Novel Symmetries in Axion-Dilaton String Cosmology

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The symmetry structure of axion-dilaton quantum string cosmology is investigated. The invariance of the string effective action under S-duality group, $SU(1,1)$, facilitates solution of Wheeler-De Witt equation from group theoretic considerations; revealing existence of a new class of wave functions. We discover the an underlying W-infinity algebra in this formulation.
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It is recognized that symmetries play a very important role in understanding diverse properties of string theories. Dualities provide foundations to study string dynamics and to construct a unique and unified theory [1]. It is well known that string effective actions in lower spacetime dimensions tend to exhibit higher symmetries. For example, the 4-dimensional heterotic string effective action derived from $D = 10$ action by toroidal compactification on T^6, is endowed with $O(6, 22)$ T-duality symmetry. The 3-form field strength may be dualized to introduce the pseudoscalar axion, χ. The dilaton, ϕ, and χ parametrize the coset $\frac{SL(2,R)}{U(1)} \sim \frac{SU(1,1)}{U(1)}$. Thus the duality group [2] of $D = 4$ theory is $O(6, 22) \times SU(1, 1)$. If we envisage the theory in $D = 3$ by compactifying the $D = 10$ theory on T^7, then an enlarged group, $O(8, 32)$, emerges as the duality group. The two dimensional effective action derived through dimensional reduction on T^8 incorporates affine algebra [3].

The string cosmological scenario, where backgrounds depend on cosmic time, is quite interesting in $D = d+1$ dimensions. The dilaton-graviton action is invariant under the scale factor duality [4] which is a subgroup of $O(d, d)$ group, d being number of spatial dimensions. The O(d,d) symmetry appears [5] if the time dependent B-field is incorporated in the action. There has been considerable amount of activities to study cosmologies in the frame work string theory and M-theory in recent years [6,7].

The pair (ϕ, χ) play a special role among a large number of massless scalars, present in 4-dimensional string theories. Whereas, $e^{<\phi>}$ determines G_N, α_{YM} and other coupling constants of the theory; χ is identified with the elusive axion. Although, it was introduced in the context of QCD, axion is recognized to play a significant role in cosmology.

In this letter we study isotropic, homogeneous dilaton-axion string cosmology and explore its symmetry content. We present a general class of solutions to the Wheeler-De Witt (WDW) equation. The S-duality invariance of the action is exploited to solve WDW equation from the group theoretic considerations. The wave function is factorized as a product of a function of the scale factor and an eigenfunction of the Casimir of $SU(1, 1)$, the S-duality group. These wave functions are new and were missed earlier works [8]. It is argued that one can construct a set of operators from the generators of $SU(1, 1)$ and show that these operators close into an infinite dimensional algebra. We show that the w_∞ algebra proposed by Bakas [9] emerges in a specific example. In recent years w_∞ and W_∞, which might be considered as deformation of the former, have played useful roles in a variety of problems in physics. They first appeared as $N \to \infty$ limit of W_N algebras [10]. These algebras were studied in the context of $c = 1$ theories [11], in large N limit of $SU(N)$gauge theories [12] and in quantum Hall effect [13]. It is quite surprizing that such algebras now resurfaces in string cosmology. In the past, symmetry content of string cosmologies have been explored from diverse perspectives [14] and our efforts are in yet another novel direction.

Let us consider an action in the Einstein frame

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{2\phi}\partial_\mu\chi\partial^\mu\chi\right) \quad (1)$$

where $g_{\mu\nu}$ is the metric, g its determinant, $\mu, \nu = 0, 1, 2, 3$ and R is the scalar curvature. One can derive such an action from four dimensional heterotic string theory, after dualising $H^{\mu\nu\rho} = e^{2\phi}\epsilon^{\mu\nu\rho\lambda}\partial_\lambda\chi$ and setting rest of the backgrounds to zero or by compactifying [15] $D = 10$ type IIB theory on T^6 and truncating action to the above form. One may reexpress (1) in a manifestly S-duality, $SU(1, 1)$, invariant form

$$S = \int d^4x\sqrt{-g}\left(R + \frac{1}{4}\mathrm{Tr}[\partial_\mu\mathbf{V}^{-1}(x)\partial^\mu\mathbf{V}(x)]\right) \quad (2)$$

where

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} A+B & 2B\chi + i(A-B) \\ 2B\chi - i(A-B) & A+B \end{pmatrix} \quad (3)$$

The elements of the 2×2 matrix $\mathbf{V}$ are defines as: $A = e^{-\phi} + \chi^2 e^{\phi}$ and $B = e^{\phi}$. Note that $\mathbf{V} \in SU(1, 1)$ and satisfies $\mathbf{V}^{-1} = \sigma_3\mathbf{V}^\dagger\sigma_3$; $\sigma_i, i = 1, 2, 3$ denote the Pauli matrices. The action (2) is invariant under

$$\mathbf{V} \to \Omega^\dagger\mathbf{V}\Omega, \quad g_{\mu\nu} \to g_{\mu\nu} \text{ and } \Omega^\dagger\sigma_3\Omega = \sigma_3, \quad (4)$$

where $\Omega \in SU(1, 1)$ and σ_3 is the SU(1,1) metric which remains invariant under the transformation. We mention *en passant* that the action is usually expressed in

$SL(2,R)$ invariant form in terms of the $\mathcal{M}$-matrix [16]. The two matrices are related, *i.e.* $\mathcal{M} = e^{-i\frac{\sigma_1 \pi}{4}} \mathbf{V} e^{i\frac{\sigma_1 \pi}{4}}$. Note that any $SU(1,1)$ matrix can be expressed as

$$\mathbf{V} = \mathbf{1} v_0 + \sum_{i=1}^{3} \sigma_i v_i \tag{5}$$

$\mathbf{1}$ being the unit matrix and the spacetime dependent coefficients satisfy: $v_0^2 - v_1^2 - v_2^2 + v_3^2 = 1$. For $\mathbf{V}$ given by (3), $v_3 = 0$; therefore, the constraint reads

$$v_0^2 - v_1^2 - v_2^2 = 1 \tag{6}$$

with $v_0 = \frac{1}{2}(A+B)$, $v_1 = B\chi$ and $v_2 = \frac{1}{2}(B-A)$ which satisfy the desired constraint (6). In fact (6) defines a space of constant Gaussian curvature in the *moduli* space. In the cosmological context, the FRW spacetime metric is

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) \tag{7}$$

$a(t)$ being the scale factor; $k = +1, 0, -1$ correspond to closed, flat and open Universes respectively. Furthermore, the $\mathbf{V}$-matrix or alternatively, v_0, v_1, v_2 depend on cosmic time, t. From now on, we consider, $k = 1$, closed Universe. The scalar curvature derived from (7) is $R = 6(-a\dot{a}^2 + a)$, up to a total derivative term. Here and everywhere overdot stands for $\frac{d}{dt}$. It is convenient to scale the metric to get rid of the factor 6 in the expression for the curvature scalar and the redefine the matter fields suitably so that action takes the following form:

$$S = \frac{1}{2} \int dt \left(-a\dot{a}^2 + a + a^3 \dot{v}_i \dot{v}_j \eta^{ij} \right) \tag{8}$$

The metric $\eta^{ij} = \text{diag}(1,-1,-1)$ and the constraint is: $\eta^{ij} v_i v_j = 1$. The three conserved quantities; $X_1 = (v_1\dot{v}_2 - v_2\dot{v}_1)$, $X_2 = (v_0\dot{v}_2 - v_2\dot{v}_0)$ and $X_3 = (v_1\dot{v}_0 - v_0\dot{v}_1)$, constructed from (8) satisfy $SU(1,1)$ algebra. Note that X_1 is compact and generates rotation around v_0 axis and other two correspond to boosts. We recall that the non-compact group $SU(1,1)$ is endowed with three generators: $J_i, i = 1,2,3$ and they satisfy the following algebra:

$$[J_1, J_2] = -iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2 \tag{9}$$

The raising/lowering operators: $J_\pm = J_1 \pm iJ_2$ obey the commutation relations: $[J_3, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = -2J_3$. The Casimir operator is $C = -J_1^2 - J_2^2 + J_3^2$. The Hamiltonian constraint reads

$$H = \frac{1}{2} \left(-\frac{1}{a} P_a^2 - a + \frac{1}{a^3} \eta_{ij} P_v^i P_v^j \right) = 0. \tag{10}$$

where H is the Hamiltonian derived from (8) and P_a and P_v^i are canonical momenta associated with a and v_i respectively. In quantum cosmology, the Hamiltonian constraint (10) translates into WDW equation. The Hamiltonian operator $\hat{H}$ is defined by identifying $\hat{P}_a = -i\frac{\partial}{\partial a}$ and $\hat{P}_v^i = -i\frac{\partial}{\partial v_i}$. The standard procedure is to derive the wave function for a given $\hat{H}$, satisfying (10). Notice, however that the scale factor remains unchanged under $SU(1,1)$ transformation and a little calculation shows that the term $\eta_{ij} P_v^i P_v^j$ in (10) is the Casimir operator of the S-duality group. The problem is analogous to that of motion of a particle in spherically symmetric potential, where the L^2 term contributes the angular momentum barrier leaving one to solve for the radial equation. The quantum Hamiltonian needs to be defined with an operator ordering prescription as is obvious from inspection of (10). We adopt the one [17] which respects coordinate invariance in the minisuperspace and the WDW equation takes form

$$\left(\frac{\partial^2}{\partial a^2} + \frac{\partial}{\partial a} - a^2 + \frac{1}{a^2} C \right) \Psi = 0. \tag{11}$$

We may factorize the wave function $\Psi = \mathcal{U}(a) Y$ where Y is the eigenfunction of the Casimir operator satisfying: $CY = j(j+1)Y$. Next, we seek simultaneous eigenfunction of C and the compact generator J_3 and exploit the well known group theoretic results [18] of $SU(1,1)$. The wave functions satisfy the requirements that

$$C|j,m>= j(j+1)|j,m>, \text{ and } J_3|j,m>= m|j,m> \tag{12}$$

We need the unitary infinite dimensional representations of $SU(1,1)$, to solve for the 'angular' part of WDW equation, which are classified as follows.

(i) The discrete series $D_j^\pm$: D_j^+ is defined such that $j = -1/2, -1, -3/2, -2,...$ and for a given j, m is unbounded from above i.e. $m = -j, -j+1, -j+2, ...$ The other discrete series, D_j^- for which j is negative (integer or half integer), is the one where m is unbounded from below i.e. $m = j, j-1, j-2, ...$ There is a symmetry between these sets of wave functions under $m \rightarrow -m$ which follows from the properties of the $d^j_{mm'}$ functions of $SU(1,1)$. Note that j is negative in our convention.

(ii) The continuous series C_k^0 and $C_k^{1/2}$ correspond to

$$j = -\frac{1}{2} + ik, \quad k > 0, \quad \text{and real} \tag{13}$$

with $m = 0, \pm1, \pm2, ..$ and $m = \pm1/2, \pm3/2, ..$ for C_k^0 and $C_k^{1/2}$ respectively; also note that $j^* = -j-1$. Since k takes continuous values, some times these states are called 'scattering states' in the literature.

(iii) The supplementary series is defined for $-1/2 < j < 0$ and $m = 0, \pm1, \pm2....$ It is well known that the set $\{D_j^\pm, C_k^0, C_k^{1/2}\}$ provide a complete set of basis functions. Therefore, any function corresponding to $\frac{SU(1,1)}{U(1)}$ may be expanded in this basis. One may omit the supplementary series while looking for the wave functions from purely group theoretic considerations. However, we shall utilize this series in an example.

Let us focus attention on solution of the WDW equation (11). We define a 'polar' coordinate system: $v_0 =$

coshα, v_1 = sinhαcosβ and v_2 = sinhαsinβ; α real and $0 \le \beta \le 4\pi$. The Casimir of the $SU(1,1)$ is given by the Lapace-Beltrami operator [19]

$$-\frac{1}{\sinh\alpha}\frac{\partial}{\partial\alpha}\sinh\alpha\frac{\partial}{\partial\alpha} - \frac{1}{\sinh^2\alpha}\frac{\partial^2}{\partial\beta^2} \quad (14)$$

and the wave function Y, satisfying: $CY = j(j+1)Y$ is given by $Y_j^m(\cosh\alpha, \beta) = e^{im\beta}P_j^m(\cosh\alpha)$. It is eigenfunction of both C and compact generator J_3. The 'magnetic' quantum number, m, is quantized and $P_j^m(\cosh\alpha)$ are associated Lengedre polynomials.
Now we proceed to solve the equation for $\mathcal{U}(a)$

$$\left(\frac{d^2}{da^2} + \frac{d}{da} - a^2 + \frac{j(j+1)}{a^2}\right)\mathcal{U}(a) = 0 \quad (15)$$

The solutions are Bessel functions [20]: $\mathcal{U}(a)_\nu = J_{\pm i\nu/2}(\frac{i}{2}a^2)$ where $\nu^2 = j(j+1)$ is introduced for notational conveniences.Thus, the solution, Ψ, is given by

$$\Psi(a,\alpha,\beta) = \mathcal{U}_\nu(a)e^{im\beta}P_j^m(\cosh\alpha) \quad (16)$$

In quantum cosmology one chooses suitable linear combinations of the solutions to derive the wave function of the Universe, depending on the boundary conditions such as the one adopted by Hartle and Hawking [21] (the no boundary proposal) or one proposed by Vilenkin [22]. In the context of string cosmology, interesting classical solutions are derived in the pre-big bang scenario [23]; alterntively in the big crunch to big bang approach [24]. These initial conditions have interesting consequences in the studies of quantum string cosmologies [6,8]. In this note, we shall not persue to derive the wave function of the Universe, in our approach, according to various proposals in quantum cosmology and differ this aspect of quantum string cosmology to a separate investigation.
We proceed to unravel a novel symmetry which, in our openion, is quite interesting and is unexplored in axion-dilaton string cosmology. The *raison de etre* of the symmetry is the invariance of the theory under the S-duality group, $SU(1,1)$ and the wave functions (16) are endowed with huge degeneracy, since m is unbounded (from below or above depending on the choice of wave function). This is an attribute of the noncompact nature of $SU(1,1)$.
Notice that one may construct tensor operators, from the set $\{J_i\}$, which transform as higher dimensional representation of $SU(1,1)$. Such operators act in a Hilbert space, where the Casimir operator takes a specific value, say $C = \lambda$, and the operators close into an infinite dimensiosal algebra. This algebra is parametrized by λ and is denoted as $\mathcal{T}(\lambda)$. Such an approach has been persued by Pope etal. [25] for $SL(2,R)$ group, in detail, to study W_∞ algebra in an abstract setting. In the present context, the wave functions are known explicitly and the operators may be constructed in terms of $J_3, J_\pm$ as given below

$$J_\pm = \mp e^{\pm i\beta}\frac{\partial}{\partial\alpha} - i\coth\alpha e^{\pm i\beta}\frac{\partial}{\partial\beta},\ J_3 = -i\frac{\partial}{\partial\beta} \quad (17)$$

Then one might adopt the prescription of ref. [25].
As an illustative example, let us consider a specific case, $C = -3/16$, to show how the w_∞ algebra emerges. A more elegant way is to express the generators in terms of a single boson creation and annihilation operators [26] which serve our purpose better.

$$J_+ = \frac{1}{2}(a^\dagger)^2,\ J_- = \frac{1}{2}a^2,\ J_3 = \frac{1}{2}(a^\dagger a + 1/2) \quad (18)$$

$[a, a^\dagger] = 1$. Defining, $|n>= (n!)^{-1/2}(a^\dagger)^n|0>$, where $a|0>= 0$ is the condition on vacuum, we find

$$J_+|n>= \frac{\sqrt{(n+1)(n+2)}}{2}|n+2>, \quad (19)$$

$$J_-|n>= \frac{\sqrt{(n(n-1)}}{2}|n-2>, \quad (20)$$

and

$$J_3|n>= (n+\frac{1}{2})|n> \quad (21)$$

We get two different representations of the Lie algebra from $|n>$: (i) For odd n, $j = -3/4$ and (ii) for even n, $j = -1/4$ which belongs to the supplementary series. When J_3 and $J_\pm$ are defined as in (18), a suitable set of operators can be constructed, in both the cases, which satisfy the w_∞ algebra [9]

$$[\mathcal{T}_{n,m}, \mathcal{T}_{k,l}] = ((m+1)(k+1) - (n+1)(l+1))\,\mathcal{T}_{n+k,m+l} \quad (22)$$

where $\mathcal{T}_{n,m} = (a^\dagger)^{n+1}a^{m+1},\ n,m \ge -1$. Here, we have computed the classical algebra, ignoring normal orderings.
In what follows, we briefly discuss the procedure to construct [25] the tensor operators satisfying the algebra $\mathcal{T}(\lambda)$ for arbitrary λ. First, one starts with a highest weight tensor, say $T^l = (J_+)^l$ and then obtains lower weight operators by successive actions of J_-. Such objects commute with the Casimir. As remarked earlier, for a given λ, $\mathcal{T}_m^l$ close into an infinite dimensional algebra. An explicit construction is

$$\mathcal{T}_m^l = \mathcal{N}(Ad_{J_-})^{l-m}(J_+)^l \quad (23)$$

with the definition $Ad_X(Y) = [X,Y]$ and $\mathcal{N}$ is a suitable normalization constant. These tensors can be expressed as a product of polynomial of order $l-m$ in J_3 and J_+^l, since $[J_-, (J_+)^l] = 2lJ_3(J_+)^{l-1} - l(l-1)(J_+)^{l-1}$. The representation (17) may be used to realize the operators defined in (23). Thus, for a given λ, one gets the $W_\infty(\lambda)$ algebra. Note that for special value of $\lambda = 0$, one obtains the subalgebra $W_\wedge$ which is contained in W_∞, as has been elaborated in [25]. The former still allows an infinite dimensional representation. In our cosmological scenario, the operators $J_\pm$ act on the wave function obtained from solutions of differtial equation associate with

$SU(1,1)$ Casimir (14). Thus, these operators acting on Ψ produce another wave function, with same $j(j+1)$ (alternatively same λ), but still it is a solution of WDW equation. In other words, the energy eigenvalue is still zero for such a wave function.

To summarize, we have derived hitherto unknown solutions to WDW equation in axion-dilaton cosmology which are highly degenerate. The implications of the such degeneracies in string quantum cosmology are not completely explored. The existence of W-infinity symmetry is a surprize in the present investigation of string cosmology which originates from the underlying S-duality invariance of the theory. String cosmology with dilaton alone is unlikely to to possess such a symmetry. It is very tempting to ask whether the symmetries presented here have any role to address the cosmological constant problem, since it is hoped that string theory, with its rich symmetry contents, might provide a resolution of the cosmological constant problem.

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