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Noncompact Symmetries in String Theory*

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Abstract

Noncompact groups, similar to those that appeared in various supergravity theories in the 1970's, have been turning up in recent studies of string theory. First it was discovered that moduli spaces of toroidal compactification are given by noncompact groups modded out by their maximal compact subgroups and discrete duality groups. Then it was found that many other moduli spaces have analogous descriptions. More recently, noncompact group symmetries have turned up in effective actions used to study string cosmology and other classical configurations. This paper explores these noncompact groups in the case of toroidal compactification both from the viewpoint of low-energy effective field theory, using the method of dimensional reduction, and from the viewpoint of the string theory world sheet. The conclusion is that all these symmetries are intimately related. In particular, we find that Chern–Simons terms in the three-form field strength $H_{\mu\nu\rho}$ play a crucial role.

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1. Introduction

The unexpected appearance of noncompact global symmetries was one of the most intriguing discoveries to emerge from the study of supergravity theories in the 1970's. The example that attracted the most attention at the time was the $E_{7,7}$ symmetry (a noncompact variant of E_7) of $N = 8, D = 4$ supergravity. More recently, noncompact groups have been found to play a significant role in string theory. Narain's analysis of the heterotic string with d toroidally compactified dimensions [1] focussed attention on the group $O(d, d+16)$. He showed that the coset space $O(d, d+16)/O(d) \times O(d+16)$ is essentially the moduli space of inequivalent compactifications. Analogous coset spaces describe the moduli spaces of certain Calabi–Yau, orbifold, and other string compactifications, as well. While Narain's $O(d, d + 16)$ group is certainly not an exact symmetry of the compactified heterotic string theory, the discrete subgroup $O(d, d + 16, Z)$ apparently is. (This is an example of a "target space duality" group, which relates distinct geometries corresponding to the same conformal field theory.) In the last couple of years, motivated by considerations of superstring cosmology, attention has turned to the study of what happens when the compactification moduli are allowed to be time-dependent. Mueller has found solutions with "rolling radii" and a time-dependent dilaton [2]. Veneziano discovered an inversion symmetry for the cosmological scale factor, or "scale factor duality," both for vacuum solutions and for the motion of classical strings in cosmological backgrounds [3]. Similar observations were made by Tseytlin [4] for the case of closed strings and compact target space. Scale factor duality was later extended to a full continuous $O(d, d)$ symmetry of time-dependent (but independent of d space dimensions) solutions to the low-energy theory both in the absence [5] and in the presence of classical string sources [6]. The purpose of this paper is to explore the relationships between these various appearances of noncompact global symmetry groups. We will find that they are all very closely related and that Chern–Simons terms play a significant role in the realization of the symmetry.

The first appearance of a noncompact symmetry was the discovery of a global

$SU(1,1)$ invariance in an appropriate formulation of $N=4,\ D=4$ supergravity [7]. The qualification "appropriate formulation" refers to the fact that duality transformations allow n-forms to be recast as $(D-n-2)$-forms in D dimensions ($d\tilde{A} = *dA$), interchanging the role of Bianchi identities and equations of motion. Only after appropriate transformations is the full noncompact symmetry exhibited. In the $SU(1,1)$ theory there are two scalar fields, which parametrize the coset space $SU(1,1)/U(1)$. A year later, Cremmer and Julia showed that $N=8,\ D=4$ supergravity could be formulated with $E_{7,7}$ symmetry [8]. In this case the 70 scalars parametrize the coset $E_{7,7}/SU(8)$. In analogous manner the $N=8,\ D=5$ theory was found to have $E_{6,6}$ symmetry, with the 42 scalars parametrizing $E_{6,6}/USp(8)$ [9]. The largest known symmetry of this type occurs in the $N=16,\ D=3$ theory, which has $E_{8,8}$ symmetry, with 128 scalars parametrizing $E_{8,8}/O(16)$ [10]. It appears to be a general feature that the scalars parametrize G/H, where H is the maximal compact subgroup of G. Two-dimensional examples, in which G and H are both infinite, have also been considered [11].

The examples listed above (except for the $SU(1,1)$ case) all refer to maximally supersymmetric theories. If they have any connection to string theory, it is with the type II superstring. Although it may be worthwhile to do so, that line of inquiry will not be pursued here. Rather, we shall focus on theories with half as much supersymmetry ($N=1$ in $D=10$ or $N=4$ in $D=4$). These should be relevant to heterotic string theories. In [12] it was shown that the $N=1,\ D+d=10$ supergravity theory, dimensionally reduced to D dimensions (by dropping the dependence of the fields on d dimensions), has global $O(d,d)$ symmetry. One exception occurs for $D=3$, where duality transformations allow the symmetry to be extended to $O(8,8)$ [10]. Moreover, when the original $N=1$ $D=10$ theory has n Abelian vector supermultiplets in addition to the supergravity multiplet, the global symmetry of the dimensionally-reduced theory becomes extended to $O(d,d+n)$, except for $d=7$, where one obtains $O(8,8+n)$ [10].

The coupling of $N=1$ $D=10$ supergravity to vector supermultiplets require the inclusion of a Chern–Simons term ($H = dB - \omega_3$) in order to achieve supersymmetry.

This was shown in the Abelian case by Bergshoeff *et al.* [13] and in the non-Abelian case by Chapline and Manton [14]. In this paper we will focus on the bosonic sector, which can be formulated in any dimension. In section 2 we show that dimensional reduction from $D+d$ dimensions to D dimensions gives rise to a theory with global $O(d,d)$ symmetry when there are no vector fields in $D+d$ dimensions. In section 4 the addition of n Abelian vector fields in $D+d$ dimensions is considered. We show that the dimensionally-reduced theory has $O(d,d+n)$ symmetry provided that the Chern–Simons term (described above) is included. Thus the desirability of such terms is deduced from purely bosonic considerations!

The $O(d,d)$ symmetric theories considered by Veneziano and collaborators [3] are special cases of the theories derived here. Hassan and Sen have considered the extension to $n \neq 0$ and arbitrary D [15]. However, for their purposes only the $O(d) \times O(d+n)$ subgroup is of interest.

In the older supergravity theories, discussed above, a beautiful technique for formulating the G/H theory was developed. One starts with a matrix V_{iA} of scalar fields belonging to the adjoint representation of G, which acts as a sort of "vielbein." The i index runs over a representation of G and the A index over the corresponding (possibly reducible) representation of the subgroup H. Then the theory is formulated with global G symmetry and an independent local H symmetry. The latter is implemented by introducing auxiliary gauge fields for the group H, without any kinetic term. These fields, which are somewhat analogous to the spin connection in a first-order formulation of general relativity, can be eliminated by solving their equations of motion (algebraically) and substituting back in the action. The local H symmetry, which still is present after this substitution, can then be used to choose a gauge in which the scalar fields belonging to the H subgroup are set to zero. In section 3 we carry out this procedure explicitly for the $O(d,d)$ symmetric theory and show that it gives the correct action for the moduli fields. The vector fields are shown to form a $2d$-dimensional vector multiplet of $O(d,d)$. It is a general feature of the supergravity theories that all bosonic fields other than the scalars are inert under the local H symmetry. To our surprise, we discovered a second construction that linearizes the action of G, which is also presented in section 3.

In section 5 we reconsider the noncompact symmetries from the viewpoint of the world-sheet (σ-model) action. The result of Narain, Sarmadi, and Witten [16] that the moduli of toroidal compactification parametrize $O(d,d)/O(d) \times O(d)$ is briefly reviewed, as is the argument that string corrections break the $O(d,d)$ symmetry to the discrete $O(d,d,Z)$ subgroup. By introducing d coordinates ($\tilde{Y}_\alpha$) that are dual to the d compact string coordinates (Y^α), we are able to obtain a set of $2d$ classical equations of motion that have manifest $O(d,d)$ symmetry. The equations of motion for the space-time embedding of the string X^μ are also recast in an $O(d,d)$ symmetric form. The symmetry is broken to $O(d,d,Z)$ by boundary conditions.

2. Dimensional Reduction Gives O(d,d) Symmetry

In the 1970's it was noted that noncompact global symmetries are a generic feature of supergravity theories containing scalar fields. One of the useful techniques that was exploited in these studies was the method of "dimensional reduction." In its simplest form, this consists of considering a theory in a spacetime M×K, where M has D dimensions and K has d dimensions, and supposing that the fields are independent of the coordinates y^α of K. For this to be a consistent procedure it is necessary that K-independent solutions be able to solve the classical field equations. Then one speaks of "spontaneous compactification" (at least when K is compact). In a gravity theory this implies that K is flat, a torus for example. Of course, in recent times more interesting possibilities, such as Calabi–Yau spaces, have received a great deal of attention. In such a case, the analog of dropping y dependence is to truncate all fields to their zero modes on K. Here we will only consider flat K, though generalizations would clearly be deserving of study.

Explicit formulas for dimensional reduction were given in a 1979 paper by Joël Scherk and JHS [17] and subsequently developed further by Cremmer [18]. The main purpose of [17] was to introduce a "generalized" method of dimensional reduction that could give rise to massive fields in the D-dimensional theory starting from massless ones in the $(D+d)$-dimensional theory. That procedure will not be utilized here. Rather we will stick to the simplest case in which the fields are taken to be independent of the K coordinates. Our notation is as follows: Local coordinates of M are $x^\mu(\mu = 0,1,...,D-1)$ and local coordinates of K are $y^\alpha(\alpha = 1,...,d)$. The tangent space Lorentz metric has signature $(-+...+)$, unlike [17], which results in a number of sign changes in the formulas given there. All fields in $D+d$ dimensions are written with hats on the fields and the indices ($\hat{\phi}$, $\hat{g}_{\hat{\mu}\hat{\nu}}$, etc.). Quantities without hats are reserved for D dimensions. Thus, for example, the Einstein action on M×K (with a dilaton field $\hat{\phi}$) is

$$S_{\hat{g}} = \int\limits_M dx \int\limits_K dy \, \sqrt{-\hat{g}} \, e^{-\hat{\phi}} \big[\hat{R}(\hat{g}) + \hat{g}^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi}\big] \tag{2.1}$$

If K is assumed to be a torus we can choose the coordinates y^α to be periodic with unit periods, so that $\int_K dy = 1$. The radii and angles that characterize the torus are then encoded in the metric tensor. As usual, the strength of the gravitational interaction is determined by the value of the dilaton field.

The formulas that follow can be read off from [17], generalized to include the dilaton field. In terms of a $(D+d)$-dimensional vielbein, we can use local Lorentz invariance to choose a triangular parametrization

$$\hat{e}_{\hat{\mu}}^{\hat{r}} = \begin{pmatrix} e_\mu^r & A_\mu^{(1)\beta} E_\beta^a \\ 0 & E_\alpha^a \end{pmatrix} \quad \text{and} \quad \hat{e}_{\hat{r}}^{\hat{\mu}} = \begin{pmatrix} e_r^\mu & -e_r^\nu A_\nu^{(1)\alpha} \\ 0 & E_a^\alpha \end{pmatrix}. \tag{2.2}$$

The "internal" metric is $G_{\alpha\beta} = E_\alpha^a \delta_{ab} E_\beta^b$ and the "spacetime" metric is $g_{\mu\nu} = e_\mu^r \eta_{rs} e_\nu^s$. As usual, $G^{\alpha\beta}$ and $g^{\mu\nu}$ represent inverses and are used to raise the appropriate indices. In terms of these quantities the complete $(D+d)$-dimensional metric is

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + A_\mu^{(1)\gamma} A_{\nu\gamma}^{(1)} & A_{\mu\beta}^{(1)} \\ A_{\nu\alpha}^{(1)} & G_{\alpha\beta} \end{pmatrix} \text{ and } \hat{g}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g^{\mu\nu} & -A^{(1)\mu\beta} \\ -A^{(1)\nu\alpha} & G^{\alpha\beta} + A^{(1)\rho\alpha} A_\rho^{(1)\beta} \end{pmatrix}. \tag{2.3}$$

A convenient property of this parametrization is that

$$\sqrt{-\hat{g}} = \det \hat{e}_{\hat{\mu}}^{\hat{r}} = \det e_\mu^r \det E_\alpha^a = \sqrt{-g}\sqrt{\det G}\,. \tag{2.4}$$

If all fields are assumed to be y independent, one finds after a tedious calculation

$$
\begin{aligned}
S_{\hat{g}} = \int\limits_M dx \sqrt{-g}\ e^{-\phi} \Big\{ & R + g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi \\
& + \frac{1}{4} g^{\mu\nu} \partial_\mu G_{\alpha\beta} \partial_\nu G^{\alpha\beta} - \frac{1}{4} g^{\mu\rho} g^{\nu\lambda} G_{\alpha\beta} F^{(1)\alpha}_{\mu\nu} F^{(1)\beta}_{\rho\lambda} \Big\} ,
\end{aligned}
\tag{2.5}
$$

where we have introduced a shifted dilaton field [19,20,3]

$$
\phi = \hat{\phi} - \frac{1}{2} \log\ \det G_{\alpha\beta} \tag{2.6}
$$

and $F^{(1)\alpha}_{\mu\nu} = \partial_\mu A^{(1)\alpha}_\nu - \partial_\nu A^{(1)\alpha}_\mu$.

Another field that is of interest in string theory is a second-rank antisymmetric tensor $\hat{B}_{\hat{\mu}\hat{\nu}}$ with field strength

$$
\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \partial_{\hat{\mu}} \hat{B}_{\hat{\nu}\hat{\rho}} + \text{cyc. perms} . \tag{2.7}
$$

The Chern–Simons terms that appear in superstring theory are not present here since we are not including $(D+d)$-dimensional vector fields (in this section). The Lorentz Chern–Simons term [21] is of higher order in derivatives than we are considering. The action for the $\hat{B}$ term is

$$
S_{\hat{B}} = -\frac{1}{12} \int\limits_M dx \int\limits_K dy\ \sqrt{-\hat{g}}\ e^{-\hat{\phi}}\ \hat{g}^{\hat{\mu}\hat{\mu}'}\ \hat{g}^{\hat{\nu}\hat{\nu}'}\ \hat{g}^{\hat{\rho}\hat{\rho}'}\ \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\ \hat{H}_{\hat{\mu}'\hat{\nu}'\hat{\rho}'} . \tag{2.8}
$$

Because of the structure of the inverse metric, a little thought is required to organize the terms in the dimensional reduction of eq. (2.8) in a useful form. A systematic procedure is to first convert $\hat{H}$ to tangent space indices $\hat{H}_{\hat{r}\hat{s}\hat{t}}$ and then use

e^r_μ and E^a_α to convert back to Greek indices. This procedure leads to the result

$$
\begin{aligned}
S_{\hat{B}} = -\int_M dx\sqrt{-g}\; e^{-\phi}\Big\{ & \frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} + \frac{1}{4}H_{\mu\alpha\beta}H^{\mu\alpha\beta} \\
& + \frac{1}{4}H_{\mu\nu\alpha}H^{\mu\nu\alpha} + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big\}\,.
\end{aligned}
\tag{2.9}
$$

Here $H_{\alpha\beta\gamma} = 0$, since $\hat{B}_{\alpha\beta} = B_{\alpha\beta}$ is y independent. Also,

$$
H_{\mu\alpha\beta} = e^r_\mu \hat{e}^{\hat{\mu}}_r \hat{H}_{\hat{\mu}\alpha\beta} = \hat{H}_{\mu\alpha\beta} = \partial_\mu B_{\alpha\beta}\,. \tag{2.10}
$$

Similarly,

$$
\begin{aligned}
H_{\mu\nu\alpha} &= e^r_\mu e^s_\nu \hat{e}^{\hat{\mu}}_r \hat{e}^{\hat{\nu}}_s \hat{H}_{\hat{\mu}\hat{\nu}\alpha} \\
&= \hat{H}_{\mu\nu\alpha} - A^{(1)\beta}_\mu \hat{H}_{\beta\nu\alpha} - A^{(1)\beta}_\nu \hat{H}_{\mu\beta\alpha} \\
&= F^{(2)}_{\mu\nu\alpha} - B_{\alpha\beta}F^{(1)\beta}_{\mu\nu}\;,
\end{aligned}
\tag{2.11}
$$

where we have used $F^{(2)}_{\mu\nu\alpha} = \partial_\mu A^{(2)}_{\nu\alpha} - \partial_\nu A^{(2)}_{\mu\alpha}$ and

$$
A^{(2)}_{\mu\alpha} = \hat{B}_{\mu\alpha} + B_{\alpha\beta}A^{(1)\beta}_\mu\;. \tag{2.12}
$$

The gauge transformations of the vector fields are simply $\delta A^{(1)\alpha}_\mu = \partial_\mu \Lambda^{(1)\alpha}$ and $\delta A^{(2)}_{\mu\alpha} = \partial_\mu \Lambda^{(2)}_\alpha$, under which $H_{\mu\nu\alpha}$ is invariant.

For $H_{\mu\nu\rho}$ one finds

$$
\begin{aligned}
H_{\mu\nu\rho} &= e^r_\mu e^s_\nu e^t_\rho \hat{e}^{\hat{\mu}}_r \hat{e}^{\hat{\nu}}_s \hat{e}^{\hat{\rho}}_t \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \\
&= \hat{H}_{\mu\nu\rho} - \big(A^{(1)\alpha}_\mu \hat{H}_{\alpha\nu\rho} + 2\text{ perms}\big) + \big(A^{(1)\alpha}_\mu A^{(1)\beta}_\nu \hat{H}_{\alpha\beta\rho} + 2\text{ perms}\big) \\
&= \partial_\mu B_{\nu\rho} - \frac{1}{2}\big(A^{(1)\alpha}_\mu\; F^{(2)}_{\nu\rho\alpha} + A^{(2)}_{\mu\alpha}F^{(1)\alpha}_{\nu\rho}\big) + \text{cyc. perms.}\,,
\end{aligned}
\tag{2.13}
$$

where

$$
B_{\mu\nu} = \hat{B}_{\mu\nu} + \frac{1}{2}A^{(1)\alpha}_\mu A^{(2)}_{\nu\alpha} - \frac{1}{2}A^{(1)\alpha}_\nu A^{(2)}_{\mu\alpha} - A^{(1)\alpha}_\mu B_{\alpha\beta}A^{(1)\beta}_\nu\;. \tag{2.14}
$$

In this case gauge invariance of the last line in eq. (2.13) requires that under the $\Lambda^{(1)}$

and $\Lambda^{(2)}$ transformations

$$\delta B_{\mu\nu} = \frac{1}{2}\big(\Lambda^{(1)\alpha}F^{(2)}_{\mu\nu\alpha} + \Lambda^{(2)}_{\alpha}F^{(1)\alpha}_{\mu\nu}\big)\ . \tag{2.15}$$

The extra terms in $H_{\mu\nu\rho}$, which have arisen as a consequence of the dimensional reduction, are abelian Chern–Simons terms. Recall that the requirement that H is globally defined implies that dH is exact and hence that $\mathrm{tr}(R\wedge R) - \mathrm{tr}(F\wedge F)$ is exact for the familiar Chern–Simons terms of $N=1$, $D=10$ supersymmetric theories [21,22]. In the present case, similar reasoning yields the requirement that $F^{(1)\alpha}\wedge F^{(2)}_{\alpha}$ be exact. Again, this is a significant restriction on possible background configurations.

To recapitulate, the dimensionally reduced form of $S = S_{\hat{g}} + S_{\hat{B}}$ has been written in the form

$$S = \int\limits_M dx\sqrt{-g}\ e^{-\phi}\mathcal{L}\ . \tag{2.16}$$

For the factor $\mathcal{L}$ we have found $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$, where

$$\begin{aligned}
\mathcal{L}_1 &= R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\
\mathcal{L}_2 &= \frac{1}{4}g^{\mu\nu}\big(\partial_\mu G_{\alpha\beta}\partial_\nu G^{\alpha\beta} - G^{\alpha\beta}G^{\gamma\delta}\partial_\mu B_{\alpha\gamma}\partial_\nu B_{\beta\delta}\big) \\
\mathcal{L}_3 &= -\frac{1}{4}g^{\mu\rho}g^{\nu\lambda}\big(G_{\alpha\beta}F^{(1)\alpha}_{\mu\nu}F^{(1)\beta}_{\rho\lambda} + G^{\alpha\beta}H_{\mu\nu\alpha}H_{\rho\lambda\beta}\big) \\
\mathcal{L}_4 &= -\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\ .
\end{aligned} \tag{2.17}$$

We now claim that there is an $O(d,d)$ global symmetry that leaves each of these four terms separately invariant. The first term ($\mathcal{L}_1$) is trivially invariant since $g_{\mu\nu}$ and ϕ are. It should be noted, however, that the individual terms in $\phi = \hat{\phi} - \frac{1}{2}\log\det G_{\alpha\beta}$ are not invariant.

To investigate the invariance of $\mathcal{L}_2$ we first rewrite it, using matrix notation, as

$$\mathcal{L}_2 = \frac{1}{4}\mathrm{tr}\big(\partial_\mu G^{-1}\partial^\mu G + G^{-1}\partial_\mu B G^{-1}\partial^\mu B\big)\ . \tag{2.18}$$

Then we introduce two $2d \times 2d$ matrices, written in $d \times d$ blocks, as follows [23]:

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{2.19}$$

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.20}$$

Since η has d eigenvalues $+1$ and d eigenvalues -1, it is a metric for the group $O(d,d)$ in a basis rotated from the one with a diagonal metric. The diagonal form will be used briefly in the next section. Next we note that $M \in O(d,d)$, since

$$M^T \eta M = \eta . \tag{2.21}$$

In fact, M is a *symmetric* $O(d,d)$ matrix, which implies that

$$M^{-1} = \eta M \eta = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} . \tag{2.22}$$

It is now a simple exercise to verify that

$$\mathcal{L}_2 = \frac{1}{8} \mathrm{tr}(\partial_\mu M^{-1} \partial^\mu M) . \tag{2.23}$$

Thus $\mathcal{L}_2$ is invariant under a global $O(d,d)$ transformation

$$M \to \Omega M \Omega^T , \tag{2.24}$$

where

$$\Omega^T \eta \Omega = \eta . \tag{2.25}$$

This transformation acts on G and B in a rather complicated nonlinear way. We will give a simple description of this action later.

One might be tempted to think that the symmetry is even larger, since $\mathcal{L}_2$ is formally invariant under $M \to AMA^T$ for any matrix $A \in GL(2d)$. However, M is not an arbitrary symmetric matrix (which would have $d(2d-1)$ parameters), but one which belongs to $O(d,d)$ and has just d^2 parameters. Thus $O(d,d)$ transformations are the most general transformations that preserve the structure of M and can be realized as transformations of G and B.

Next we consider the $\mathcal{L}_3$ term:

$$\begin{aligned}\mathcal{L}_3 &= -\frac{1}{4}\big[F^{(1)\alpha}_{\mu\nu} G_{\alpha\beta} F^{(1)\mu\nu\beta} + \big(F^{(2)}_{\mu\nu\alpha} - B_{\alpha\gamma} F^{(1)\gamma}_{\mu\nu}\big) G^{\alpha\beta} \big(F^{(2)\mu\nu}_{\beta} - B_{\beta\delta} F^{(1)\mu\nu\delta}\big)\big] \\ &= -\frac{1}{4}\mathcal{F}^i_{\mu\nu}(M^{-1})_{ij}\mathcal{F}^{\mu\nu j} \ ,\end{aligned} \tag{2.26}$$

where $\mathcal{F}^i_{\mu\nu}$ is the $2d$-component vector of field strengths

$$\mathcal{F}^i_{\mu\nu} = \begin{pmatrix} F^{(1)\alpha}_{\mu\nu} \\ F^{(2)}_{\mu\nu\alpha} \end{pmatrix} = \partial_\mu \mathcal{A}^i_\nu - \partial_\nu \mathcal{A}^i_\mu \ . \tag{2.27}$$

Then $\mathcal{L}_3$ is seen to be $O(d,d)$ invariant provided that the vector fields transform linearly according to the vector representation of $O(d,d)$, i.e., $\mathcal{A}^i_\mu \to \Omega^i{}_j \mathcal{A}^j_\mu$.

Finally we turn to $\mathcal{L}_4$. In this case $H_{\mu\nu\rho}$ can be written in the form

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2}\mathcal{A}^i_\mu \eta_{ij} \mathcal{F}^j_{\nu\rho} + (\text{cyc. perms.}) \ . \tag{2.28}$$

This is $O(d,d)$ invariant if we require that $B_{\nu\rho}$ not transform. The second term is invariant since $\Omega^T \eta \Omega = \eta$.

To make contact with string theory, the formulas we have presented here are appropriate to the massless fields of the closed oriented bosonic string with $D+d = 26$. In that case the $O(d,d)$ symmetry is certainly broken by higher mass and higher dimension terms that have been dropped. An $O(d,d,Z)$ subgroup is believed to survive as an exact symmetry of the theory, though it is broken spontaneously when a particular background is selected. This discrete group and its relationship to the continuous groups described here will be explored in section 5. To make contact with the heterotic string, Yang–Mills gauge fields should be introduced in the original $(D+d)$-dimensional theory. This extension will be explored in section 4.

3. Coset Space Reformulations

The realization of $O(d,d)$ symmetry found in the last section, $M \to \Omega M \Omega^T$, is not very transparent as a rule for the transformation of the d^2 scalar fields

$$X = G + B\,. \tag{3.1}$$

Let us explore this in a little detail. To start with, consider an infinitesimal $O(d,d)$ transformation given by*

$$\Omega = \begin{pmatrix} 1+\alpha & \beta \\ \gamma & 1-\alpha^T \end{pmatrix}, \tag{3.2}$$

where α, β, γ are infinitesimal $d \times d$ matrices and $\beta = -\beta^T$, $\gamma = -\gamma^T$. Then

$$\delta M = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^T \end{pmatrix} M + M \begin{pmatrix} \alpha^T & -\gamma \\ -\beta & -\alpha \end{pmatrix}, \tag{3.3}$$

which is easily seen to correspond to [24]

$$\delta X = \gamma - \alpha^T X - X\alpha - X\beta X\,. \tag{3.4}$$

The similarity of the last formula to one for an infinitesimal $SL(2,C)$ transformation, which exponentiates to $z \to (a\,z+b)(c\,z+d)^{-1}$, suggests the following. Write

$\star$ The $GL(d,R)$ subalgebra parametrized by the matrix α corresponds to constant (global) general coordinate transformations of the internal manifold K. Clearly, in view of the toroidal topology, only the $SL(d,Z)$ subgroup belongs to Diff(K) (see section 5). The remaining generators of $O(d,d,Z)$ correspond to integer shifts of the moduli $B_{\alpha\beta}$.

an arbitrary $O(d,d)$ matrix Ω in block form

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} . \tag{3.5}$$

Then the finite transformation

$$X \to (\Omega_{22} X + \Omega_{21})(\Omega_{11} + \Omega_{12} X)^{-1} \tag{3.6}$$

reproduces the infinitesimal transformation formula obtained above [25]. Moreover, it has the correct group property, and so must be correct in general. The matrix that appears here is actually

$$\tilde{\Omega} = \eta \Omega \eta = \begin{pmatrix} \Omega_{22} & \Omega_{21} \\ \Omega_{12} & \Omega_{11} \end{pmatrix} , \tag{3.7}$$

which is an equivalence transformation. It is the matrix X^{-1} that undergoes a linear fractional transformation controlled by the matrix Ω. This transformation law of X is reminiscent of that for period matrices matrices under symplectic modular transformations in the theory of Riemann surfaces.

How should we utilize these facts? A possible goal is to rewrite the action in terms of the matrix X rather than the matrix M. Another possible goal is to introduce auxiliary gauge fields and extra scalar fields such that the $O(d,d)$ symmetry is realized linearly. Towards these ends let us introduce a second real $d \times d$ matrix of scalar fields, called Y, and generalize eq. (3.6) to

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \Omega_{22} & \Omega_{21} \\ \Omega_{12} & \Omega_{11} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} , \tag{3.8}$$

which corresponds to the previous nonlinear transformation rule for the matrix XY^{-1}. In other words, (3.6) corresponds to (3.8) in the "gauge" $Y = 1$. It is convenient to introduce a $2d \times d$ matrix V consisting of the blocks X and Y (as in eq. (3.8)), such that the above transformation is

$$V_{i\alpha} \to \tilde{\Omega}_{ij}\ V_{j\alpha}\ .$$

The rectangular matrix $V_{i\alpha}$ transforms as d copies (labeled by α) of the vector representation of $O(d,d)$.

In order to have enough gauge freedom to eliminate Y, which is an arbitrary real nonsingular matrix, we need local $GL(d,R)$ gauge symmetry. If $m_{\alpha\beta}$ is a matrix belonging to $GL(d,R)$, we require that $V_{i\alpha}$ transform as $2d$ copies (labeled by i) of the vector representation of $GL(d,R)$

$$V_{i\alpha} \to m_{\alpha\beta}\ V_{i\beta} = (Vm^T)_{i\alpha}\ . \tag{3.9}$$

Next we introduce auxiliary gauge fields, belonging to the $GL(d,R)$ algebra, called $(A_\mu)_{\alpha\beta}$, and we define a covariant derivative

$$D_\mu V_{i\alpha} = \partial_\mu V_{i\alpha} + (A_\mu)_{\alpha\beta} V_{i\beta}\ . \tag{3.10}$$

Now let us try to write a V kinetic term with global $O(d,d)$ symmetry and local $GL(d,R)$ symmetry. Two $O(d,d)$ invariant $d \times d$ matrices are $(V^T\eta V)_{\alpha\beta}$ and $(D_\mu V^T\eta\ D^\mu V)_{\alpha\beta}$. Under local $GL(d,R)$ transformations

$$\begin{aligned} V^T\eta V &\to m(V^T\eta V)m^T \\ D_\mu V^T\eta D^\mu V &\to m(D_\mu V^T\eta D^\mu V)m^T\ . \end{aligned} \tag{3.11}$$

Therefore the natural guess with the desired symmetries is

$$\mathcal{L}'_2 = \frac{1}{4}\mathrm{tr}\big[(V^T\eta V)^{-1}(D_\mu V^T\eta D^\mu V)\big]\ . \tag{3.12}$$

It is straightforward to solve the classical field equation implied by this Lagrangian for $(A_\mu)_{\alpha\beta}$ in the $Y=1$ gauge with the result

$$(A_\mu)_{\alpha\beta} = -\frac{1}{2}(G^{-1})_{\beta\gamma}\partial_\mu(G+B)_{\gamma\alpha}\ . \tag{3.13}$$

Substituting this back into ${\cal L'}_2$, one obtains the desired result found in Section 2:

$${\cal L'}_2 = {\cal L}_2 = \frac{1}{4}tr(\partial_\mu G^{-1}\partial^\mu G + G^{-1}\partial_\mu B G^{-1}\partial^\mu B)\ . \tag{3.14}$$

To complete this part of the story we still need to recast ${\cal L}_3$ in terms of V in an $O(d,d)\times GL(d,R)$ invariant form. Since ${\cal F}^i$ is an $O(d,d)$ vector, $(V^T{\cal F})_\beta$ is $O(d,d)$ invariant and a $GL(d,R)$ vector. Thus an invariant combination is

$$({\cal F}^T V)_\alpha (V^T\eta V)^{-1}_{\alpha\beta}(V^T{\cal F})_\beta\ . \tag{3.15}$$

It is straightforward to show that in the $Y=1$ gauge this reduces to

$$\frac{1}{2}\big({\cal F}^T\eta{\cal F} + {\cal F}^T M^{-1}{\cal F}\big)\ . \tag{3.16}$$

Thus the desired result is

$${\cal L}_3 = -\frac{1}{2}({\cal F}^T V)(V^T\eta V)^{-1}(V^T{\cal F}) + \frac{1}{4}{\cal F}^T\eta{\cal F}\ . \tag{3.17}$$

The result found above is not what was expected. Experience from supergravity theories leads one to expect that it should be possible to linearize the $O(d,d)$ symmetry transformations by introducing a complete adjoint multiplet of scalar fields and gauging the maximal compact subgroup $O(d)\times O(d)$, so that the d^2 scalar fields $X=G+B$ would parametrize the coset space $O(d,d)/O(d)\times O(d)$. What we have done above is quite different – it is not a coset construction, since the starting multiplet of scalars V does not parametrize the adjoint representation of any group. Rather it belongs to vector representations of both $O(d,d)$ and $GL(d,R)$, the latter being gauged. This raises a question. Does the usual G/H construction give an equivalent result or does it give a wrong result?

To construct an $O(d,d)/O(d)\times O(d)$ theory we follow the procedure used in various supergravity theories. The way to do this is to introduce a $2d\times 2d$ matrix V_{Ai} which plays the role of a "vielbein" for the matrix M_{ij} [24], in the sense that

$$M_{ij} = (V^T V)_{ij} = \delta^{AB} V_{Ai} V_{Bj} \ . \tag{3.18}$$

A matrix that solves this equation is

$$V = \begin{pmatrix} E^{-1} & -E^{-1}B \\ 0 & E \end{pmatrix} \tag{3.19}$$

where E is a $d\times d$ vielbein satisfying $E^T E = G$. It should be noted that the matrix V belongs to $O(d,d)$, i.e., $V^T \eta V = \eta$.

The obvious guess, then, for an action with global $O(d,d)$ symmetry and local $O(d)\times O(d)$ symmetry is

$$\mathcal{L} = \frac{1}{4}\eta^{ij}\eta^{AB}(D_\mu V)_{Ai}(D^\mu V)_{Bj} \ , \tag{3.20}$$

with V an arbitrary $O(d,d)$ matrix (not yet of the special form in eq. (3.19)) and auxiliary gauge fields for local $O(d)\times O(d)$, which are incorporated in the covariant derivatives. The covariant derivative is a little awkward to formulate in the basis with the off-diagonal metric η. Therefore we make a change of basis that diagonalizes it. Introducing $\rho^T \eta \rho = \sigma$, where

$$\rho = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{3.21}$$

we rotate the matrix V by defining $W = \rho^T V \rho$. Since V is an $O(d,d)$ matrix satisfying $V^T \eta V = \eta$, W is an $O(d,d)$ matrix satisfying $W^T \sigma W = \sigma$. Now the covariant derivative takes the form

$$(D_\mu W)_{Ai} = \partial_\mu W_{Ai} + \omega_{\mu AB}\sigma^{BC} W_{Ci} \ , \tag{3.22}$$

where the auxiliary $O(d)\times O(d)$ gauge fields are given by

$$\omega_\mu = \begin{pmatrix} \omega_\mu^{(1)} & 0 \\ 0 & \omega_\mu^{(2)} \end{pmatrix} \ . \tag{3.23}$$

In this expression $\omega_\mu^{(1)}$ and $\omega_\mu^{(2)}$ are independent $O(d)$ gauge fields (antisymmetric). The Lagrangian now takes the form

$$\mathcal{L} = \frac{1}{4}\, \sigma^{ij}\sigma^{AB}(D_\mu W)_{Ai}(D^\mu W)_{Bj} \ . \tag{3.24}$$

To make contact with $\mathcal{L}_2$ one varies with respect to the gauge fields, solves their classical equations, and substitutes back into $\mathcal{L}$. This procedure is certainly valid in the present context. One finds that

$$\omega^{(1)}_{\mu ab} = \frac{1}{2}\eta^{ij}(W_{ai}\partial_\mu W_{bj} - W_{bi}\partial_\mu W_{aj}) \tag{3.25}$$

where a, b run over the first d values of the indices A, B. $\omega^{(2)}_{\mu ab}$ is given by an analogous formula using the second d values of the indices. Substituting back into eq. (3.24) we find that

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}\big[(W\sigma\partial_\mu W^T)_{12}\ (W\sigma\partial^\mu W^T)_{21}\big] \, . \tag{3.26}$$

The notation is that the numerical indices represent $d\times d$ blocks of the $2d\times 2d$ matrix $W\sigma\partial_\mu W^T$. At this point W is an arbitrary $O(d,d)$ matrix. However, $\mathcal{L}$ still has local $O(d)\times O(d)$ symmetry even though the gauge fields have been eliminated. This local symmetry allows us to choose a gauge in which W takes the form $W = \rho^T V \rho$, with V the matrix given in eq. (3.19).

Now we must compare the result above to $\mathcal{L}_2 = \frac{1}{8}\mathrm{tr}(\eta\ \partial_\mu M\ \eta\ \partial^\mu M)$. Substituting $M = V^T V = \rho W^T W \rho^T$ and using $\rho^T \eta \rho = \sigma$ gives

$$\mathcal{L}_2 = \frac{1}{8} tr(\sigma\ \partial_\mu (W^T W) \sigma\ \partial^\mu (W^T W))\ . \tag{3.27}$$

Expanding out the derivatives and using $W^T \sigma W = \sigma$ gives

$$\mathcal{L}_2 = \frac{1}{4}\mathrm{tr}\left[(W\sigma\ \partial_\mu W^T)(W\sigma\ \partial^\mu W^T) - (W\sigma\ \partial_\mu W^T)\sigma(W\sigma\partial^\mu W^T)\sigma\right]\ . \tag{3.28}$$

Expanding in $d \times d$ blocks one finds that eqs. (3.26) and (3.28) are identical, and hence $\mathcal{L} = \mathcal{L}_2$, as desired! Thus the conventional wisdom that G and B parametrize an $O(d,d)/O(d) \times O(d)$ coset is correct. The somewhat surprising fact is that there is an alternative interpretation utilizing local $GL(d,R)$ described in the first part of this section.

4. Generalization to O(d, d+n) Symmetry

Previous work in supergravity [10] and superstring theory [1] suggests that if we add n Abelian $U(1)$ gauge fields to the original $(D+d)$-dimensional theory, that $O(d,d+n)$ symmetry should result from dimensional reduction to D dimensions. In this section we explore whether this is the case. The additional term to be added to the action is

$$S_{\hat{A}} = -\frac{1}{4} \int_M dx \int_K dy\ \sqrt{-\hat{g}}\ e^{-\hat{\phi}}\ \hat{g}^{\hat{\mu}\hat{\rho}} \hat{g}^{\hat{\nu}\hat{\lambda}} \delta_{IJ} \hat{F}^I_{\hat{\mu}\hat{\nu}}\ \hat{F}^J_{\hat{\rho}\hat{\lambda}}\ , \tag{4.1}$$

where $\hat{F}^I_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}} \hat{A}^I_{\hat{\nu}} - \partial_{\hat{\nu}} \hat{A}^I_{\hat{\mu}}$ and the index I takes the values $I = 1,\ 2, \cdots, n$.

The most important point to note is that the original $(D+d)$-dimensional theory should have $O(n)$ symmetry described by the formulas of section 2 with $M_{IJ} = \eta_{IJ} = \delta_{IJ}$. Looking at the various pieces of the Lagrangian, we see that $\mathcal{L}_1$ has the usual form, $\mathcal{L}_2 = 0$, and $\mathcal{L}_3$ gives $S_{\hat{A}}$. The crucial observation concerns $\mathcal{L}_4$, which is built from the square of

$$\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \partial_{\hat{\mu}}\hat{B}_{\hat{\nu}\hat{\rho}} - \frac{1}{2}\hat{A}^I_{\hat{\mu}}\delta_{IJ}\hat{F}^J_{\nu\rho} + (\text{cyc. perms.}) \ . \tag{4.2}$$

This contains the Chern–Simons term (for the $U(1)$ gauge fields), a feature that is clearly crucial for the symmetries we wish to implement.[⋆] Once this point is understood, the analysis is a fairly straightforward generalization of that presented in section 2, though some of the algebra is more complicated.

The dimensional reduction can now be carried out by the same methods introduced in section 2. The reduction of $S_{\hat{g}}$ is unchanged from before. For the vectors we obtain

$$S_{\hat{A}} = -\frac{1}{4} \int dx\sqrt{-g}\ e^{-\phi}\Big\{F^I_{\mu\nu}F^{I\mu\nu} + 2F^I_{\mu\alpha}F^{I\mu\alpha}\Big\} \ , \tag{4.3}$$

where we define

$$\begin{aligned} A^{(3)I}_\mu &= \hat{A}^I_\mu - a^I_\alpha A^{(1)\alpha}_\mu \\ F^{(3)I}_{\mu\nu} &= \partial_\mu A^{(3)I}_\nu - \partial_\nu A^{(3)I}_\mu \\ a^I_\alpha &= \hat{A}^I_\alpha \\ F^I_{\mu\nu} &= F^{(3)I}_{\mu\nu} + F^{(1)\alpha}_{\mu\nu}a^I_\alpha \\ F^I_{\mu\alpha} &= \partial_\mu a^I_\alpha \ . \end{aligned} \tag{4.4}$$

The reduction of the various H terms includes additional pieces beyond those of section 2, because of the presence of the Chern–Simons term. We find the following:

$$\begin{aligned} H_{\mu\alpha\beta} &= \partial_\mu B_{\alpha\beta} + \frac{1}{2}(a^I_\alpha\partial_\mu a^I_\beta - a^I_\beta\partial_\mu a^I_\alpha) \\ H_{\mu\nu\alpha} &= -C_{\alpha\beta}F^{(1)\beta}_{\mu\nu} + F^{(2)}_{\mu\nu\alpha} - a^I_\alpha F^{(3)I}_{\mu\nu} \\ H_{\mu\nu\rho} &= \partial_\mu B_{\nu\rho} - \frac{1}{2}\mathcal{A}^i_\mu\eta_{ij}\mathcal{F}^j_{\nu\rho} + \text{cyc. perms.} \ , \end{aligned} \tag{4.5}$$

⋆ It is remarkable that the necessity of the Chern–Simons terms is deduced from purely bosonic considerations. This has been argued previously, in the σ-model description of strings, based on anomaly effects arising from gauge fields that couple chirally to the world sheet [26]. Such a chiral coupling is not assumed in the present analysis.

where we have used the definitions

$$A^{(2)}_{\mu\alpha} = \hat{B}_{\mu\alpha} + B_{\alpha\beta} A^{(1)\beta}_\mu + \frac{1}{2} a^I_\alpha A^{(3)I}_\mu \tag{4.6}$$

$$C_{\alpha\beta} = \frac{1}{2} a^I_\alpha a^I_\beta + B_{\alpha\beta} \ . \tag{4.7}$$

As usual, $H_{\mu\nu\rho}$ is gauge invariant for $\delta\mathcal{A}^i_\mu = \partial_\mu \Lambda^i$ and $\delta B_{\mu\nu} = \frac{1}{2}\Lambda^i \eta_{ij} \mathcal{F}^j_{\mu\nu}$. In matrix notation we write $C = \frac{1}{2} a^T a + B$. We have introduced a $(2d+n)$-component vectors $\mathcal{A}^i_\mu = (A^{(1)\alpha}_\mu,\ A^{(2)}_{\mu\alpha},\ A^{(3)I}_\mu)$ and $\mathcal{F}^i_{\mu\nu} = \partial_\mu \mathcal{A}^i_\nu - \partial_\nu \mathcal{A}^i_\mu$ and the $O(d, d+n)$ metric η, which, when written in blocks, takes the form

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{4.8}$$

With these definitions, $H_{\mu\nu\rho}$ has manifest $O(d, d+n)$ symmetry.

Next, we look at all terms that are quadratic in field strengths F. The contributions to

$$\mathcal{L}_3 = -\frac{1}{4}\ \mathcal{F}^i_{\mu\nu} (M^{-1})_{ij}\ \mathcal{F}^{j\mu\nu} \tag{4.9}$$

come from $S_{\hat{g}}$ (as before), from $\frac{1}{4} F^I_{\mu\nu} F^{I\mu\nu}$, and from $\frac{1}{4} H_{\mu\nu\alpha} H^{\mu\nu\alpha}$. From these we read off the result

$$M^{-1} = \begin{pmatrix} G + C^T G^{-1} C + a^T a & -C^T G^{-1} & C^T G^{-1} a^T + a^T \\ -G^{-1} C & G^{-1} & -G^{-1} a^T \\ a G^{-1} C + a & -a G^{-1} & 1 + a G^{-1} a^T \end{pmatrix} \tag{4.10}$$

To check whether this is an $O(d, d+n)$ matrix we form

$$\eta M^{-1} \eta = \begin{pmatrix} G^{-1} & -G^{-1} C & -G^{-1} a^T \\ -C^T G^{-1} & G + C^T G^{-1} C + a^T a & C^T G^{-1} a^T + a^T \\ -a G^{-1} & a G^{-1} C + a & 1 + a G^{-1} a^T \ . \end{pmatrix} \tag{4.11}$$

Multiplying these, we find that $M^{-1} \eta M^{-1} \eta = 1$. Hence M^{-1} and M are symmetric $O(d, d+n)$ matrices, as expected.

Motivated by the results of section 3, we next seek a matrix V belonging to $O(d,d+n)$ such that $V^TV = \eta M^{-1}\eta = M$. It is very easy at this point to discover that a suitable choice is

$$V = \begin{pmatrix} E^{-1} & -E^{-1}C & -E^{-1}a^T \\ 0 & E & 0 \\ 0 & a & 1 \end{pmatrix}, \tag{4.12}$$

which is remarkably simple.

The last remaining check of $O(d,d+n)$ symmetry is to verify that we recover $\mathcal{L}_2 = \frac{1}{8}\mathrm{tr}(\partial_\mu M^{-1}\partial^\mu M)$, with the matrix M given above. Relevant contributions come from $S_{\hat{g}}$, $-\frac{1}{2}(F^I_{\mu\alpha})^2$, and $-\frac{1}{4}(H_{\mu\alpha\beta})^2$. The calculation is a bit tedious, but the desired result is obtained. Clearly this term can also be understood using an $O(d,d+n)/O(d)\times O(d+n)$ analysis generalizing that presented in section 3. In this case $\mathcal{L}_2$ would be obtained, as before, if one uses the local $O(d)\times O(d+n)$ symmetry to bring an arbitrary $O(d,d+n)$ matrix V to the form given above.

It is natural to inquire what happens if the $(D+d)$-dimensional theory contains a non-Abelian Yang–Mills group. After all, the heterotic string in 10 dimensions can have $O(32)$ or $E_8\times E_8$. In general, compactification with nontrivial moduli breaks these symmetries. The only thing that is easy to do, and which makes contact with Narain's analysis [1], is to set to zero all the gauge fields except those belonging to a Cartan subalgebra – $[U(1)]^{16}$ in the case of the heterotic string. Then the problem reduces to the Abelian theory, and the analysis of this section becomes applicable. In this way one obtains the noncompact group $O(d,d+16)$ considered by Narain.

5. O(d,d) Symmetric World Sheet Equations

In this section, we discuss how $O(d,d)$ symmetry appears in the σ-model description of string theory. The generalization to the $O(d,d+n)$ case will not be presented in detail here, but the result will be stated at the end of the section. First, we review the description of a string in the presence of constant background fields when d coordinates are compactified on a torus, following refs. [16,23,27]. Then we consider the extension to spacetime-dependent background fields, generalizing previous studies in refs. [28,29,30,24,31,32]. Only closed string theories are considered here. For a recent discussion of toroidal compactification of open string theories see [33].

To be specific, let us consider the two-dimensional σ-model description of a bosonic string in a space with d compactified coordinates $Y^\alpha(\sigma,\tau)$. The portion of the action containing these coordinates is

$$S_K = \frac{1}{2}\int d^2\sigma\ \left[G_{\alpha\beta}\eta^{ab}\partial_a Y^\alpha \partial_b Y^\beta + \epsilon^{ab} B_{\alpha\beta}\partial_a Y^\alpha \partial_b Y^\beta\right], \tag{5.1}$$

where $G_{\alpha\beta}$ and $B_{\alpha\beta}$ are constants. The coordinates are taken to satisfy the periodicity conditions $Y^\alpha \simeq Y^\alpha + 2\pi$.⋆ For closed strings it is necessary that

$$Y^\alpha(2\pi,\tau) = Y^\alpha(0,\tau) + 2\pi m^\alpha\,, \tag{5.2}$$

where the integers m^α are called winding numbers. It follows from the single-valuedness of the wave function on the torus that the zero modes of the canonical momentum, $P_\alpha = G_{\alpha\beta}\partial_\tau Y^\beta + B_{\alpha\beta}\partial_\sigma Y^\beta$, are also integers n_α. Therefore the zero modes of Y^α are given by

$$Y_0^\alpha = y^\alpha + m^\alpha\sigma + G^{\alpha\beta}(n_\beta - B_{\beta\gamma}n^\gamma)\tau\,, \tag{5.3}$$

⋆ We apologize for switching conventions from section 2, where the y^α's were taken to have unit periods.

where $G^{\alpha\beta}$ is the inverse of $G_{\alpha\beta}$ as before. The Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2} G_{\alpha\beta} (\dot{Y}^\alpha \dot{Y}^\beta + Y'^\alpha Y'^\beta) \,, \tag{5.4}$$

where $\dot{Y}^\alpha$ and Y'^β are derivatives with respect to τ and σ, respectively.

Since $Y^\alpha(\sigma, \tau)$ satisfies the free wave equation, we can decompose it as the sum of left- and right-moving pieces. The zero mode of $P^\alpha = G^{\alpha\beta} P_\beta$ is given by $p_L^\alpha + p_R^\alpha$ where

$$p_L^\alpha = \frac{1}{2}[m^\alpha + G^{\alpha\beta}(n_\beta - B_{\beta\gamma} m^\gamma)] \quad \text{and} \quad p_R^\alpha = \frac{1}{2}[-m^\alpha + G^{\alpha\beta}(n_\beta - B_{\beta\gamma} m^\gamma)] \tag{5.5}$$

The mass-squared operator, which corresponds to the zero mode of $\mathcal{H}$, is given (aside from a constant) by

$$(mass)^2 = G_{\alpha\beta} \big(p_L^\alpha p_L^\beta + p_R^\alpha p_R^\beta \big) + \sum_{m=1}^{\infty} \sum_{i=1}^{d} (\alpha_{-m}^i \alpha_m^i + \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i) \,, \tag{5.6}$$

As usual, $\{\alpha_m\}$ and $\{\tilde{\alpha}_m\}$ denote oscillators associated with right- and left-moving coordinates, respectively. Substituting the expressions for p_L and p_R, the mass squared can be rewritten as

$$(mass)^2 = \frac{1}{2} G_{\alpha\beta} m^\alpha m^\beta + \frac{1}{2} G^{\alpha\beta} (n_\alpha - B_{\alpha\gamma} m^\gamma)(n_\beta - B_{\beta\delta} m^\delta) + \sum (\alpha_{-m}^i \alpha_m^i + \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i) \,. \tag{5.7}$$

It is significant that the zero mode portion of eq. (5.7) can be expressed in the form

$$(M_0)^2 = \frac{1}{2} (m \;\; n) M^{-1} \begin{pmatrix} m \\ n \end{pmatrix} , \tag{5.8}$$

where M is the $2d \times 2d$ matrix introduced in section 2, which we display once again:

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{5.9}$$

In order to satisfy σ-translation symmetry, the contributions of left- and right-moving sectors to the mass squared must agree ($L_0 = \tilde{L}_0$ in the usual notation). The zero mode contribution to their difference is

$$G_{\alpha\beta}(p_L^\alpha p_L^\beta - p_R^\alpha p_R^\beta) = m^\alpha n_\alpha \,. \tag{5.10}$$

Since this is an integer, it always can be compensated by oscillator contributions, which are also integers.

Equation (5.10) is invariant under interchange of the winding numbers m^α and the discrete momenta n_α. Indeed, the entire spectrum remains invariant if we interchange $m^\alpha \leftrightarrow n_\alpha$ and simultaneously let [23]

$$(G - BG^{-1}B) \leftrightarrow G^{-1} \quad \text{and} \quad BG^{-1} \leftrightarrow -G^{-1}B \,. \tag{5.11}$$

These interchanges precisely correspond to inverting the $2d \times 2d$ matrix M. This is the spacetime duality transformation generalizing the well-known duality $R \leftrightarrow \alpha'/R$ in the $d = 1$ case [34,35,36,37]. The general duality symmetry implies that the $2d$-dimensional Lorentzian lattice spanned by the vectors $\sqrt{2}(p_L^\alpha,\, p_R^\alpha)$ with inner product

$$\sqrt{2}\;(p_L,\, p_R)\cdot\sqrt{2}\;(p'_L,\, p'_R) \equiv 2G_{\alpha\beta}(p_L^\alpha p_L'^\beta - p_R^\alpha p_R'^\beta) = (m^\alpha n'_\alpha + m'^\alpha n_\alpha)\,, \tag{5.12}$$

is even and self-dual [1].

The moduli space parametrized by $G_{\alpha\beta}$ and $B_{\alpha\beta}$ is locally the coset $O(d,d)/O(d)\times O(d)$ [16], just as we found in section 3. The global geometry requires also modding out the group of discrete symmetries generated by $B_{\alpha\beta} \to B_{\alpha\beta} + N_{\alpha\beta}$ and $G + B \to (G+B)^{-1}$. These symmetries generate the $O(d,d,Z)$ subgroup of $O(d,d)$. An $O(d,d,Z)$ transformation is given by a $2d \times 2d$ matrix A having integral entries and satisfying $A^T\eta A = \eta$, where η consists of off-diagonal unit matrices as before. Under an $O(d,d,Z)$ transformation

$$\begin{pmatrix} m \\ n \end{pmatrix} \to \begin{pmatrix} m' \\ n' \end{pmatrix} = A \begin{pmatrix} m \\ n \end{pmatrix} \quad \text{and} \quad M \to AMA^T \,. \tag{5.13}$$

It is evident that

$$m \cdot n = \frac{1}{2}(m \;\; n)\eta \begin{pmatrix} m \\ n \end{pmatrix} , \tag{5.14}$$

which appears in eq. (5.10), and M_0^2 in eq. (5.8) are preserved under these transformations. The crucial fact, already evident from the spectrum, is that toroidally compactified string theory certainly does not share the full $O(d,d)$ symmetry of the low energy effective theory. It is at most invariant under the discrete $O(d,d,Z)$ subgroup. However, as emphasized by Sen [15], if the Y coordinates are *not* compactified, but still flat, so that $K = R^d$, there is a continuous $O(d) \times O(d)$ symmetry (the compact part of $O(d,d)$) corresponding to independent rotations of Y_L and Y_R. The diagonal subgroup describes ordinary rotations of K.

Now we turn our attention to the case when the $(D+d)$-dimensional massless background fields $\hat{g}_{\hat{\mu}\hat{\nu}}$ and $\hat{B}_{\hat{\mu}\hat{\nu}}$ depend on D coordinates. The $D+d$ string coordinates $X^{\hat{\mu}}$ decompose into two sets $\{X^\mu\}$ and $\{Y^\alpha\}$ where $\mu = 0, 1, \ldots, D-1$ and $\alpha = 1, 2, \ldots, d$. The world sheet action is

$$S = \frac{1}{2}\int d^2\sigma (\hat{g}_{\hat{\mu}\hat{\nu}}\eta^{ab} + \hat{B}_{\hat{\mu}\hat{\nu}}\epsilon^{ab})\partial_a X^{\hat{\mu}}\partial_b X^{\hat{\nu}} . \tag{5.15}$$

Varying this with respect to $X^{\hat{\mu}}(\sigma,\tau)$ gives the classical equation of motion for the string

$$\begin{aligned} \frac{\delta S}{\delta X^{\hat{\mu}}} = & -\hat{\Gamma}_{\hat{\mu}\hat{\nu}\hat{\rho}}\partial^a X^{\hat{\nu}}\partial_a X^{\hat{\rho}} - \hat{g}_{\hat{\mu}\hat{\nu}}\partial^a\partial_a X^{\hat{\nu}} \\ & + \frac{1}{2}\epsilon^{ab}(\partial_{\hat{\mu}}\hat{B}_{\hat{\nu}\hat{\rho}} + \partial_{\hat{\nu}}\hat{B}_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\rho}}\hat{B}_{\hat{\mu}\hat{\nu}})\partial_a X^{\hat{\nu}}\partial_b X^{\hat{\rho}} = 0 , \end{aligned} \tag{5.16}$$

where

$$\hat{\Gamma}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \frac{1}{2}(\partial_{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\rho}} + \partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}} - \partial_{\hat{\mu}}\hat{g}_{\hat{\nu}\hat{\rho}}) . \tag{5.17}$$

To analyze these equations it is convenient to consider X^μ and Y^α separately. Since the Y^α equation is somewhat simpler we begin with that. Indeed for that case, let us back up and focus on those terms in S that are Y dependent. These are

$$S_Y = \int d^2\sigma \left\{ \frac{1}{2} \big(\eta^{ab} G_{\alpha\beta}(X) \partial_a Y^\alpha \partial_b Y^\beta + \epsilon^{ab} B_{\alpha\beta}(X) \partial_a Y^\alpha \partial_b Y^\beta \big) + \Gamma^a_\alpha(X) \partial_a Y^\alpha \right\}, \tag{5.18}$$

where

$$\begin{aligned} \Gamma^a_\alpha &\equiv \eta^{ab} \hat{g}_{\mu\alpha} \partial_b X^\mu - \epsilon^{ab} \hat{B}_{\mu\alpha} \partial_b X^\mu \\ &= \eta^{ab} G_{\alpha\beta} A^{(1)\beta}_\mu \partial_b X^\mu - \epsilon^{ab} \big(A^{(2)}_{\mu\alpha} - B_{\alpha\beta} A^{(1)\beta}_\mu \big) \partial_b X^\mu \end{aligned} \tag{5.19}$$

encodes information about the gauge fields $A^{(1)\alpha}_\mu$ and $A^{(2)}_{\mu\alpha}$. This action generalizes eq. (5.1), both by including background vector fields and by allowing X dependence for all the background fields.

The goal now is to study the resulting Y equations of motion, to recast them into a form with manifest $O(d,d)$ symmetry, and to understand why the symmetry breaks to $O(d,d,Z)$. The $O(d,d)$ symmetry cannot be explicitly realized on the action. Rather, it is necessary to combine the equations of motion for Y with those for dual coordinates $\tilde{Y}$, in order to make the symmetry manifest [29,24].* In the absence of nontrivial backgrounds, Y and $\tilde{Y}$ would correspond to the sum and difference of left-moving and right-moving components. In more general settings, the interpretation is not quite so simple. It has been suggested on occasion [39,24,40] that this doubling of coordinates has some deep significance. However one feels about that, the mathematics is indisputable.

Since the backgrounds are independent of Y^α, the Euler–Lagrange equations take the form

$$\partial_a \left(\frac{\delta S}{\delta \partial_a Y^\alpha} \right) = 0\,. \tag{5.20}$$

$\star$ Previous studies of four-dimensional examples illustrate that, when duality transformations are required, the equations of motion can be made manifestly invariant under the "hidden" symmetries even though the action cannot be [38]. Actually, hidden symmetries sometimes can be made manifest in the action it one is willing to give up some other symmetry. For example, in ref. [40], a duality invariant action that does not have manifest world sheet Lorentz invariance is formulated.

Therefore, locally, we can write

$$\frac{\delta S}{\delta \partial_a Y^\alpha} = \eta^{ab}\partial_b Y^\beta G_{\alpha\beta} + \epsilon^{ab}\partial_b Y^\beta B_{\alpha\beta} + \Gamma^a_\alpha = \epsilon^{ab}\partial_b \tilde{Y}_\alpha \ , \tag{5.21}$$

where $\tilde{Y}_\alpha$ are the dual coordinates. They clearly have the same periodicities as the Y^α. Introducing auxiliary fields U^α_a, let us now define a dual action

$$\tilde{S} = \int d^2\sigma \left\{ \frac{1}{2}\left(\eta^{ab}U^\alpha_a U^\beta_b G_{\alpha\beta} + \epsilon^{ab}U^\alpha_a U^\beta_b B_{\alpha\beta}\right) + \epsilon^{ab}\partial_a \tilde{Y}_\alpha U^\alpha_b + \Gamma^a_\alpha U^\alpha_a \right\}. \tag{5.22}$$

Varying this action with respect to $\tilde{Y}_\alpha$ gives $\partial_a(\epsilon^{ab}U^\alpha_b) = 0$, while the U^α_a equation of motion

$$\eta^{ab}U^\beta_b G_{\alpha\beta} + \epsilon^{ab}U^\beta_b B_{\alpha\beta} - \epsilon^{ab}\partial_b \tilde{Y}_\alpha + \Gamma^a_\alpha = 0 \tag{5.23}$$

agrees with eq. (5.21) when one identifies U^α_a with $\partial_a Y^\alpha$. This can be used to solve for U^α_a in terms of $\partial_a \tilde{Y}_\alpha$ and Γ^a_α. The result is

$$U^\alpha_a = \left(\epsilon_a{}^b \mathcal{G}^{\alpha\beta} + \delta^b_a \mathcal{B}^{\alpha\beta}\right)\left(\partial_b \tilde{Y}_\beta - \epsilon_{bc}\Gamma^c_\beta\right), \tag{5.24}$$

where we have introduced

$$\mathcal{G} = (G - BG^{-1}B)^{-1} \tag{5.25}$$

and

$$\mathcal{B} = -G^{-1}B(G - BG^{-1}B)^{-1}\,. \tag{5.26}$$

Note that $(G+B)(\mathcal{G}+\mathcal{B}) = 1$, so that $\mathcal{G}$ and $\mathcal{B}$ are the symmetric and antisymmetric parts of $(G+B)^{-1}$, respectively.

Substituting for U^α_a, the dual action (5.22) takes the form

$$\begin{aligned} \tilde{S} = \int d^2\sigma \bigg\{ & \frac{1}{2}\left(\eta^{ab}\partial_a \tilde{Y}_\alpha \partial_b \tilde{Y}_\beta \mathcal{G}^{\alpha\beta} + \epsilon^{ab}\partial_a \tilde{Y}_\alpha \partial_b \tilde{Y}_\beta \mathcal{B}^{\alpha\beta}\right) - \epsilon^a{}_b \partial_a \tilde{Y}_\alpha \Gamma^b_\beta \mathcal{G}^{\alpha\beta} \\ & - \partial_a \tilde{Y}_\alpha \Gamma^a_\beta \mathcal{B}^{\alpha\beta} - \frac{1}{2}\left(\eta_{ab}\Gamma^a_\alpha \Gamma^b_\beta \mathcal{G}^{\alpha\beta} + \epsilon_{ab}\Gamma^a_\alpha \Gamma^b_\beta \mathcal{B}^{\alpha\beta}\right) \bigg\}. \end{aligned} \tag{5.27}$$

Since $\mathcal{G}^{\alpha\beta}$ and $\mathcal{B}^{\alpha\beta}$ are determined in terms of $G_{\alpha\beta}$ and $B_{\alpha\beta}$, they depend only on X^μ, as does Γ^a_α. As before, the equation of motion derived from $\tilde{S}$ is $\partial_a\left(\frac{\delta \tilde{S}}{\delta \partial_a \tilde{Y}_\alpha}\right) = 0$.

The two Lagrangians S and $\tilde{S}$ give a pair of equivalent equations of motion (at least locally), which are obtained by applying ∂_a to eq. (5.21) and

$$\epsilon^{ab}\partial_b Y^\alpha = \frac{\delta\tilde{S}}{\delta\partial_a\tilde{Y}_\alpha} = \eta^{ab}\partial_b\tilde{Y}_\beta\mathcal{G}^{\alpha\beta} + \epsilon^{ab}\partial_b\tilde{Y}_\beta\mathcal{B}^{\alpha\beta} - \epsilon^a{}_b\mathcal{G}^{\alpha\beta}\Gamma^b_\beta - \mathcal{B}^{\alpha\beta}\Gamma^a_\beta\,. \tag{5.28}$$

In order to express equations (5.21) and (5.28) in an $O(d,d)$ covariant form, let us multiply them by G^{-1} and $\mathcal{G}^{-1}$, respectively, as well as by ϵ^{ab}, as follows:

$$G^{\alpha\beta}\partial_a\tilde{Y}_\beta - (G^{-1}B)^\alpha{}_\beta\partial_a Y^\beta = \epsilon_a{}^b\partial_b Y^\alpha + \epsilon_{ab}G^{\alpha\beta}\Gamma^b_\beta\ . \tag{5.29}$$

$$(\mathcal{G}^{-1})_{\alpha\beta}\partial_a Y^\beta - (\mathcal{G}^{-1}\mathcal{B})_\alpha{}^\beta\partial_a\tilde{Y}_\beta = \epsilon_a{}^b\partial_b\tilde{Y}_\alpha - \eta_{ab}\Gamma^b_\alpha - \epsilon_{ab}(\mathcal{G}^{-1}\mathcal{B})_\alpha{}^\beta\Gamma^b_\beta\,. \tag{5.30}$$

If we define an enlarged manifold combining the coordinates Y^α and $\tilde{Y}_\alpha$ such that $\{Z^i\} = \{Y^\alpha,\, \tilde{Y}_\alpha\}$, $i = 1, 2, \ldots, 2d$, then eqs. (5.29) and (5.30) can be combined as the single equation

$$M\eta\partial_a Z = \epsilon_a{}^b\partial_b Z + M\eta\Sigma_a\,. \tag{5.31}$$

Here Σ_a is an $O(d,d)$ vector (for each value of a) given by the column vector

$$\Sigma^i_a = \begin{pmatrix} -\eta_{ab}G^{\alpha\beta}\Gamma^b_\beta \\ \epsilon_{ab}\Gamma^b_\alpha - \eta_{ab}B_{\alpha\gamma}G^{\gamma\beta}\Gamma^b_\beta \end{pmatrix}\,. \tag{5.32}$$

Substituting eq. (5.19) into eq. (5.32) gives

$$\Sigma^i_a = -\partial_a X^\mu\mathcal{A}^i_\mu + \epsilon_a{}^b\partial_b X^\mu(M\eta\mathcal{A}_\mu)^i\ , \tag{5.33}$$

where $\mathcal{A}^i_\mu$ is comprised of $A^{(1)\alpha}_\mu$ and $A^{(2)}_{\mu\alpha}$, as in section 2. Inserting this into eq. (5.31) then gives the first-order equation

$$M\eta(\partial_a Z + \mathcal{A}_\mu\partial_a X^\mu) = \epsilon_a{}^b(\partial_b Z + \mathcal{A}_\mu\partial_b X^\mu)\ . \tag{5.34}$$

(This equation appears in ref. [24] for the special case $\mathcal{A}_\mu = 0$.) One can eliminate $\tilde{Y}$, of course, obtaining a second-order equation for Y, but then the noncompact symmetry is no longer evident. This is reminiscent of the issue of making Lorentz invariance manifest for the Dirac equation. Unlike that case, there is no obvious action principle that gives the desired first-order equation for the Z coordinates. In terms of light-cone components on the world sheet, (5.34) is equivalent to the pair of equations

$$\begin{aligned}(1+M\eta)(\partial_+ Z + \mathcal{A}_\mu \partial_+ X^\mu) &= 0 \\ (1-M\eta)(\partial_- Z + \mathcal{A}_\mu \partial_- X^\mu) &= 0\ .\end{aligned} \tag{5.35}$$

These equations have nontrivial solutions, since $(M\eta)^2 = 1$. Furthermore, they have manifest $O(d,d)$ invariance provided the transformation rules $M \to \Omega M \Omega^T$ and $\mathcal{A}_\mu \to \Omega \mathcal{A}_\mu$, obtained in section 2, are supplemented with $Z \to \Omega Z$.

Using the identity $\eta V \eta V^T = 1$, and recalling that $M = V^T V$, we can rewrite (5.35) in the form

$$(\eta \pm 1) V \eta (\partial_\pm Z + \mathcal{A}_\mu \partial_\pm X^\mu) = 0\ . \tag{5.36}$$

Written this way, it is clear that the plus and minus cases each consist of d linearly independent equations. Defining

$$(D_a Z)^i = \partial_a Z^i + \mathcal{A}^i_\mu \partial_a X^\mu\ , \tag{5.37}$$

the component equations for Y and $\tilde{Y}$ are

$$\begin{aligned}(G-B) D_+ Y + D_+ \tilde{Y} &= 0 \\ (G+B) D_- Y - D_- \tilde{Y} &= 0\ ,\end{aligned} \tag{5.38}$$

which is quite a bit simpler than the second order equation for Y that we started from. Even though these equations have continuous $O(d,d)$ invariance, the symmetry is broken to the discrete subgroup $O(d,d,Z)$ by the boundary conditions $Y^\alpha \simeq Y^\alpha + 2\pi$ and $\tilde{Y}_\alpha \simeq \tilde{Y}_\alpha + 2\pi$. The fundamental point is that all geometries related by $O(d,d,Z)$ transformations correspond to the same conformal field theory and are physically equivalent. The moduli space of conformally inequivalent (and hence physically inequivalent) classical solutions is given by the coset space $O(d,d)/O(d)\times O(d)\times O(d,d,Z)$ and is parametrized locally by the scalar fields $G_{\alpha\beta}$ and $B_{\alpha\beta}$.

The combination $D_a Z^i = \partial_a Z^i + \mathcal{A}^i_\mu \partial_a X^\mu$, which appears above, can be given a covariant interpretation under gauge transformations. For this purpose it is necessary to redefine the internal coordinates Y^α and $\tilde{Y}_a$ in an X^μ dependent way. Namely, a gauge transformation $\delta\mathcal{A}^i_\mu(X) = \partial_\mu \Lambda^i(X)$ should be accompanied by $\delta Z^i = -\Lambda^i(X)$. Despite superficial appearances, this does not allow the internal coordinates to be eliminated as part of a gauge choice. In particular, the winding numbers m^α and discrete moment n_α are encoded in $Y^\alpha(2\pi,\tau) = Y^\alpha(0,\tau) + 2\pi m^\alpha$ and $\tilde{Y}_\alpha(2\pi,\tau) = \tilde{Y}_\alpha(0,\tau) + 2\pi n_\alpha$. They cannot be changed by a gauge transformation, since $X^\mu(2\pi,\tau) = X^\mu(0,\tau)$.

Let us turn now to the X^μ equation of motion. This requires considering eq. (5.16) for the case of $\hat{\mu} = \mu$ and substituting the various definitions given in section 2. After a certain amount of algebra one finds, separating different powers of Y, that

$$\frac{\delta S}{\delta X^\mu} = E^2_\mu + E^1_\mu + E^0_\mu \ , \tag{5.39}$$

where

$$E^2_\mu = \partial_\mu (G+B)_{\alpha\beta}\partial_+ Y^\alpha \partial_- Y^\beta \tag{5.40}$$

$$\begin{aligned} E^1_\mu = &- (A^{(1)}_\mu G)_\alpha \partial^a \partial_a Y^\alpha + \epsilon^{ab}\partial_a X^\nu F^{(2)}_{\mu\nu\alpha}\partial_b Y^\alpha \\ &+ (\partial_\mu [A^{(1)}_\nu (G+B)]_\alpha - (\mu\nu))\partial_+ X^\nu \partial_- Y^\alpha \\ &+ (\partial_\mu [A^{(1)}_\nu (G-B)]_\alpha - (\mu\nu))\partial_- X^\nu \partial_+ Y^\alpha \end{aligned} \tag{5.41}$$

$$\begin{aligned} E^0_\mu = & -\hat{\Gamma}_{\mu\nu\rho}\partial^a X^\nu \partial_a X^\rho - \hat{g}_{\mu\nu}\partial^a \partial_a X^\nu \\ &+ \frac{1}{2}\epsilon^{ab}(\partial_\mu \hat{B}_{\nu\rho} + \partial_\nu \hat{B}_{\rho\mu} + \partial_\rho \hat{B}_{\mu\nu})\partial_a X^\nu \partial_b X^\rho \ . \end{aligned} \tag{5.42}$$

In the expression for E^0_μ one must still substitute (see section 2)

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + A^{(1)}_\mu G A^{(1)}_\nu \tag{5.43}$$

and

$$\hat{B}_{\mu\nu} = B_{\mu\nu} - \frac{1}{2} A^{(1)}_\mu A^{(2)}_\nu + \frac{1}{2} A^{(1)}_\nu A^{(2)}_\mu + A^{(1)}_\mu B A^{(1)}_\nu \ . \tag{5.44}$$

Now we must try to reexpress all this in an $O(d,d)$ invariant form. As a first step consider the manifestly $O(d,d)$ invariant expression

$$F^2_\mu = \frac{1}{2} D_+ Z^i \big(\partial_\mu M^{-1}\big)_{ij} D_- Z^j \ . \tag{5.45}$$

Inserting the matrix M^{-1} and expanding out the terms, one can show (by using the equations of motion (5.38)) that

$$F^2_\mu = D_+ Y \ \partial_\mu (G+B) \ D_- Y \ . \tag{5.46}$$

Therefore, comparing with eq. (5.40), we see that F^2_μ is an $O(d,d)$ invariant term containing E^2_μ. To proceed we must compensate for the terms linear in Y and independent of Y in F^2_μ as additions $E^{1'}_\mu$ and $E^{0'}_\mu$ to E^1_μ and E^0_μ. The difference of eqs. (5.46) and (5.40) gives

$$\begin{aligned} E^{1'}_\mu = & - \partial_+ X^\nu A^{(1)}_\nu \partial_\mu (G+B) \ \partial_- Y \\ & - \partial_- X^\nu A^{(1)}_\nu \partial_\mu (G-B) \ \partial_+ Y \end{aligned} \tag{5.47}$$

$$E^{0'}_\mu = -\partial_+ X^\nu A^{(1)}_\nu \partial_\mu (G+B) \ A^{(1)}_\rho \partial_- X^\rho \ . \tag{5.48}$$

Next, we need to find $O(d,d)$ invariant terms that contain $E^1_\mu + E^{1'}_\mu$. Making repeated use of eq. (5.38), we find that these terms are completely contained in the manifestly invariant term

$$F^1_\mu = \epsilon^{ab} \partial_a X^\nu \mathcal{F}_{\mu\nu} \ \eta D_b Z \ . \tag{5.49}$$

Compensating for the additional Y-independent terms that have been introduced gives

$$
\begin{aligned}
E_\mu^{0''} = & -\partial_+ X^\nu \partial_- X^\rho [A_\mu^{(1)} F_{\nu\rho}^{(2)} + A_\rho^{(1)} F_{\mu\nu}^{(2)} + A_\nu^{(1)} F_{\rho\mu}^{(2)} \\
& + A_\rho^{(1)} (G-B) F_{\mu\nu}^{(1)} + A_\nu^{(1)} (G+B) F_{\mu\rho}^{(1)}] \\
& + \partial_+ [A_\nu^{(1)} \partial_- X^\nu (G-B)] A_\mu^{(1)} + \partial_- [A_\nu^{(1)} \partial_+ X^\nu (G+B)] A_\mu^{(1)} \ .
\end{aligned}
\tag{5.50}
$$

To complete this part of the story $E_\mu^0 + E_\mu^{0'} + E_\mu^{0''}$ must still be recast in $O(d,d)$ invariant form. Remarkably, there is a great deal of cancellation and one ends up with

$$
F_\mu^0 = -\Gamma_{\mu\nu\rho} \partial^a X^\nu \partial_a X^\rho - g_{\mu\nu} \partial^a \partial_a X^\nu + \frac{1}{2} \epsilon^{ab} H_{\mu\nu\rho} \partial_a X^\nu \partial_b X^\rho \ , \tag{5.51}
$$

with $H_{\mu\nu\rho}$ as defined in section 2.

To summarize, we have found that the X^μ equation of motion can be written in the manifestly $O(d,d)$ invariant form

$$
\begin{aligned}
& \frac{1}{2} D_+ Z \big(\partial_\mu M^{-1}\big) D_- Z + \epsilon^{ab} \partial_a X^\nu \mathcal{F}_{\mu\nu} \ \eta D_b Z \\
& - \Gamma_{\mu\nu\rho} \partial^a X^\nu \partial_a X^\rho - g_{\mu\nu} \partial^a \partial_a X^\nu + \frac{1}{2} \epsilon^{ab} H_{\mu\nu\rho} \partial_a X^\nu \partial_b X^\rho = 0 \ .
\end{aligned}
\tag{5.52}
$$

Together with eq. (5.36) or eq. (5.38) this gives the classical dynamics of strings moving in an arbitrary X-dependent background. The equations are remarkably simple considering all the information they encode. Clearly, $O(d,d)$ is a useful guide for making them intelligible.

It should come as no surprise to the reader to learn that eqs. (5.36) and (5.52) continue to hold for the $O(d,d+n)$ generalization, provided that M, η, and $\mathcal{A}_\mu^i$ are defined as in section 4. Also, Z^i now becomes a $(2d+n)$-component vector made by combining $Y^\alpha, \tilde{Y}_\alpha$, and Y^I, where Y^I are n additional internal coordinates. It is natural to require that

$$
\partial_- Y^I + A_\mu^{(3)I} \partial_- X^\mu = 0 \ , \tag{5.53}
$$

as a "gauge invariant" generalization of what we know to be true for the heterotic string with vanishing $A_\mu^{(3)I}$ background fields, *viz.* that the Y^I are left-moving. (The second term in eq. (5.53) was omitted in sect. 6 of ref. [25].) Once eq. (5.53) is imposed, the number of unknowns and equations for the Y coordinates matches up properly.

6. **Discussion**

This work has explored the noncompact $O(d,d)$ group that appears in toroidal compactification of oriented closed bosonic strings as well as the $O(d,d+n)$ generalization that is required for the heterotic string. In sections 2 and 4 we showed, using methods of dimensional reduction, that these noncompact groups are exact symmetries of the (classical) low-energy effective field theory that is obtained when one truncates the dependence on the internal coordinates y^α keeping zero modes only.

In section 5 we explored noncompact symmetries from the world-sheet viewpoint, extending the analysis of previous authors [16,29,24] to a somewhat more general setting. We found that the classical string dynamics that results from toroidal compactification and zero-mode truncation is also described by equations of motion that can be written in a manifestly $O(d,d+n)$-invariant form. Only global boundary conditions break the symmetry to the discrete subgroup $O(d,d+n,Z)$. Therefore the moduli space that arises in toroidal string compactification is given by the $O(d,d+n)$ group manifold modded out by $O(d,d+n,Z)$ as well as by the maximal compact subgroup $O(d)\times O(d+n)$.

Logically, the analysis of section 5 should perhaps come first, since it describes the noncompact symmetry at tree-level of the σ model, *i.e.*, to leading order in the α' expansion. The low-energy effective field theory analysis of sections 2 and 4 corresponds to the requirement of conformal invariance of the sigma model at the one-loop order [41,42]. In particular, at this order the sigma model action must be modified to include a term coupling the dilaton to the world-sheet curvature [41]. We have not investigated the higher-loop corrections, which generate additional higher-dimension terms in the field equations of the massless fields. They could in principle be generated by enforcing conformal symmetry of the world-sheet action to higher orders in α'. It seems very plausible that the noncompact symmetries would continue to hold for them as well. For example, a strong case could probably be made by using formal path-integral manipulations along the lines described by Fradkin and Tseytlin [43]. In fact, some evidence that the $O(d,d)$ symmetry is present at the two-loop order has been presented by Panvel [44], and more general arguments have been advanced in refs. [30,25].

One result that seems interesting to us is that the need for Chern–Simons terms in the $H_{\mu\nu\rho}$ field strength was deduced from purely bosonic considerations. One wonders whether two-loop conformal invariance implies the necessity of Lorentz Chern–Simons terms, again from purely bosonic considerations.

The noncompact symmetries transform the moduli fields in complicated nonlinear ways. In section 4 we reviewed techniques (well-known from previous supergravity studies) for realizing these symmetries linearly. Two distinct constructions to achieve this were presented. The first one was a bit of a surprise, whereas the second was the standard coset construction in which one introduces auxiliary scalar fields to fill out the adjoint representation of the noncompact group and then compensates by introducing a local gauge symmetry corresponding to the maximal compact subgroup. This is implemented using a generalized 'vielbein' formalism, which we saw gives rise to a better understanding of some of the otherwise mysterious matrices that appear.

In the special (but physically interesting) case $D=4$, it is well known that there is an additional $SU(1,1)$ or $SL(2,R)$ symmetry of the low-energy effective field theory. The special feature of four dimensions is that by making a duality transformation it is possible to replace the antisymmetric tensor $B_{\mu\nu}$ by a scalar field, usually called the 'axion'. The axion and dilaton together then magically parametrize the coset space $SL(2,R)/SO(2)$. The full $SL(2,R)$ symmetry in the presence of vector fields $\mathcal{A}_\mu$ cannot be realized on the action, but can be understood in terms of the classical field equations [7]. (This involves duality transformations of the vector fields.) The way this works is rather analogous to the way the $O(d,d+n)$ symmetry is realized on the world sheet. There too the symmetry could only be made manifest for the field equations. Despite these common features, the $SL(2,R)$ symmetry appears to be of a qualitatively different character than the $O(d,d+n)$ symmetry. The evidence for this is that it is apparently impossible to realize it on the classical string equations of section 5. However, this question still deserves further investigation.

Veneziano and collaborators [3] have considered the $O(d,d)$ effective theory with background fields $B_{\alpha\beta}(t)$ and $G_{\alpha\beta}(t)$ depending on "time" only. The action they arrive at is

$$S = \int dte^{-\phi}[\Lambda + (\dot{\phi})^2 + \frac{1}{8}tr(\dot{M}\eta\dot{M}\eta)] \ , \tag{6.1}$$

where Λ is a cosmological constant proportional to $D - D_{crit}$. Solutions to the classical field equations obtained from (6.1) describe spatially homogeneous cosmological models. They exploit the global symmetry of the theory to generate new solutions that would have been difficult to discover by other methods. Transformations involving dimensions that are not compactified should correspond to exact symmetries, even in the string case. However, for uncompactified dimensions, one has to mod out $O(d,d)$ by $GL(d,R)$ and by constant shifts in B. The resulting coset was identified by Sen [15] with $O(d) \times O(d)/O(d)$. Sen has considered more general models of this type in order to obtain new black hole and black string solutions [45]. His techniques appear to be quite powerful.

The emphasis in our work has been to understand the common origin of noncompact groups in string theory and field theory, both as symmetry groups of low energy effective action and for the characterization of string theory moduli spaces. For toroidally compactified dimensions, the only case studied in detail, this has been achieved. Clearly, it would be desirable to explore extensions and generalizations appropriate to other internal spaces K. For example, we know from the work of Seiberg that K3 compactification of the heterotic string should give a six-dimensional theory with a $O(20,4)$ coset structure [46]. (This is remarkably similar to what one gets from T^4 compactification, though the two cases do seem to be somewhat different [47].)

Calabi–Yau spaces are of particular interest in string theory, since in the context of heterotic string compactification they can lead to many realistic features [48]. In addition to the $SU(1,1)/U(1)$ associated with the axion-dilaton system, the moduli space of the heterotic string compactified on a Calabi–Yau manifold⋆ consists of two factors, $\mathcal{M}_{11} \times \mathcal{M}_{21}$, where $\mathcal{M}_{11}$ corresponds to Kähler form deformations and has complex dimension h_{11}, while $\mathcal{M}_{21}$ describes complex structure deformations and has complex dimension h_{21}. (This factorization was established in refs. [50,51].) The integers h_{11} and h_{21} are Hodge numbers of the Calabi–Yau space. In general, each factor should have a discrete symmetry group analogous to the $O(d, d+n, Z)$ of toroidal compactification. It is quite difficult to compute the groups for specific examples, but it is known that they must be subgroups of $Sp(2b_{11}+2, Z)$ and $Sp(2b_{21}+2, Z)$, respectively. (One specific CY example has been worked out in detail in ref. [52]. An orbifold example is given in ref. [53].)

The spaces $\mathcal{M}_{11}$ and $\mathcal{M}_{21}$ are themselves Kähler manifolds of a special type for which the Kähler potential can be derived from a holomorphic prepotential [54]. Homogeneous spaces of this type have been classified. Presumably, at least in certain cases, Calabi–Yau moduli spaces are given by such homogeneous spaces modded out by the discrete group. Whether or not this is the general case (we do not know), it may be interesting to try to classify Calabi–Yau spaces whose moduli spaces are of this type. For this class, the techniques described in this paper for tori should have the most straightforward generalizations.

In conclusion, there is much more still to be learned by pursuing the study of noncompact groups of the type described here. In string theory they are broken to discrete subgroups. These subgroups are, in fact, "discrete gauge symmetries," [55] which means that they should be quite robust, surviving the plethora of phenomena that typically break global symmetries. By thinking hard about them, it may be possible to draw some very powerful general conclusions about compactified dimensions, as well as the implications for physical four-dimensional spacetime.

⋆ For a description of the geometry of Calabi–Yau moduli space see ref. [49] and references therein. This subject has been very active in recent years, and we will not attempt to give a complete set of references here.

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