Dynamic and Static Excitations of a Classical Discrete Anisotropic Heisenberg Ferromagnetic Spin Chain

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Abstract

Using Jacobi elliptic function addition formulas and summation identities we obtain several static and moving periodic soliton solutions of a classical anisotropic, discrete Heisenberg spin chain with and without an external magnetic field. We predict the dispersion relations of these nonlinear excitations and contrast them with that of magnons and relate these findings to the materials realized by a discrete spin chain. As limiting cases, we discuss different forms of domain wall structures and their properties.

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I. INTRODUCTION

Establishing integrability and obtaining exact solutions of discrete nonlinear physical systems are important issues of current interest. Starting with the integrable discrete model of Ablowitz and Ladik [1], for several other discrete nonlinear evolution equations exact elliptic function and soliton solutions have been obtained in recent years [2]. These include certain discrete versions of nonlinear Schrödinger (NLS) equation [3, 4], ϕ^4 equation [5, 6], derivative NLS equation [7], coupled ϕ^4 equation [8], coupled asymmetric double well and coupled ϕ^6 equation [9], complex modified Korteweg-de Vries equation [10], etc., where effective use of summation relations of Jacobian elliptic functions was made and periodic and solitary wave solutions of moving and static types obtained.

In this connection, a physically important discrete nonlinear dynamical system which has been of considerable interest in diverse areas of physics for a long time is the anisotropic Heisenberg ferromagnetic (and antiferromagnetic) spin system with or without an external magnetic field. It has been studied for various aspects in magnetism, condensed matter physics/materials science, statistical physics, nonlinear dynamics, etc. both from classical and quantum points of view [11]. For example, the one-dimensional quantum spin-1/2 XYZ chain has been shown to be an exactly solvable system either through Bethe ansatz procedure or through quantum inverse scattering method [12, 13] and the eigenvalue spectrum and eigenfunctions have been obtained. For large value of spins, however, a classical/quasiclassical description has been known to be an adequate description so that spins can be treated as unit vectors and classical equations of motion for the spin vectors can be obtained as limiting forms of the quantum equation of motion or as dynamical equations derived from postulated spin Poisson bracket relations [14]. Another area of considerable physical interest in which such classical anisotropic spin systems have been studied in the presence of Gilbert damping is the microscopic behavior of spin waves in magnetic bodies of arbitrary shape [15] and the study of spin-torque effect in ferromagnetic layers with spin currents [16] on spin waves and domain walls. Recently it has also been pointed out that discrete breathers can exist in anisotropic spin chains with additional onsite anisotropy [17]. In any case, the resultant equations of motion describe an extremely interesting class of discrete nonlinear dynamical systems and exploration of the underlying dynamical properties is of interest both from theoretical and applied physics points of view.

The one-dimensional Heisenberg ferromagnetic spin system with nearest neighbor exchange interaction has been shown to possess several completely integrable soliton bearing systems in its *continuum limit*: (i) the pure isotropic case [18, 19], (ii) the uniaxial anisotropic case [20], and (iii) the biaxial anisotropic case [21]. These systems also have a strong connection with the nonlinear Schrödinger equation [22]. However, till date no exactly integrable *discrete* dynamical Heisenberg spin system has been identified in the literature, although a variant of the system, namely the Ishimori spin chain, is known to be completely integrable [23]. It is generally expected that the discrete anisotropic Heisenberg spin chain is a nonintegrable nonlinear dynamical system. Yet, as we show in this article, a number of interesting exact periodic and stationary structures, including domain wall type structures, for the fully anisotropic system (XYZ case as well as the limitng XYY and planar XY cases) can be obtained and their properties analyzed using standard techniques. In fact, Roberts and Thompson [24] and Granovskii and Zhedanov in a series of papers [25, 26, 27], have obtained special classes of solutions for the anisotropic spin system. In particular, the latter authors have shown that the time-independent case of the XYZ anisotropic spin chain is an integrable map by relating it to a Neumann type discrete system [26], see also Ref. [28].

In this paper, by parameterizing the unit spin vector in terms of the basic Lamé polynomials of lower order [29] or their derivatives and by a judicious use of various addition theorems and summation relations obeyed by Jacobi elliptic functions [30] we point out that several classes of explicit dynamical and static structures can be obtained. In the limiting cases we obtain linear spin wave solutions and different nonlinear domain wall type solutions in a natural way. We study the physical implications of these solutions like the energy spectrum, effect of discreteness such as the Peierls-Nabarro barrier [31, 32, 33], linear stability and so on.

The plan of the paper is as follows. In Sec. II, we introduce the dynamical equations of motion and introduce certain natural parametrizations of the unit spin vector. In Sec. III, we obtain two classes of periodic solutions, investigate the associated dispersion relations and energy expressions and indicate a semiclassical quantization of these solutions. In Sec. IV we report various classes of static solutions for the XYZ, XYY and XY planar models. In Sec. V, we obtain the total energy expressions associated with the various static solutions and discuss the effect of discreteness including the Peierls-Nabarro potential barrier. In Sec. VI, the isotropic case is considered, while in Sec. VII the linear stability of both time periodic and static solutions is investigated. Then in Sec. VIII, we present some explicit time dependent solutions for the case when the onsite anisotropy or an external magnetic field is introduced. Finally, in Sec. IX we summarize our results. In the Appendix A we include some of the relevant addition theorems and summation relations obeyed by the Jacobi elliptic functions required for our analysis, while in Appendix B some details on semiclassical quantization are given.

II. THE HEISENBERG ANISOTROPIC SPIN CHAIN

A. Equation of Motion

We consider a one dimensional anisotropic Heisenberg ferromagnetic spin chain with the spin components $\vec{S_n} = (S_n^x, S_n^y, S_n^z)$, satisfying the constraint of unit length

$$(S_n^x)^2 + (S_n^y)^2 + (S_n^z)^2 = 1, \tag{1}$$

modeled by the Hamiltonian

$$H = -\sum_{\{n\}}(AS_n^x S_{n+1}^x + BS_n^y S_{n+1}^y + CS_n^z S_{n+1}^z) - D\sum_n (S_n^z)^2 - \vec{\mathcal{H}} \cdot \sum_n \vec{S}_n, \tag{2}$$

where the sum is over the nearest neighbors, A, B and C are the (exchange) anisotropy parameters, D is the onsite anisotropy parameter and $\vec{\mathcal{H}} = (\mathcal{H}, 0, 0)$ is the external magnetic field along the x direction (for convenience). For the XYZ model, $A \neq B \neq C, D = 0$ and for the XY model $C = 0$ and $D = 0$. Using the spin Poisson bracket relation [14]

$$\{S_i^\alpha, S_j^\beta\}_{PB} = \delta_{ij}\epsilon_{\alpha\beta\gamma}S_j^\gamma, \quad \alpha,\ \beta,\ \gamma = 1, 2, 3, \tag{3}$$

where δ_{ij} is the Kronecker delta and $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor, for any two functions $\mathcal{A}$ and $\mathcal{B}$ of spins one has

$$\{\mathcal{A}, \mathcal{B}\}_{PB} = \sum_{\alpha,\beta,\gamma}\sum_{i=1}^{N} \epsilon_{\alpha\beta\gamma}\frac{\partial\mathcal{A}}{\partial S_i^\alpha}\frac{\partial\mathcal{B}}{\partial S_i^\beta}S_i^\gamma, \tag{4}$$

and the equation of motion becomes

$$\begin{aligned}\frac{d\vec{S}_n}{dt} = \vec{S}_n \times [A(S_{n+1}^x + S_{n-1}^x)\vec{i} + B(S_{n+1}^y + S_{n-1}^y)\vec{j} + C(S_{n+1}^z + S_{n-1}^z)\vec{k} \\ +2DS_n^z\vec{k}] + \vec{S}_n \times \vec{\mathcal{H}}, \quad n = 1, 2, ..., N,\end{aligned} \tag{5}$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ form a triad of Cartesian unit vectors. Explicitly, in component form the above equation reads

$$\frac{dS_n^x}{dt} = CS_n^y(S_{n+1}^z + S_{n-1}^z) - BS_n^z(S_{n+1}^y + S_{n-1}^y) - 2DS_n^y S_n^z, \tag{6}$$

$$\frac{dS_n^y}{dt} = AS_n^z(S_{n+1}^x + S_{n-1}^x) - CS_n^x(S_{n+1}^z + S_{n-1}^z) + 2DS_n^x S_n^z + \mathcal{H}S_n^z, \tag{7}$$

$$\frac{dS_n^z}{dt} = BS_n^x(S_{n+1}^y + S_{n-1}^y) - AS_n^y(S_{n+1}^x + S_{n-1}^x) - \mathcal{H}S_n^y. \tag{8}$$

Equations (5) or (6)-(8) can also be obtained as the limiting case of the corresponding quantum dynamical equation of motion for the spin operators when $\hbar \to 0$ or $S \to \infty$. In either case, the dynamics is obtained by solving the initial value problem of the system of coupled nonlinear ordinary differential equations (5) or (6)-(8) along with the constraint (1) on the spin vectors, subject to appropriate boundary conditions like $S_n \to_{n\to\infty} (\pm 1, 0, 0)$ or $S_n \to_{n\to\infty} (0, 0, \pm 1)$. However, it appears that the system of differential equations (6)-(8) is in general nonintegrable. Even then one can obtain several special classes of solutions of physical interest by making use of the properties of (Jacobian) elliptic functions and parametrizing the spin vector to satisfy the unit length condition (1). As noted in the Introduction, some of these solutions were reported earlier by Roberts and Thompson [24], and by Granovskii and Zhedanov [27], which are to be discussed in the following sections; however, as we point out in this paper a much larger class of explicit exact solutions can be found in a rather transparent manner through appropriate parametrizations of the spin vectors.

Before dwelling upon the discrete chain, it is also of interest to note as pointed out in the Introduction that the long wavelength/low temperature continuum limit of Eqs. (6)-(8), when the lattice parameter $a \to 0$, takes the form (in the $D = 0$ limit)

$$\frac{\partial \vec{S}(x,t)}{\partial t} = \vec{S} \times \vec{J}\,\frac{\partial^2 \vec{S}}{\partial x^2} + \vec{S} \times \vec{\mathcal{H}}, \tag{9}$$

where $\vec{\mathcal{H}} = (\mathcal{H}, 0, 0)$, $\vec{J}\vec{S} = AS_x\vec{i} + BS_y\vec{j} + CS_z\vec{k}$, $S_x^2 + S_y^2 + S_z^2 = 1$ and A, B, and C are the anisotropy parameters. The isotropic case $A = B = C$ is a completely integrable soliton system and is equivalent to a nonlinear Schrödinger equation in a geometrical [18] and gauge equivalence sense [19]. So are the uniaxial anisotropic spin chain ($A = B \neq C$) in the presence of a longitudinal magnetic field [20] and the biaxial anisotropic spin chain without the magnetic field [21] integrable soliton systems. In spite of the existence of these integrable continuum spin systems, the discrete chain (5) remains as a rather difficult problem to analyze.

B. The Parametrization of the Unit Spin Vector

One way to proceed with the analysis is to start with an appropriate parametrization of the unit sphere of spin given by Eq. (1). Obviously natural parametrizations are in terms of elliptic functions. For this purpose one can start with the eigenfunctions of the Lamé equation

$$\frac{d^2\psi(u)}{du^2} + [E - n(n+1)k^2\mathrm{sn}^2(u,k)]\psi(u) = 0 \tag{10}$$

for positive integer n, which are given in terms of Lamé polynomials. The lowest order $(n = 1)$ polynomials are [29]

$$\psi_{11} \propto \mathrm{sn}(u,k), \quad \psi_{12} \propto \mathrm{cn}(u,k), \quad \psi_{13} \propto \mathrm{dn}(u,k), \tag{11}$$

while the next order ones are $(n = 2)$

$$\psi_{21} \propto \mathrm{sn}(u,k)\mathrm{cn}(u,k), \quad \psi_{22} \propto \mathrm{cn}(u,k)\mathrm{dn}(u,k), \quad \psi_{23} \propto \mathrm{sn}(u,k)\mathrm{dn}(u,k), \tag{12}$$

and so on. Here $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ are the standard Jacobian elliptic functions [30] characterized by the modulus parameter k [see also Appendix A for the relevant properties of the Jacobian elliptic functions]. Consequently we can choose, for example, an appropriate set of parametrization for the unit spin vectors as

$$S_n^x = \alpha\ \mathrm{sn}(u,k), \quad S_n^y = \beta\ \mathrm{cn}(u,k), \quad S_n^z = \gamma\ \mathrm{dn}(u,k). \tag{13}$$

where α, β and γ are constant parameters to be fixed. The requirement that condition (1) should be satisfied requires

$$\alpha^2 = 1 - \gamma^2 + \gamma^2 k^2 = 1 - \gamma^2 k'^2, \quad \beta^2 = 1 - \gamma^2, \tag{14}$$

where γ is a free parameter $(0 \leq \gamma \leq 1)$ and k is the modulus parameter and $k' = \sqrt{1-k^2}$ is the complementary modulus. One can easily check that a parametrization

$S_n^x = \alpha\mathrm{sn}(u,k)\mathrm{cn}(u,k)$, $S_n^y = \beta\mathrm{cn}(u,k)\mathrm{dn}(u,k)$ and $S_n^z = \gamma\mathrm{sn}(u,k)\mathrm{dn}(u,k)$ does not satisfy the condition (1) for any set of real values of α, β and γ. So one can proceed to higher order Lamé polynomials [29] for other possible parametrizations.

One can even proceed with more general parametrizations in terms of two variables such as

$$S_n^x = \mathrm{cn}(u,k_1), \quad S_n^y = \mathrm{sn}(u,k_1)\mathrm{cn}(v,k_2), \quad S_n^z = \mathrm{sn}(u,k_1)\mathrm{sn}(v,k_2), \tag{15}$$

with two different moduli k_1 and k_2 or even more general forms such as

$$\begin{aligned} S_n^x &= \frac{\alpha\mathrm{cn}(u,k_1)}{1-\gamma\mathrm{sn}(u,k_1)\mathrm{sn}(v,k_2)}, \quad S_n^y = \frac{\alpha\mathrm{sn}(u,k_1)\mathrm{sn}(v,k_2)}{1-\gamma\mathrm{sn}(u,k_1)\mathrm{sn}(v,k_2)}, \\ S_n^z &= \frac{\mathrm{sn}(u,k_1)\mathrm{sn}(v,k_2)-\gamma}{1-\gamma\mathrm{sn}(u,k_1)\mathrm{sn}(v,k_2)}, \quad \alpha = \sqrt{1-\gamma^2}, \end{aligned} \tag{16}$$

both of which satisfy condition (1). We will also make use of these parametrizations in our analysis.

III. MOVING SOLUTIONS: ANISOTROPIC CASE ($D = 0$, $\vec{\mathcal{H}} = 0$)

We now look for time dependent moving solutions of Eqs. (6) - (8) when the onsite anisotropy and magnetic field are absent ($D = 0$ and $\vec{\mathcal{H}} = 0$) in the form (13) and (14) with the substitution $u = pn - \omega t + \delta$, so that

$$S_n^x = \alpha\ \mathrm{sn}(pn-\omega t+\delta,k), \quad S_n^y = \beta\ \mathrm{cn}(pn-\omega t+\delta,k), \quad S_n^z = \gamma\ \mathrm{dn}(pn-\omega t+\delta,k), \tag{17}$$

along with the relations (14). Here p and ω are the wave vector and angular frequency, respectively, which are to be fixed in conjunction with Eqs. (6) - (8) and δ is a phase constant. On substituting the expressions (13) for the components of the spin vector $\vec{S}_n(t)$ into the Eqs. (6) - (8) and making use of the addition theorems for the Jacobian elliptic functions (see Appendix A), one requires the following conditions to be satisfied:

$$-\omega\alpha = \frac{2\beta\gamma[C\mathrm{dn}(p,k)-B\mathrm{cn}(p,k)]}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(p,k)}, \tag{18}$$

$$\omega\beta = \frac{2\alpha\gamma\mathrm{dn}(p,k)[A\mathrm{cn}(p,k)-C]}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(p,k)}, \tag{19}$$

$$\omega\gamma k^2 = \frac{2\alpha\beta\mathrm{cn}(p,k)[B-A\mathrm{dn}(p,k)]}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(p,k)}, \quad u = pn-\omega t+\delta. \tag{20}$$

In each of the above expressions, we note that the variable u occurs explicitly on the right hand sides. Consequently the u-dependent terms in the above relations can be avoided iff p and ω are chosen in one of the following three ways:

1. $\omega = 0$, $p \neq 0$,
2. modulus parameter $k = 0$ (linear spin wave solution, see below),
3. $p = 4K(k)$, or $p = 2K(k)$ (or integral multiples of the right hand sides), where $K(k)$ is the complete elliptic integral of the first kind [30].

In the above, case (1), $\omega = 0$, corresponds to the existence of static solutions. These are discussed in detail in sec. IV below.

Case (2), $k = 0$, corresponds to linear spin wave (or magnon) solutions. More details are given in the subsection IIIc below.

For case (3), note that $\mathrm{sn}(2K,k) = 0 = \mathrm{sn}(4K,k)$, $\mathrm{cn}(2K,k) = -1$, $\mathrm{dn}(2K,k) = 1 = \mathrm{cn}(4K,k) = \mathrm{dn}(4K,k)$. Further, since $\frac{\pi}{2} \leq K(k) < \infty$ as $0 \leq k < 1$, p in Eq. (21) is bounded below by π or 2π as the case may be. Correspondingly, we can have two families of periodic solutions, each of which we will consider separately:

$$p = 4K(k), \quad \textit{or} \quad p = 2K(k), \tag{21}$$

A. Spatially homogeneous time-dependent solutions

For the choice $p = 4K(k)$, the conditions (18) - (20) reduce to

$$\omega\alpha = -2\beta\gamma(C - B), \quad \omega\beta = 2\alpha\gamma(A - C), \quad \omega\gamma k^2 = 2\alpha\beta(B - A). \tag{22}$$

Solving (22), we obtain

$$\omega = 2\gamma\sqrt{(B - C)(A - C)}, \quad k^2 = \frac{1 - \gamma^2}{\gamma^2}\frac{(B - A)}{(A - C)}, \quad (B > A > C), \tag{23}$$

where γ is a free parameter ($0 \leq \gamma \leq 1$). The corresponding solutions are

$$S_n^x = \sqrt{1 - \gamma^2 k'^2}\ \mathrm{sn}(4Kn - \omega t + \delta, k) = -\sqrt{1 - \gamma^2 k'^2}\ \mathrm{sn}(\omega t + \delta, k), \tag{24}$$

$$S_n^y = \sqrt{1 - \gamma^2}\ \mathrm{cn}(4Kn - \omega t + \delta, k) = \sqrt{1 - \gamma^2}\ \mathrm{cn}(\omega t + \delta, k), \tag{25}$$

$$S_n^z = \gamma\ \mathrm{dn}(4Kn - \omega t + \delta, k) = \gamma\ \mathrm{dn}(\omega t + \delta, k). \tag{26}$$

The above spatially homogeneous and time periodic solution is nothing but the description of Poinsot's motion of a rigid body pointed out by Roberts and Thompson [24]. Each of the spins in the lattice precesses about one of the axes with nutation in the same manner. This is depicted schematically in Fig. 1, where all the spins precess parallel to each other. Note that the axes stand for the three spin components S^x, S^y and S^z.

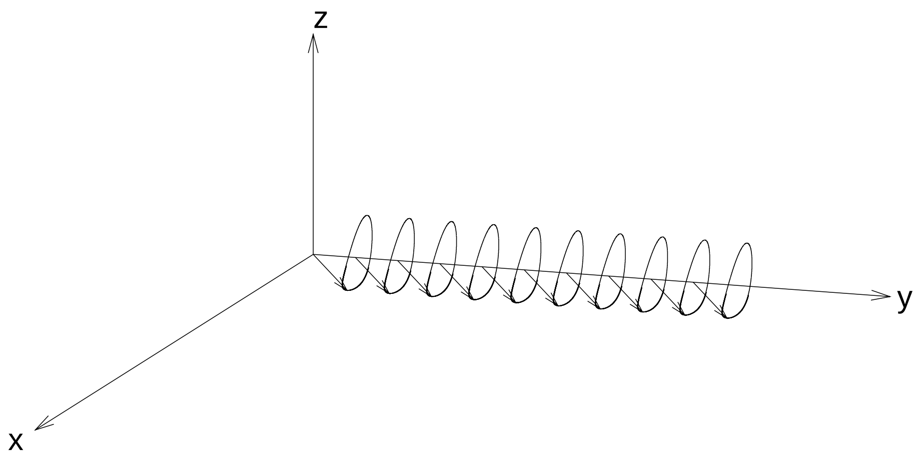

FIG. 1: Spatially homogeneous time-dependent solution for the spin vectors.

The total energy associated with the spin precession of N nearest neighbor spins in a periodic lattice can be evaluated by substituting the solutions (24) - (26) into the energy expression (2) (with $D = 0$ and $\vec{\mathcal{H}} = 0$) and making use of the summation relations of the Jacobi elliptic functions (Appendix A):

$$\begin{aligned} E &= -\sum_{n=1}^{N}[AS_n^x(S_{n+1}^x + S_{n-1}^x) + BS_n^y(S_{n+1}^y + S_{n-1}^y) + CS_n^z(S_{n+1}^z + S_{n-1}^z)] \\ &= -2N(B\beta^2 + C\gamma^2) = -N[B + (C - B)\gamma^2] \\ &= -2N[B - (B - C)\gamma^2], \quad (B > C, \ \ 0 \le \gamma \le 1) \end{aligned} \tag{27}$$

so that the energy per site becomes

$$\epsilon = \frac{E}{N} = -2[B - (B - C)\gamma^2]. \tag{28}$$

Further, since each of these spins evolves identically, the spin chain may be treated to be equivalent to N independent rigid bodies executing synchronous periodic motions of the form (24) - (26). Consequently, each of the above spin motion can be quantized semi-classically using the Bohr-Sommerfeld quantization condition

$$\oint p_i dq_i = \left(n_i + \frac{1}{2}\right) h, \qquad n_i = 0, 1, 2, ..., \quad i = 1, 2, ..., N \tag{29}$$

where the canonically conjugate variables

$$p_n = S_n^z, \quad q_n = \arctan\left(\frac{S_n^y}{S_n^x}\right), \quad n = 1, 2, ..., N. \tag{30}$$

Using the explicit forms for S_n^x, S_n^y and S_n^z given in (24) - (26), q_n's and p_n's may be expressed in terms of elliptic functions (see Appendix B for details). Carrying out the integral over a cycle of period $4K$, one obtains the following transcendental equation for the quantization of the amplitude γ:

$$\frac{4}{\gamma}\sqrt{\frac{1-\gamma^2 k'^2}{1-\gamma^2}}\left[\Pi\left(\frac{-\gamma^2 k^2}{(1-\gamma^2)}, k\right) - (1-\gamma^2)K(k)\right] = \left(n_i + \frac{1}{2}\right) h,$$
$$n_i = 0, 1, 2, ..., \quad i = 1, 2, \ldots, N. \tag{31}$$

Here $\Pi(x, k)$ is the complete elliptic integral of the third kind [30]. Equation (31) is a transcendental equation in the amplitude parameter γ. For each value of the quantum number n_i (=0, 1, 2, ...), the solution γ_{n_i} can be found by solving numerically the transcendental equation (31). It may be noted that such a semiclassical quantization procedure by solving transcendental equations involving all the three complete elliptic integrals has been carried out successfully for isotropic anharmonic oscillators with two and three degrees of freedom [34] and for the two center Coulomb problem [35]. Then using the resultant allowed set of values of the amplitude $\{\gamma_{n_i}\}$, n_i=0,1,2,... , $i = 1, 2, ..., N$, in the classical energy expression per site (28), the corresponding quantized energy spectrum can be evaluated. Consequently, the full spectrum associated with the solutions (24) - (26) of the total lattice can be evaluated by associating quantum numbers as $\{n_1, n_2, ..., n_N\}$ with the full lattice. Complete details will be published elsewhere.

B. Spatially oscillatory time periodic solutions

Now taking the possibility $p = 2K(k)$ in Eq. (18) for the wave vector, and using it in the conditions (19) and (20), we obtain the relations connecting the unknowns ω, γ and k as

$$\omega\alpha = -2\beta\gamma(B+C), \quad \omega\beta = -2\alpha\gamma(A+C), \quad \omega\gamma k^2 = -2\alpha\beta(B-A). \tag{32}$$

Solving these equations, we obtain

$$\omega = 2\gamma\sqrt{(A+C)(B+C)}, \quad k^2 = \frac{1-\gamma^2}{\gamma^2}\left(\frac{B-A}{A+C}\right), \quad p = 2K(k). \tag{33}$$

The corresponding spatially alternating time periodic solutions are

$$S_n^x = \sqrt{1-\gamma^2 k'^2}\ \mathrm{sn}(2Kn - \omega t + \delta, k) = (-1)^{n+1}\sqrt{1-\gamma^2 k'^2}\ \mathrm{sn}(\omega t + \delta, k), \tag{34}$$

$$S_n^y = \sqrt{1-\gamma^2}\ \mathrm{cn}(2Kn - \omega t + \delta, k) = (-1)^{n}\sqrt{1-\gamma^2}\ \mathrm{cn}(\omega t + \delta, k), \tag{35}$$

$$S_n^z = \gamma\ \mathrm{dn}(2Kn - \omega t + \delta, k) = \gamma\ \mathrm{dn}(\omega t + \delta, k). \tag{36}$$

The solution (34) - (36) is depicted schematically in Fig. 2. Note that the x and y components of the alternate spins flip and next nearest neighbors evolve in parallel. Also, these solutions have no counterpart in the continuum limit of the lattice.

Again from the energy expression in Eq. (2), using the above solution (34) - (36), we obtain the total energy of the system for a periodic lattice of N spins (with N even)

$$E = N[B - (B+C)\gamma^2]. \tag{37}$$

Correspondingly the energy per lattice site is

$$\epsilon = [B - (B+C)\gamma^2], \tag{38}$$

which is greater than the energy of the uniform periodic solution, see Eq. (28). Consequently, the present solution constitutes an excited state of the system.

Since every other spin evolves identically, a lattice of N spins (N even) may be split into two sublattices of $N/2$ spins, and each member of the sublattice evolves identically. Consequently, the semiclassical quantization condition may be given separately for each member of the two sublattices:

$$\begin{aligned} \oint p_{1,i} dq_{1,i} &= \left(n_{1,i} + \frac{1}{2}\right)h, \quad n_{1,i} = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots \frac{N}{2} \\ \oint p_{2,i} dq_{2,i} &= \left(n_{2,i} + \frac{1}{2}\right)h, \quad n_{2,i} = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots \frac{N}{2} \end{aligned} \tag{39}$$

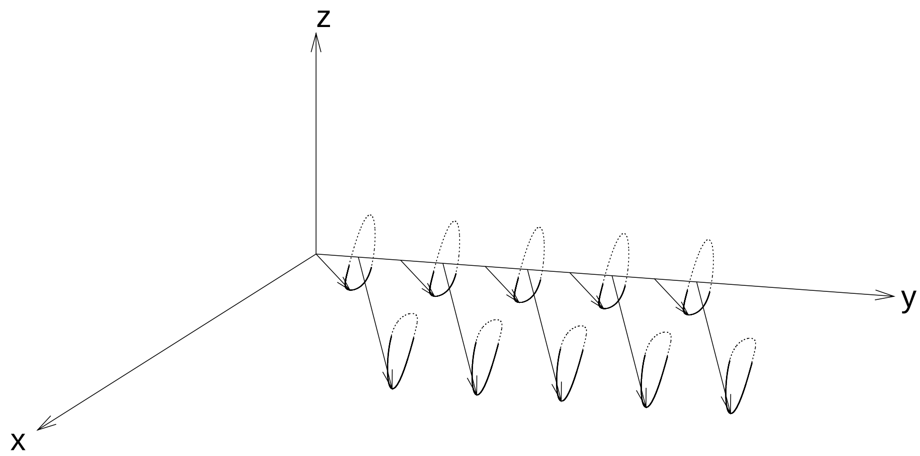

FIG. 2: Spatially oscillatory time periodic solution.

and the energy expression may be correspondingly quantized with q_i and p_i chosen in the form (30). From (34) - (36) one can choose for the first sublattice the spin solutions corresponding to n odd and for the second sublattice corresponding to n even in the solution (34) - (36). Then evaluating the integrals (39), which result in expressions essentially of the form (31), the energy expression (37) can be quantized, as discussed earlier.

C. The linear and nonlinear magnon solutions and dispersion relations

In the uniaxial anisotropic case $A = B < C$, from the expression (33) for k^2, we find that $k = 0$. Consequently, we have the magnon solution. Considering the case (2), $k^2 = 0$ in Eqs (18) - (20), we find that here one has the standard dispersion relation for the uniaxial anisotropic case ($A = B < C$)

$$\omega = 2\gamma(C - A\cos p), \quad A = B < C,$$

corresponding to the linear magnon solution

$$\begin{aligned} S_n^x &= \sqrt{1-\gamma^2}\ \sin(pn-\omega t+\delta), \\ S_n^y &= \sqrt{1-\gamma^2}\ \cos(pn-\omega t+\delta), \\ S_n^z &= \gamma, \end{aligned} \tag{40}$$

which was noted in ref. [18].

It is also now instructive to analyze the nature of dispersion relations (33) underlying the nonlinear magnon or elliptic function propagating spin wave solutions (34) - (36).

First we note that these solutions (given by (34) - (36)) in the limit $k \to 0$, reduce to the above linear spin wave solutions (40) with the specific value $p = \pi$ and $\omega = 2\gamma(A+C)$. However, localized solitary wave solutions (for $k = 1$) do not appear in the moving case because $p \to \infty$ (In fact, in the limit $k \to 1$, one gets the ground state solution $(\pm 1, 0, 0)$. From the expression (33), we note that for $\gamma = 1$, $k = 0$ and for $\gamma = \gamma_{min} = \sqrt{(B-A)/(B+C)}$ one gets $k = 1$. In other words, $1 \geq \gamma > \gamma_{min}$. Defining $\zeta = (B-A)/(A+C)$ with $B > A$ implies

$$k^2 = \left(\frac{1}{\gamma^2} - 1\right)\zeta, \quad \gamma^2 = \frac{\zeta}{\zeta + k^2}. \tag{41}$$

This leads to the dispersion relation [see Eq. (33)]

$$\omega = 2\sqrt{\frac{(B-A)(B+C)}{\zeta + k^2}}, \tag{42}$$

with $p = 2K(k)$. Note that $\pi \leq p < \infty$ when $0 \leq k < 1$. For $k \simeq 0$, $K(k) = (\pi/2)(1 + k^2/4)$ and we get the dispersion relation for the magnons (40) as

$$\omega = 2\sqrt{\frac{(B-A)(B+C)}{(\zeta - 4) + 4p/\pi}}, \tag{43}$$

which is finite at $p = \pi$ and zero as $p \to \infty$. Similarly, for $k \simeq 1$ (but not equal to one), we have $K(k) = ln(4/k')$ and we get the dispersion for the soliton like structure as

$$\omega = 2\sqrt{\frac{(B-A)(B+C)}{(1+\zeta) - 16\exp(-p)}}, \tag{44}$$

which is finite at both $p = \pi$ and as $p \to \infty$. The above dispersion relations in the limiting cases of the parameter k in the allowed region of p, $\pi \leq p < \infty$, are plotted in Fig. 3 for a simple choice of the anisotropy parameters. Dispersion curves for other values of k lie between these two limiting curves.

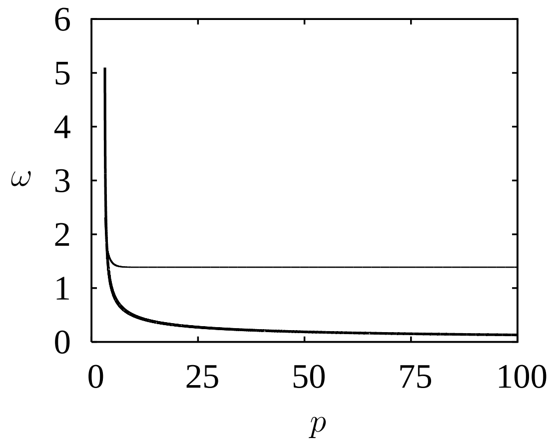

FIG. 3: Nonlinear spin wave dispersion relations in the limiting cases (i) $k \approx 0$: thick line and (ii) $k \approx 1$: thin line, for the parameter values $A = 2.45, B = 2.65$ and $C = 0$.

IV. STATIC SOLUTIONS

Considering Eqs. (18) - (20), one can easily note that the term $[1 - k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(p,k)]$ in the denominator on the right hand sides is "harmless" provided $\omega = 0$ [case(1)] on the left hand sides. This actually corresponds to the static case (or time independent case) of the equation of motion (6) - (8) for $D = 0$ and $\vec{\mathcal{H}} = 0$. In this case the conditions (18) - (20) become

$$\begin{aligned} \beta\gamma[C\mathrm{dn}(p,k) - B\mathrm{cn}(p,k)] &= 0, \\ \gamma\alpha[A\mathrm{cn}(p,k) - C)] &= 0, \\ \alpha\beta[B - A\mathrm{dn}(p,k)] &= 0. \end{aligned} \tag{45}$$

Then there are three possibilities.

A. Nonplanar static structures for XYZ and XYY models

In this case $S_n^x, S_n^y, S_n^z \neq 0$, which implies that $\alpha, \beta, \gamma \neq 0$. The static, periodic solutions of such a discrete Heisenberg chain are then obtained from (13) and (14) with $u = pn + \delta$ as

$$S_n^x = \sqrt{1-\gamma^2 k'^2}\ \mathrm{sn}(pn+\delta,k),$$
$$S_n^y = \sqrt{1-\gamma^2}\ \mathrm{cn}(pn+\delta,k),$$
$$S_n^z = \gamma\ \mathrm{dn}(pn+\delta,k), \tag{46}$$

with the fixed modulus k and fixed wave vector p given by

$$k^2 = \frac{A^2-B^2}{A^2-C^2}, \quad \mathrm{dn}(p,k) = \frac{B}{A}. \tag{47}$$

In (46), δ is a constant phase factor. Note that the above expressions (47) follow from Eq. (45) for $\alpha \neq \beta \neq 0$. Here p denotes a wave vector. In the special case of an XYY model, i.e. $A \neq B = C$, which implies $k = 1$ [from Eq. (47)], we get the localized single soliton (kink- and pulse-like) solutions or domain wall structures:

$$S_n^x = \tanh(pn+\delta), \quad S_n^y = \sqrt{1-\gamma^2}\ \mathrm{sech}(pn+\delta) \quad S_n^z = \gamma\ \mathrm{sech}(pn+\delta), \tag{48}$$

with $\mathrm{sech}(p) = B/A$. The above domain wall structure is depicted schematically in Fig. (4a). Alternatively, for the XXY model, $A = B \neq C$ (i.e. $k = 0$), we get the linear excitations (frozen magnons):

$$S_n^x = \sqrt{1-\gamma^2}\ \sin(pn+\delta), \quad S_n^y = \sqrt{1-\gamma^2}\ \cos(pn+\delta), \quad S_n^z = \gamma. \tag{49}$$

B. Planar (XY) case

There are two other possibilities from Eq. (45) corresponding to the planar (XY) case.

Case (i): In this case $\alpha = \beta = 1$ and $\gamma = 0$ which implies $S_n^z = 0$ and $S_n^x, S_n^y \neq 0$. Again from Eqs. (13) and (14) the solution is given by

$$S_n^x = \mathrm{sn}(pn+\delta,k), \quad S_n^y = \mathrm{cn}(pn+\delta,k), \tag{50}$$

provided $\mathrm{dn}(p,k) = B/A$, where the modulus k ($0 \leq k \leq 1$) and the constant δ are arbitrary, which follows from Eq. (45). As in the XYZ case one can obtain domain wall structure in the XY case also by taking the limit $k \to 1$, namely $S_n^x = \tanh(pn+\delta)$ and $S_n^y = \mathrm{sech}(pn+\delta)$. The above solution was already reported by Roberts and Thompson [24] and Granovskii and Zhedanov [27].

Case (ii): In this case $\alpha = k$, $\beta = 0$ and $\gamma = 1$ which implies $S_n^y = 0$ and $S_n^x, S_n^z \neq 0$. The solution is now given by

$$S_n^x = k\ \mathrm{sn}(pn+\delta,k), \quad S_n^z = \mathrm{dn}(pn+\delta,k), \tag{51}$$

provided $\mathrm{cn}(p,k) = C/A$. Here also the modulus $0 < k < 1$ and the constant δ are arbitrary. The domain structure in the $k \to 1$ limit is again given by $S_n^x = \tanh(pn+\delta)$ and $S_n^z = \mathrm{sech}(pn+\delta)$. This solution was also reported in references [24] and [27].

C. Another class of Nonplanar XYY structures

Now making use of the more general parametrization (15), one can easily check that in the XYY case, that is $B = C \neq A$ with the second variable v fixed as a constant, one can identify the following static solution:

$$S_n^x = \mathrm{cn}(pn+\delta,k), \quad S_n^y = \gamma\ \mathrm{sn}(pn+\delta,k), \quad S_n^z = \sqrt{1-\gamma^2}\ \mathrm{sn}(pn+\delta,k), \tag{52}$$

provided $\mathrm{dn}(p,k) = A/B$, while the modulus parameter k $(0 \leq k \leq 1)$, and the constants $\gamma = \mathrm{cn}(v,k_2)$ and δ are arbitrary. In the limit $k \to 1$, one obtains the domain wall structure

$$S_n^x = \mathrm{sech}(pn+\delta), \quad S_n^y = \gamma\ \tanh(pn+\delta), \quad S_n^z = \sqrt{1-\gamma^2}\ \tanh(pn+\delta), \tag{53}$$

with $\tanh(p) = A/B$. The above type of domain wall structure is depicted schematically in Fig. (4b). It is also of interest to note that none of the above static solutions survive in the continuum limit of the lattice and they are all patently structures belonging to discrete lattices.

D. Integrability of the static case

In their important work [27], Granovskii and Zhedanov have shown that the static case of Eq. (5), namely

$$\vec{S}_n \times \vec{J}(\vec{S}_{n+1} + \vec{S}_{n-1}) = 0, \quad (\vec{J}\vec{S}_n) = AS_n^x + BS_n^y + CS_n^z, \tag{54}$$

or equivalently

$$\vec{S}_{n+1} + \vec{S}_{n-1} = \lambda_n J^{-1}\vec{S}_n, \tag{55}$$

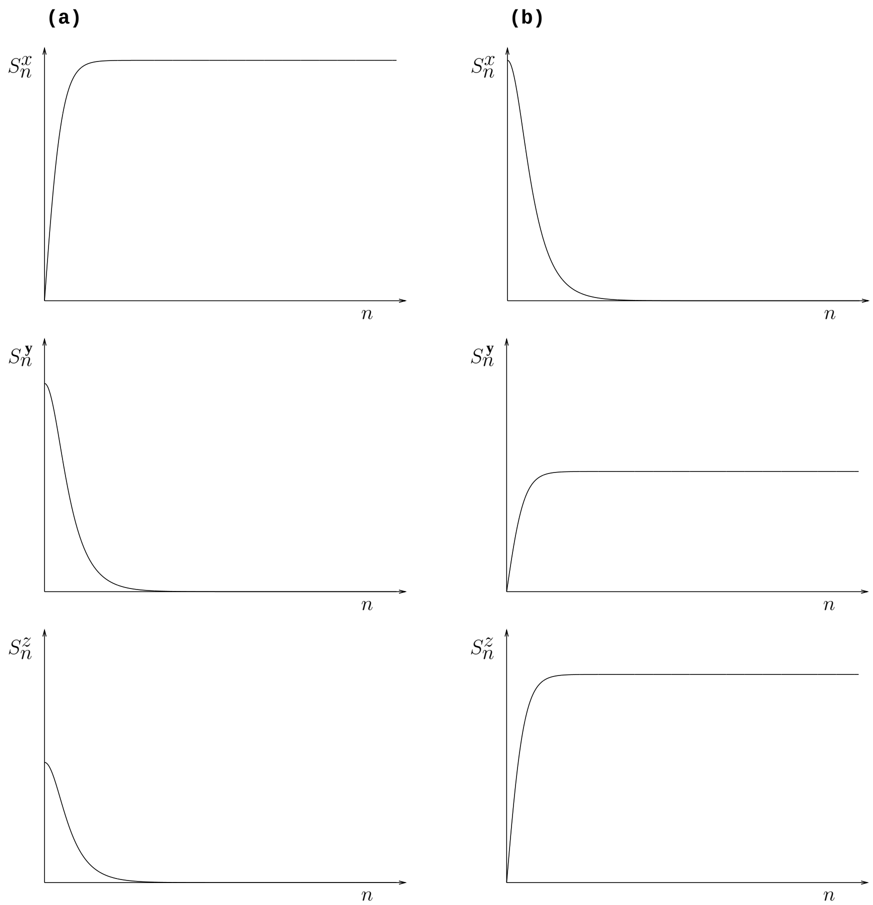

FIG. 4: Localized static single soliton/domain wall structures in the nonplanar case of XYY model with $\gamma = 0.5$: (a) Spin structure given by Eq. (48), (b) Spin structure given by Eq. (53).

where the Lagrange multiplier

$$\lambda_n = \frac{2(\vec{S}_n \cdot J^{-1}\vec{S}_{n+1})}{\vec{S}_n \cdot J^{-2}\vec{S}_n}, \tag{56}$$

is equivalent to a discretized version of the Schrödinger equation with two-level Bargmann type potential or a discrete analog of a Neumann system [28]. Some explicit solutions have also been reported in their works.

V. ENERGY VALUES OF STATIC STRUCTURES AND PEIERLS-NABARRO POTENTIAL BARRIER

For each of the static structures discussed in Sec. IV, the total energy can be obtained explicitly by making use of the various summation formulas given in the Appendix. In the following we indicate how the total energy for the lattice can be obtained for the nonplanar static structure (46) and then present the final results only for the other cases.

A. Total energy of the nonplanar static structure (46)

For the nonplanar static structure (46), the total energy

$$\begin{aligned} E &= -\sum_{\{n\}}[AS_n^xS_{n+1}^x + BS_n^yS_{n+1}^y + CS_n^zS_{n+1}^z] \\ &= -\frac{1}{2}\sum_n\{A\alpha^2\mathrm{sn}(u,k)[\mathrm{sn}(u+p,k)+\mathrm{sn}(u-p,k)] + B\beta^2\mathrm{cn}(u,k)[\mathrm{cn}(u+p,k) \\ &\quad +\mathrm{cn}(u-p,k)] + C\gamma^2\mathrm{dn}(u,k)[\mathrm{dn}(u+p,k)+\mathrm{dn}(u-p,k)]\}. \end{aligned} \tag{57}$$

Now using the identities derived recently by Khare et al. [36] for the products of elliptic functions, summarized in Appendix A, Eq. (57) can be rewritten as

$$\begin{aligned} E &= -\sum_{n=1}^{N}[B\beta^2\mathrm{cn}(p,k) + C\gamma^2\mathrm{dn}(p,k)] + \sum_{n=1}^{N}\left(-\frac{A\alpha^2}{k^2\mathrm{sn}(p,k)} + \frac{B\beta^2\mathrm{dn}(p,k)}{k^2\mathrm{sn}(p,k)} + \frac{C\gamma^2\mathrm{cn}(p,k)}{\mathrm{sn}(p,k)}\right) \\ &\quad \times[Z(p(n+1)+\delta,k) - Z(pn+\delta,k) - Z(p,k)]. \end{aligned} \tag{58}$$

Since the elliptic zeta function [30] satisfies

$$\sum_{n=1}^{N}[Z(p(n+1)+\delta,k) - Z(pn+\delta,k)] = 0, \tag{59}$$

see Ref. [36], Eq. (58) reduces to

$$E = -N[B\beta^2\mathrm{cn}(p,k)+C\gamma^2\mathrm{dn}(p,k)] - N\frac{Z(p,k)}{k^2\mathrm{sn}(p,k)}[-A\alpha^2+B\beta^2\mathrm{dn}(p,k)+C\gamma^2\mathrm{cn}(p,k)], \tag{60}$$

which is independent of the constant phase factor δ. Consequently, the Peierls-Nabarro (P-N) potential barrier vanishes. Since $\alpha = \sqrt{1-\gamma^2k'^2}$, $\beta = \sqrt{1-\gamma^2}$, $k^2 = (A^2-B^2)/(A^2-C^2)$ and $\mathrm{dn}(p,k) = B/A$, see Eqs. (46) - (47), the energy expression (60) can be rewritten as

$$E = -N\frac{BC}{A} - N\frac{\sqrt{A^2-C^2}}{(A^2-B^2)}Z(p,k)\left[-(A^2-B^2) + \gamma^2C^2\frac{(B^2-C^2)}{(A^2-C^2)}\right]. \tag{61}$$

Note that in the above expression γ is a free parameter, while all the other quantitites are fixed and the energy is a quadratic function of the free parameter γ.

In general, this energy may depend on the location of the soliton, i.e. δ. There is an energy cost associated with moving a localized mode, e.g. a soliton or breather, by half a lattice constant in a discrete lattice. Alternatively, there is a periodic dependence of the energy of a soliton on its position with respect to the lattice sites. This is called the Peierls-Nabarro barrier [31, 32]. The effects of discreteness such as the P-N barrier and the spin barrier (i.e., the total spin, which is an integral of motion, depends on the location of the soliton) may be studied from the total energy expressions [33]. In the present case the P-N barrier vanishes as the energy expression is independent of the location of the soliton δ.

B. Total energy of other static structures

In the following, we only give the final form of the total energy expressions for the other static structures.

(i) Planar XY case:

(a) Structure (50):

$$E = -NBcn(p,k) + N\frac{Z(p,k)}{k^2 sn(p,k)}[A - Bdn(p,k)]. \tag{62}$$

Here $\mathrm{dn}(p,k) = B/A$, while the modulus parameter k is arbitrary. Consequently the energy expression (62) can be expressed as

$$E = -N\frac{B}{A}\frac{\sqrt{B^2 - A^2 + A^2k^2}}{k} + N\sqrt{A^2 - B^2}\frac{Z(p,k)}{k}, \tag{63}$$

where k $(0 \le k \le 1)$ is the free parameter.

(b) Structure (51):

$$E = -NCdn(p,k) + \frac{NZ(p,k)}{k^2 sn(p,k)}[Ak^2 - Ccn(p,k)]. \tag{64}$$

Here $\mathrm{cn}(p,k) = C/A$, while k is arbitrary. Then the expression (64) can be written as

$$E = -N\frac{C}{A}\sqrt{A^2 - k^2(A^2 - C^2)} + N\frac{(A^2k^2 - C^2)}{\sqrt{A^2 - C^2}}\frac{Z(p,k)}{k^2}. \tag{65}$$

(ii) Nonplanar XYY structure: For the solution (52), the total energy expression can be deduced as

$$E = -NAcn(p,k) + N\left[Ak^2\frac{dn(p,k)}{sn(p,k)} - [B\gamma^2 + C(1-\gamma^2)]\frac{1}{k^2 sn(p,k)}\right]Z(p,k). \tag{66}$$

In Eq. (66), $\mathrm{dn}(p,k) = A/B$ while k is arbitrary. Then we have

$$E = -N\frac{A}{B}\frac{\sqrt{A^2 - B^2k^2}}{k} + \frac{N}{\sqrt{B^2 - A^2}}\left[A^2k^3 - \{C + (B-C)\gamma^2\}\frac{B}{k}\right] Z(p,k). \tag{67}$$

Note that in the above equation both k and γ are free parameters. All the above energy expressions are independent of the phase constant δ and so the P-N potential barrier in these cases is also absent.

VI. ISOTROPIC MODEL ($A = B = C = 1$)

No moving/time dependent nonlinear structures can be found in this case except for the spin wave (magnon) solutions (see below). This can be easily checked by looking at the conditions that must hold to satisfy Eqs. (18) - (20) which are for $k \neq 0$, (for $k = 0$, see below)

$$\omega\alpha = -4\beta\gamma, \quad \omega\beta = -4\alpha\gamma, \quad \omega\gamma k^2 = 0. \tag{68}$$

Then the only possible structures are the static structures of the following form.

A. Planar model: static solutions

In the special case of an isotropic planar model ($A = B = C$; $S_n^z = 0$) the limiting elliptic function solutions for $\gamma = 0$ (and $\omega = 0$) are

$$S_n^x = \mathrm{sn}\,(2Kn + \delta, k)\,, \quad S_n^y = \mathrm{cn}\,(2Kn + \delta, k)\,, \quad S_n^z = 0. \tag{69}$$

The modulus parameter k and the phase constant δ are arbitrary. These solutions were obtained previously [24]. However, the static solutions analogous to Eq. (50) do not exist in the isotropic case as $K(k) \to \infty$ as $k \to 1$ and the solution (69) reduces to the uniform solution $S_n = (\pm 1, 0, 0)$. However, if instead $\beta = 0$, the solutions are given by

$$S_n^x = k\ \mathrm{sn}\,(2Kn + \delta, k)\,, \quad S_n^y = 0, \quad S_n^z = \mathrm{dn}\,(2Kn + \delta, k)\,. \tag{70}$$

B. Nonplanar model: propagating solutions

For $k = 0$, $\omega = 2\gamma(1 - \cos p) = 4\gamma\sin^2(\frac{p}{2})$, the propagating linear excitations (i.e. magnons) are given by

$$S_n^x = \sqrt{1-\gamma^2}\ \sin(pn - \omega t), \quad S_n^y = \sqrt{1-\gamma^2}\ \cos(pn - \omega t), \quad S_n^z = \gamma. \tag{71}$$

VII. LINEAR STABILITY

Next, we consider the stability of both the time periodic solutions, discussed in Sec. III, and the static solutions in terms of Jacobi elliptic functions, obtained in Sec. IV[1]. Linear stability of the time periodic solutions in Eqs. $(24-26)$ and Eqs. $(34-36)$ is studied using the period map. For the homogeneous time periodic solution, it suffices to study the stability of spin at any one site. For the spatially oscillatory solution in Eqs. $(34-36)$, we individually study the stability of two spin vectors, one at an odd and another at an even site. Rewriting the spin equation (5) using the complex stereographic variable

$$\Omega_n = \frac{S_n^x + iS_n^y}{1+S_n^z}, \tag{72}$$

we get

$$\begin{aligned} \frac{d\Omega_n}{dt} &= iC\Omega_n\Big(\frac{1-|\Omega_{n+1}|^2}{1+|\Omega_{n+1}|^2} + \frac{1-|\Omega_{n-1}|^2}{1+|\Omega_{n-1}|^2}\Big) - i\frac{A}{2}\Omega_n^2\Big(\frac{\Omega_{n+1}+\bar{\Omega}_{n+1}}{1+|\Omega_{n+1}|^2} + \frac{\Omega_{n-1}+\bar{\Omega}_{n-1}}{1+|\Omega_{n-1}|^2}\Big) \\ &+i\frac{B}{2}\Omega_n^2\Big(\frac{\Omega_{n+1}-\bar{\Omega}_{n+1}}{1+|\Omega_{n+1}|^2} + \frac{\Omega_{n-1}-\bar{\Omega}_{n-1}}{1+|\Omega_{n-1}|^2}\Big) + 2iD\frac{1-|\Omega_n|^2}{1+|\Omega_n|^2}\Omega_n + i\frac{H_x}{2}(1+\Omega_n^2), \end{aligned} \tag{73}$$

where $\bar{\Omega}_n$ denotes the complex conjugate of the stereographic variable. After linearizing using the expansion

$$\Omega_n = \Omega_{0n} + \delta\Omega, \tag{74}$$

around the time periodic solution $\Omega_{0n}(t)$, we compute the Floquet matrix $\hat{\mathcal{M}}$ such that

$$\begin{pmatrix} \delta\Omega_n(T) \\ \delta\bar{\Omega}_n(T) \end{pmatrix} = \hat{\mathcal{M}} \begin{pmatrix} \delta\Omega_n(0) \\ \delta\bar{\Omega}_n(0) \end{pmatrix}. \tag{75}$$

Here, $T = 2\pi/\omega$ is the inherent time period in the two sets of solutions, Eqs. (24) - (26) and (34) - (36). If γ_n is the eigenvalue of $\mathcal{M}$, then the solution is unstable if $|\gamma_n| > 1$. Figure 5 shows the instability regions in the $(k-A)$ plane for the homogeneous and spatially oscillatory time periodic solutions. Here k is the modulus of the Jacobian elliptic function.

In order to numerically study the stability of the static solutions of Sec. IV, we perturb the solution $\mathbf{S}_n^0$ in Eqs. (46) - (47) by a small amount $\delta\mathbf{S}_n(t)$ such that $\mathbf{S}_n \cdot \delta\mathbf{S}_n = 0$, $|\delta\mathbf{S}_n| \ll 1$. Upon substituting the perturbed vector

$$\mathbf{S}_n^p = \mathbf{S}_n^0 + \delta\mathbf{S}_n \tag{76}$$

[1] The work in this section was carried out in collaboration with S. Murugesh

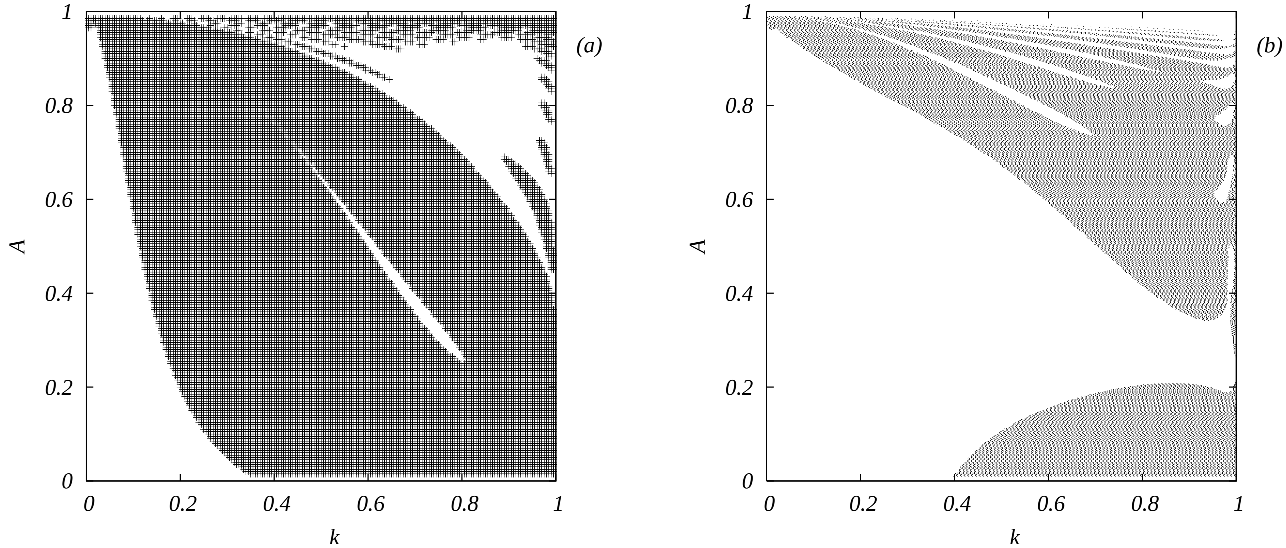

FIG. 5: Instability regions (shaded regions) in the $(k - A)$ plane for the time periodic solutions obtained using the period map. Instability region for the (a) homogeneous time periodic solution, Eqs. (24) - (26), with $B = 0.9$ and $C = 0.01$, $B > A > C$, and (b) for the spatially oscillatory solution, Eqs. (34) - (36), for the same parametric values. The instability diagram is identical for spins at both odd and even sites.

in Eq. (5), the time evolution is computed numerically. As an illustration, it is found that the static solution (46) - (47) is indeed stable for long times for small values of the modulus parameter k of the Jacobi elliptic function, and that the solution (46) - (47) is less stable with increasing value of k, i.e. the instability sets in at earlier times. Figure 6 shows the time profile of the static solution (46) - (47) under a small perturbation. Figure 7 depicts the initial and final profiles for easy comparison. Fuller details will be presented elsewhere.

Finally, it is also of interest to investigate whether the time periodic solutions (24) - (26) and (34) - (36) are modulationally stable or not. Recently such modulational stability analysis has been performed for special solutions of a number of discrete nonlinear dynamical systems, including discrete nonlinear Schrödinger equations, see for example refs. [37, 38]. Such an analysis for the time dependent elliptic function solutions (24) - (26) and (34) - (36) is being pursued at present and will be reported separately along with the details of the linear stability analysis mentioned above.

VIII. ANISOTROPIC SPIN CHAIN IN THE PRESENCE OF ON-SITE ANISOTROPY AND A CONSTANT EXTERNAL MAGNETIC FIELD

Finally, in this section we wish to point out that explicit spin solutions can be constructed for even more general situations (i) with additional on-site anisotropy and (ii) external constant magnetic field. Brief details are as follows.

A. On-site anisotropy

Many of the results discussed in the previous sections are valid even in the presence of the on-site anisotropy ($D \neq 0$) in Eq.(5) or Eqs. (6) - (8). In this case, the parametrization (17) in terms of Jacobian elliptic functions leads to the following conditions instead of (18) - (20):

$$-\omega\alpha = 2\beta\gamma\left\{\frac{[C\mathrm{dn}(p,k) - B\mathrm{cn}(p,k)]}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(p,k)} - D\right\}, \tag{77}$$

$$\omega\beta = 2\alpha\gamma\left\{\frac{\mathrm{dn}(p,k)[A\mathrm{cn}(p,k) - C]}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(p,k)} + D\right\}, \tag{78}$$

$$\omega\gamma k^2 = \frac{2\alpha\beta\mathrm{cn}(p,k)[B - A\mathrm{dn}(p,k)]}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(p,k)}, \quad u = pn - \omega t + \delta. \tag{79}$$

Unlike the $D = 0$ case (sec. III A), the above relations can be free from the space-time variable (u) only for two choices: (i) modulus parameter $k = 0$ (linear spin wave solution) and (ii) $p = 4K(k)$ (spatially homogeneous time-dependent solution) or $p = 2K(k)$ (spatially oscillatory time periodic solution).

(i) For the case $k = 0$, from Eq. (77c), we have $B = A$ and from Eqs. (77a), (77b) we obtain the dispersion relation

$$\omega = 2\gamma(C - D - A\cos\, p), \quad A = B < (C - D), \tag{80}$$

associated with the spin wave solution (40) for the present case where the on-site anisotropy $D \neq 0$.

(ii-a) For the choice $p = 4K(k)$, Eqs. (77) degenerate to

$$\omega = 2\gamma\sqrt{(C - B - D)(A - C + D)}, \quad k^2 = \frac{(1-\gamma^2)}{\gamma^2}\frac{(B - A)}{(B + D - C)}, \tag{81}$$

where the form of the associated spatially homogeneous time-dependent solutions are given by Eqs. (24) - (26) for the spin components S_n^x, S_n^y and S_n^z but with the above expressions for ω and k^2.

(ii-b) Similarly for the choice $p = 2K(k)$, Eqs. (77) lead to the expressions

$$\omega = 2\gamma\sqrt{(A+C-D)(B+C-D)}, \quad k^2 = \frac{(1-\gamma^2)}{\gamma^2}\frac{(B-A)}{(A+C-D)} \tag{82}$$

for the spatially oscillatory time periodic solutions (34) - (36) with the above expressions for ω and k^2.

It may be noted from the relation (77) that no static solution (that is $\omega = 0$) is possible for general values of k when $D \neq 0$.

B. Constant external magnetic field

Considering now the full anisotropic chain (6) - (8) with the external magnetic field also present, we have not succeeded in finding any explicit solution which generalizes the nonlinear structures discussed in the previous sections for the anisotropic spin chain in the absence of the magnetic field. However, in specific instances classes of exact solutions can be obtained. For example, for the XYY spin chain, with $B = C \neq A$, $D = 0$, Eq. (5) or Eqs. (6) - (8) admit(s) the following exact solutions. In the presence of an external magnetic field $\vec{H} = (H_x, 0, 0)$ the relevant contribution to the equation of motion for the anisotropic spin chain comes from $\vec{S}_n \times \vec{H}$:

$$\frac{dS_n^x}{dt} = C[S_n^y(S_{n+1}^z + S_{n-1}^z) - S_n^z(S_{n+1}^y + S_{n-1}^y)], \tag{83}$$

$$\frac{dS_n^y}{dt} = AS_n^z(S_{n+1}^x + S_{n-1}^x) - CS_n^x(S_{n+1}^z + S_{n-1}^z) + H_x S_n^z, \tag{84}$$

$$\frac{dS_n^z}{dt} = CS_n^x(S_{n+1}^y + S_{n-1}^y) - AS_n^y(S_{n+1}^x + S_{n-1}^x) - H_x S_n^y. \tag{85}$$

These equations have an exact solution with $\mathrm{dn}(p,k) = C/A$ and $\omega = H_x$ in the form

$$\begin{aligned} S_n^x &= \mathrm{sn}(pn+\delta,k), \\ S_n^y &= \sin(\omega t+\gamma)\mathrm{cn}(pn+\delta,k), \\ S_n^z &= \cos(\omega t+\gamma)\mathrm{cn}(pn+\delta,k). \end{aligned} \tag{86}$$

In Eq. (84), the modulus parameter k ($0 \le k \le 1$) is arbitrary, while γ and δ are arbitrary phase factors. These solutions can be inferred by generalizing the static spin structures of the XYY model discussed in Sec. IV. C. Similar solutions can also be written down when the magnetic field is along the y or z direction.

For the present case the total energy becomes

$$
\begin{aligned}
E &= -A\sum_n S_n^x S_{n+1}^x - C\sum_n (S_n^y S_{n+1}^y + S_n^z S_{n+1}^z) - H_x \sum_n S_n^x \\
&= -A\sum_n \mathrm{sn}(pn+\delta)\,\mathrm{sn}(p(n+1)+\delta) \\
&\qquad -C\sum_n \mathrm{cn}(pn+\delta)\,\mathrm{cn}(p(n+1)+\delta) - H\sum_n \mathrm{sn}(pn+\delta) \\
&= -NC\,\mathrm{cn}(p,k) - \frac{NZ(p,k)}{k^2\,\mathrm{sn}(p,k)}[-A + C\,\mathrm{dn}(p,k)] - H_x \sum_n \mathrm{sn}(pn+\delta,k) \\
&= -N\frac{C}{A}\frac{\sqrt{C^2 - A^2 + k^2A^2}}{k} + N\frac{\sqrt{A^2-C^2}}{k}Z(p,k) + H_x\sum_n \mathrm{sn}(pn+\delta,k). \qquad (87)
\end{aligned}
$$

The sum in the last term above, namely $\sum_n \mathrm{sn}(pn+\delta)$, is represented as $\sigma_3(\delta)$ in Ref. [36] by Khare and Sukhatme and is dependent on the location of the soliton δ. This ensures that the Peierls-Nabarro potential barrier is present in the anisotropic spin chain in the presence of an external magnetic field.

IX. SUMMARY AND CONCLUSIONS

Using the summation identities [36] for Jacobi elliptic functions [30] we obtained several classes of static and propagating exact solutions for the classical, anisotropic Heisenberg chain. In the special case of the isotropic planar model we recovered the previously known solutions [24]. We explicitly obtained the nontrivial dispersion relations (ω vs. p) for the propagating solutions and predicted the contrasting features of magnons and solitons. Specifically, as $p \to \infty$ the magnon frequency goes to zero whereas the soliton frequency reaches a nonzero value. These dispersion relations can be measured via neutron scattering in the quasi-one dimensional materials realized by anisotropic Heisenberg chains. It would be instructive to explore whether similar exact solutions can be obtained for the analogous quantum Heisenberg models. The effects of discreteness, e.g. Peierls-Nabarro barrier [31, 32, 33] and spin barrier [33], may be important in anisotropic spin chains. The discrete equation of motion is non-integrable in general. However, the static version in the absence of an external field is an integrable system [25]. It is instructive to explore semiclassical quantization of the $2K$ versus $4K$ periodic solutions in terms of N anharmonic oscillators [39]. The solutions expressed in terms of $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$ and $\mathrm{dn}(x,k)$ correspond to the $n=1$ Lamé functions. We have explicitly checked that $n=2$ Lamé functions do not give exact solutions. However, it is conceivable that $n=3$ Lamé functions [29] may lead to a new class of exact solutions. In addition, there may be another class of exact solutions with a denominator also containing elliptic functions. We are presently exploring these solutions also. To conclude, we wish to state that the classical anisotropic Heisenberg spin chain admits very interesting static and dynamic structures and more work is needed in this direction to identify all of them.

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APPENDIX A: JACOBIAN ELLIPTIC FUNCTIONS

Basic elliptic function properties:

$$\mathrm{sn}^2(u,k)+\mathrm{cn}^2(u,k)=1, \quad \mathrm{dn}^2(u,k)+k^2\mathrm{sn}^2(u,k)=1. \tag{A1}$$

Addition theorems:

$$\mathrm{sn}(u+v,k)+\mathrm{sn}(u-v,k)=\frac{2\mathrm{sn}(u,k)\mathrm{cn}(v,k)\mathrm{dn}(v,k)}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(v,k)}, \tag{A2}$$

$$\mathrm{cn}(u+v,k)+\mathrm{cn}(u-v,k)=\frac{2\mathrm{cn}(u,k)\mathrm{cn}(v,k)}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(v,k)}, \tag{A3}$$

$$\mathrm{dn}(u+v,k)+\mathrm{dn}(u-v,k)=\frac{2\mathrm{dn}(u,k)\mathrm{dn}(v,k)}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(v,k)}. \tag{A4}$$

Product relations:

$$m\mathrm{sn}(x,k)\mathrm{sn}(x+a,k)=-\mathrm{ns}(a,k)[Z(x+a)-Z(x)-Z(a)], \tag{A5}$$

$$m\mathrm{cn}(x,k)\mathrm{cn}(x+a,k) = m\mathrm{cn}(a,k) + \mathrm{ds}(a,k)[Z(x+a) - Z(x) - Z(a)], \tag{A6}$$

$$\mathrm{dn}(x,k)\mathrm{dn}(x+a,k) = \mathrm{dn}(a,k) + \mathrm{cs}(a,k)[Z(x+a) - Z(x) - Z(a)], \tag{A7}$$

where $Z(x) = Z(x,k)$ is the Jacobi or elliptic zeta function, and $\mathrm{ns}(x,k) = 1/\mathrm{sn}(x,k)$, $\mathrm{ds}(x,k) = \mathrm{dn}(x,k)/\mathrm{sn}(x,k)$, $\mathrm{cs}(x,k) = \mathrm{cn}(x,k)/\mathrm{sn}(x,k)$.

Summation relation:

$$\sum_{n=1}^{N} \big\{ Z[\beta\epsilon(n+1) + \delta, k] - Z[n\beta\epsilon + \delta, k] \big\} = 0. \tag{A8}$$

Integration formula:

$$\int_0^K \frac{\mathrm{sn}^2 u du}{1 - \alpha^2 \mathrm{sn}^2 u} = \frac{1}{\alpha^2}[\Pi(\alpha^2, k) - K(k)], \tag{A9}$$

where $K(k)$ and $\Pi(\alpha^2, k)$ are the complete elliptic integrals of the first and third kind, respectively.

APPENDIX B: SEMICLASSICAL QUANTIZATION

For the spatially homogeneous time-dependent solution (24) - (26), the canonically conjugate variables q_i and p_i given by Eq. (30) can be expressed in terms of elliptic functions as

$$p_i = S_i^z = \gamma \mathrm{dn} u \ ; \quad u = \omega t + \delta, \tag{B1}$$

$$q_i = \arctan\left(\frac{S_i^y}{S_i^x}\right) = \arctan\left(-\sqrt{\frac{1-\gamma^2}{1-\gamma^2 k'^2}} \frac{\mathrm{cn} u}{\mathrm{sn} u}\right). \tag{B2}$$

Then the left hand side of the semiclassical quantization condition (29) becomes [30]

$$\oint p_i dq_i = \gamma\sqrt{1-\gamma^2}\sqrt{1-\gamma^2 k'^2} \int_0^{4K(k)} \frac{\mathrm{dn}^2 u \; du}{1 - \gamma^2 \mathrm{dn}^2 u} \tag{B3}$$

$$= \frac{4}{\gamma}\sqrt{\frac{1-\gamma^2 k'^2}{1-\gamma^2}} \left[\Pi\left(\frac{-\gamma^2 k^2}{(1-\gamma^2)}, k\right) - (1-\gamma^2) K(k)\right], \tag{B4}$$

where $K(k)$ and $\Pi(\nu, k)$ are the complete elliptic integrals of the first and third kind, respectively.

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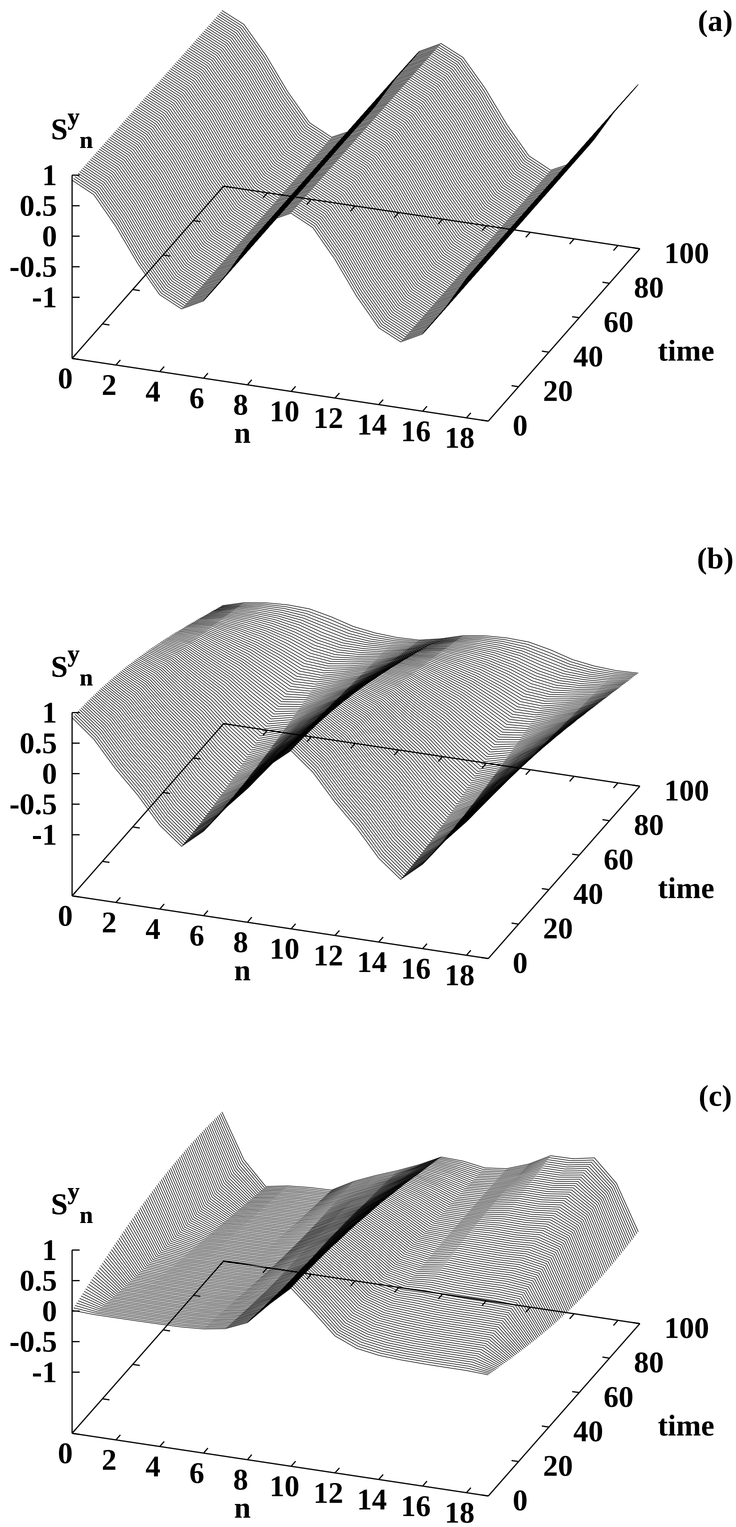

FIG. 6: Time evolution of the S_n^y component of the static solution, Eqs. (46) - (47), under perturbation, (a) $k = 0.3$, (b) $k = 0.9$ and (c) $k = 1$. As can be noticed, for small values of k, the solution tends to be more stable for long periods of time.

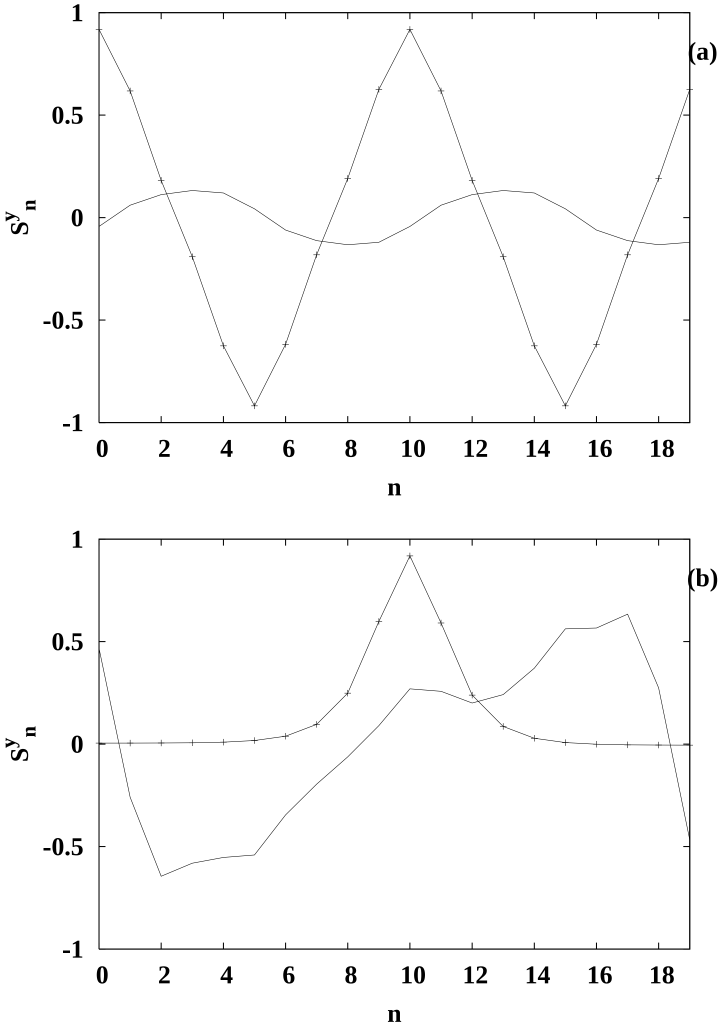

FIG. 7: Initial (line-points) and final (line) profiles of the static solution (46) - (47) under perturbation (Eq. (76)) in (a) Figure 6(b) with $k = 0.9$ and (b) Figure 6(c) with $k = 1$. Instabilities start to appear at earlier time as k approaches 1.