab} the antisymmetric tensor, such that $\epsilon^{01} = 1, \mu, \nu$ are D-dimensional target space indices and a, b are world sheet indices respectively. L_D describes the coupling of a dilaton background with string, whose explicit form will be discussed later. All the background fields $g_{\mu\nu}, B_{\mu\nu}$ and ϕ are allowed to depend on the time coordinate X^0 only. It is convenient to bring $g_{\mu\nu}$ and $B_{\mu\nu}$ to the following special form by implementing general coordinate transformation and Abelian gauge transformation respectively.

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & G_{ij}(t) \end{bmatrix} \quad \text{and} \quad B_{\mu\nu} = \begin{bmatrix} 0 & 0 \\ 0 & B_{ij}(t) \end{bmatrix} \ , \tag{2}$$

$i, j = 1, d$ are indices of the spatial coordinates. Then the Lagrangian $\bar{L}$ can be re-expressed as

$$\begin{aligned}\bar{L} = &-\frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\partial_a X^0 \partial_b X^0 + \frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\partial_a X^i \partial_b X^j G_{ij} \\ &+ \frac{1}{2}\epsilon^{ab}\partial_a X^i \partial_b X^j B_{ij} + L_D \ .\end{aligned} \tag{3}$$

The equations of motion,

$$\frac{\partial \bar{L}}{\partial X^\mu} - \partial_a \frac{\partial \bar{L}}{\partial \partial_a X^\mu} = 0 \ , \tag{4}$$

take the following form for the spatial components of the string coordinates $\{X^i\}$:

$$\partial_a \mathcal{A}^a_i = 0 \ , \tag{5}$$

with

$$\mathcal{A}^a_i \equiv \frac{\partial \bar{L}}{\partial \partial_a X^i} = \sqrt{-\gamma}\gamma^{ab}\partial_b X^j G_{ij} + \epsilon^{ab}\partial_b X^j B_{ij} \ , \tag{6}$$

since there is no explicit X^i dependence in $\bar{L}$. However, the equation of motion for the X^0 coordinate is much more complicated.

$$\frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\partial_a X^i \partial_b X^j \frac{\partial}{\partial X^0} G_{ij} + \frac{1}{2}\epsilon^{ab}\partial_a X^i \partial_b X^j \frac{\partial}{\partial X^0} B_{ij}$$

$$+\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b X^0) + \frac{\partial L_D}{\partial X^0} - \partial_a \left(\frac{\partial L_D}{\partial \partial_a X^0}\right) = 0 \ . \tag{7}$$

Note that (7) contains terms with derivatives with respect to X^0. Therefore, it cannot be written in the form of eq. (5). In the BRST quantization of string theory it is necessary to introduce ghost fields in the gauge fixed action. In this framework in stead of the usual dilaton coupling

$$\int d^2\sigma \sqrt{-\gamma} R^{(2)} \phi(X) \ , \tag{8}$$

the dilation is coupled to the ghost current[12,13] (when ON gauge fixing is adopted).

$$L_D = \frac{1}{2}\int d^2\sigma [\bar{\partial}\phi(X) b_{++} c^+ + \partial\phi b_{--} c^-] \ , \tag{9}$$

where ϕ is the dilaton and $b_{\pm\pm}$ and $c^\pm$ are the ghost fields. Notice that since we assume that ϕ depends on X^0 only, $\bar{\partial}\phi(X) = \frac{\partial}{\partial X^0}\phi(X^0)\bar{\partial}X^0$ and $\partial\phi(X) = \frac{\partial}{\partial X^0}\phi(X^0)\partial X^0$.

In what follows we shall utilize the property of $\bar{L}$ given by (1) that the equations of motion for $\{X^i\}$ is a divergence (5). Thus we are led to construct a new Lagrangian, L_1, in the first order formalism where we deal with $\{X^i\}$ coordinates. Introducing a field u^i_a, we write

$$L_1 = -\frac{1}{2}\sqrt{-\gamma}\gamma^{ab}u^i_a u^j_b G_{ij} - \frac{1}{2}\epsilon^{ab}u^i_a u^j_b B_{ij}$$

$$+\partial_a X^i(\sqrt{-\gamma}\gamma^{ab}u^j_b G_{ij} + \epsilon^{ab}u^j_b B_{ij}) \ . \tag{10}$$

The u^i_a variation of L_1 gives

$$\frac{\partial L_1}{\partial u^i_a} = (\partial_b X^j - u^j_b)(\sqrt{-\gamma}\gamma^{ab}G_{ij} + \epsilon^{ab}B_{ij}) = 0 \ , \tag{11}$$

as the u^i_a equations of motion since L_1 does not contain any derivative of the field; whereas $\partial_a X^i$ variation gives

$$\frac{\partial L_1}{\partial \partial_a X^i} = \sqrt{-\gamma}\gamma^{ab}u^j_b G_{ij} + \epsilon^{ab}u^j_b B_{ij} \ , \tag{12}$$

with the equations of motion $\partial_a\left(\frac{\partial L_1}{\partial\partial_a X^i}\right) = 0$. If we solve for $\partial_a X^i = u^i_a$ from (10) and substitute in (8), we receive the expression for $\mathcal{A}^a_i$, eq. (5) and the equations of motion for $\{X^i\}$.

Let us consider another first order Lagrangian L_2 with a variables Y_i, and an auxiliary field (again denoted by u^i_a)

$$L_2 = \frac{1}{2}\sqrt{-\gamma}\gamma^{ab}u^i_a u^j_b G_{ij} + \frac{1}{2}\epsilon^{ab}u^i_a u^j_b B_{ij}$$

$$+\epsilon^{ab}\partial_a Y_i u^j_b \ . \tag{13}$$

Variation of L_2 with respect to u^i_a gives the relation

$$\epsilon^{ab}\partial_b Y_i = \sqrt{-\gamma}\gamma^{ab}u^j_b G_{ij} + \epsilon^{ab}u^j_b B_{ij} \ , \tag{14}$$

and the Y_i equation of motion is

$$\partial_a \left(\frac{\partial L_2}{\partial \partial_a Y_i} \right) = \partial_a \left(\epsilon^{ab} u_b^i \right) = 0 \ . \tag{15}$$

Solving for u_i^a in terms of Y_i in (14) gives us

$$u_a^i = \frac{1}{\sqrt{-\gamma}} \gamma_{ab} \epsilon^{bc} \mathbf{A}^{ij} \partial_c Y_j + \mathbf{F}^{ij} \partial_a Y_j \ , \tag{16}$$

where $\mathbf{A} = B^{-1}(GB^{-1} - BG^{-1})^{-1}$ and $\mathbf{F} = -G^{-1}(GB^{-1} - BG^{-1})^{-1}$ are symmetric and antisymmetric time-dependent matrices respectively as is evident from the symmetry properties of backgrounds G and B.

The equations of motion derived from L_1 suggests that we can unite $u_i^a = \epsilon^{ab} \partial_b Y_i$ locally; whereas the Y_i equation of motion derived from L_2 allows us to write $\frac{\partial L_2}{\partial \partial_a Y_i} = \epsilon^{ab} \partial_b X^i$. Thus the $\partial_a X^i$ and $\partial_a Y_i$ variations of L_1 and L_2 (after substituting the auxiliary fields) locally take the form

$$\epsilon^{ab} \partial_b Y_i = \frac{\partial L_1}{\partial \partial_a X^i} = \sqrt{-\gamma} \gamma^{ab} \partial_b X^j G_{ij} + \epsilon^{ab} \partial_b X^j B_{ij} \tag{17}$$

$$\epsilon^{ab} \partial_b X^i = \frac{\partial L_2}{\partial \partial_a Y_i} = \sqrt{-\gamma} \gamma^{ab} \partial_b Y_j \mathbf{A}^{ij} + \epsilon^{ab} \partial_b Y_j \mathbf{F}^{ij} \ . \tag{18}$$

As always happens with such dual reformulation, the field equations derived for the string coordinates $\{X^i\}$ are the Bianchi identities for the dual variables $\{Y_i\}$, whereas the equation of motion for $\{Y_i\}$ are the Bianchi identities for $\{X^i\}$. Moreover, the matrices $\mathbf{A}$ and $\mathbf{F}$ play the role of metric and antisymmetric tensor field for the dual string coordinates. It follows from (17) and (18) that the canonical momenta $\{P_i\}$ of $\{X^i\}$ are identified with Y_i' whereas those of $\{Y_i\}$ are with $\{X^{'i}\}$. We may recall that, for constant background fields, the role of P and X' are interchanged under duality. In order to reveal the hidden symmetries associated with the equations of motion, we enlarge the manifold where X^i and Y_i are treated as independent coordinates (this is analogous to the phase space in a Hamiltonian formulation of dynamics). Let us first rewrite equations (17) and (18) as

$$\sqrt{-\gamma}\epsilon_{ab}\gamma^{bc}\partial_c X^i = G^{ij}\partial_a Y_i - G^{ij}B_{jk}\partial_a X^k \tag{19}$$

$$\sqrt{-\gamma}\epsilon_{ab}\gamma^{bc}\partial_c Y_i = (\mathbf{A}^{-1})_{ij}\partial_a X^j - (\mathbf{A}^{-1})_{ij}F^{jk}\partial_a Y_k \ . \tag{20}$$

Let W denote the $2d$ coordinates $\{X^i, Y_i\}$ collectively; then eqs. (19) and (20) can be written in a compact form as the single equation

$$\mathbf{M}\eta\partial_a W = \sqrt{-\gamma}\epsilon_{ab}\gamma^{bc}\partial_c W \ , \tag{21}$$

where the symmetric $2d \times 2d$ matrix

$$\mathbf{M} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{22}$$

is the same one as appears in the discussion of the duality properties of string effective action,[9] and

$$\eta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \ , \tag{23}$$

is the $O(d,d)$ metric, $\mathbf{1}$ being $d \times d$ unit matrix. It is easy to check that

$$\mathbf{M}\eta\mathbf{M} = \eta \ \text{ and } \ \eta\mathbf{M}\eta = \mathbf{M}^{-1} \ . \tag{24}$$

Thus we conclude that $\mathbf{M} \ \epsilon \ O(d,d)$. Equation (21) can be rewritten as

$$\epsilon^{ab}\partial_b W = \eta\mathbf{M}^{-1}\sqrt{-\gamma}\gamma^{ab}\partial_b W$$

leading to the manifest $O(d,d)$ invariant integrability equation

$$\partial_a(\eta\mathbf{M}^{-1}\sqrt{-\gamma}\gamma^{ab}\partial_b W) = 0 \ . \tag{25}$$

We note that the equations of motion are invariant under $O(d,d)$ transformations although the action is not invariant under duality transformation. This is one of the characteristics of hidden symmetries associated with duality transformations as has been emphasized by Gaillard and Zumino.[11]

Let us turn our attention to the conformal invariance of these theories and the equations of motion satisfied by the background fields. The Hamiltonian associated with L_1 together with the contributions of the X^0 coordinates and the dilaton term is

$$H = \frac{1}{2}(P_0 + X^{'0})^2 + \frac{1}{2}(P\ X')\mathbf{M}(P\ X')^T + H_D\ , \tag{26}$$

where the first term is the contribution of the X^0 coordinates, P_0 being momentum conjugate to X^0 whereas $\{P_i\}$ are the conjugate momentum of $\{X^i\}$ and $\mathbf{M}$ is the matrix defined in eq. (22). H_D is the Hamiltonian associated with the dilaton coupling to the string.[13] The other constraint, which generates σ-reparametrization, can be written as

$$P_0 X^{'0} + \frac{1}{2}(P\ X')\eta(P\ X')^T\ . \tag{27}$$

If we now demand conformal invariance of the theory we derive the equations of motion for the background fields which ensure the vanishing of the associated β-functions. These conditions can also be derived from the variation of the tree level string effective action. Indeed, such an effective action has been obtained in a compact form for time-dependent background fields by Meissner and Veneziano[9]

$$S_E = \int dt e^{-\varphi}[\Lambda + \dot{\varphi}^2 + \frac{1}{8}Tr(\partial_t\mathbf{M}\eta\partial_t\mathbf{M}\eta)]\ , \tag{28}$$

where $\varphi = \phi - \ell n\sqrt{\det G}$, is the shifted dilaton, G is as defined in eq. (2), and Λ is the cosmological term proportional to (D - 26) that appears for a noncritical bosonic string. $\mathbf{M}$ and η are defined in eqs. (22) and (23).

Let us now implement the duality transformation. Note that the X^0 coordinate and H_D remain unaffected since $\{X^i\}$ transform to $\{Y_i\}$ under duality. Thus the generator of σ-reparametrization transformation has the form

$$P_0 X^{'0} + \frac{1}{2}(\tilde{P}\ Y')\eta(\tilde{P}\ Y')^T \ , \tag{29}$$

and the dual Hamiltonian is

$$\tilde{H} = \frac{1}{2}(P_0 + X^{'0})^2 + \frac{1}{2}(\tilde{P}\ Y')\tilde{\mathbf{M}}(\tilde{P}\ Y')^T + H_D \ , \tag{30}$$

where $\{\tilde{P}^i\}$ are conjugate momenta of $\{Y_i\}$. $\tilde{\mathbf{M}}$ can be computed to be

$$\tilde{\mathbf{M}} = \begin{bmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{bmatrix} = \mathbf{M}^{-1} \ . \tag{31}$$

It is easy to write down the Meissner-Veneziano[9] effective action for the dual theory with $\tilde{\mathbf{M}}$ and it reads (φ remains unchanged)

$$\tilde{S}_E = \int dt e^{-\varphi}\left[\Lambda + \dot{\varphi}^2 + \frac{1}{8}Tr(\partial_t\tilde{\mathbf{M}}\eta\partial_t\tilde{\mathbf{M}}\eta)\right] \ . \tag{32}$$

It follows from the properties of $\mathbf{M}$ eq. (24) and $\eta^2 = 1$ that

$$\tilde{S}_E = S_E \ . \tag{33}$$

Therefore, the tree level string effective actions for the two theories are the same.

It is natural to ask what symmetries the string evolution equations will exhibit if the background fields depend on some of the spatial coordinates in addition to time. For example, one could envisage a situation where background fields are independent of coordinates $X^\alpha, \alpha = 1, ..m, m < D-1$. It has been shown by Sen,[14] in the framework of string field theory, that the string effective action in this case has an $O(m) \otimes O(m)$ symmetry. However, in our approach, it is not easy to demonstrate such invariance properties of the string equations of motion, since it is not possible to transform the metric and antisymmetric tensor to a simple form as in eq. (2) in the general case. It might be possible to show the invariance of the string equations of motion by introducing a more general duality transformation than the one used here.

To summarize: we considered a closed bosonic string in the time-dependent background of its massless modes. A duality transformation involving only spatial string coordinates was utilized to obtain a new Lagrangian. Then it was shown that the string evolution equations are $O(d,d)$ invariant. It was argued that the tree level string effective action and the dual effective action are equivalent. Therefore, the β-functions associated with the original string theory and the dual theory are identical at least to lowest order in α'.

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