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Evanescent waves and the van Cittert Zernike theorem in cylindrical geometry

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Abstract. The cylindrical angular spectrum of the wavefield is introduced. In this representation the field consists of homogeneous as well as evanescent waves. The representation is applied to propagation problems and an analogue of van Cittert Zernike theorem is obtained in cylindrical geometry.

Keywords. Evanescent waves; van Cittert Zernike theorem.

The angular momentum representation has been found extremely useful while considering the propagation problem of wavefield from a plane boundary (Asby and Wolf 1971; Marchand and Wolf 1972; Jaiswal *et al* 1973; Agrawal and Mehta 1975). It is observed that in such a representation the evanescent waves always accompany the homogeneous waves. The evanescent waves, which are exponentially decaying surface waves are indeed observed while considering, for example, the total internal reflection of plane waves. A natural question arises: what happens when the plane boundary is replaced by a non-planar one? The normal mode solutions of the wave equation in cases such as cylindrical and spherical geometries are fairly well known (Stratton 1941). While studying the surface waves, Rupin and Englman (1970) observed the existence of evanescent waves (also called the non-radiative modes) along with the homogeneous waves (or radiative modes) in cylindrical geometry but no such waves are observed for spherical geometry. In this paper we briefly discuss the existence and non-existence of evanescent waves in cylindrical and spherical geometries respectively and suggest that the question of the existence of evanescent waves is associated with the condition that the corresponding eigenvalue spectrum is continuous or discrete. This in turn depends on whether the surface is open (extends to infinity) or is bounded in the particular direction. We also consider the application of the cylindrical or spherical wave representation to wave propagation when the field is specified over a cylindrical or spherical boundary. An analogue of van Cittert-Zernike theorem is derived in the cylindrical case.

For obtaining the angular spectrum representation when the field is initially specified over a plane boundary $z=0$, the Helmholtz equation $(\nabla^2+k^2)v(\mathbf{r})=0$ satisfied by the wavefield $v(\mathbf{r})$ is solved using the method of separation of variables in cartesian coordinates. This involves two separation constants which are to be determined from the boundary conditions. These conditions allow only certain restricted values, called the eigenvalues, for the separation constants and the corresponding solutions of the Helmholtz equation are called the eigenfunctions. In Cartesian geometry the eigenfunction is exp $\{ik\,(px+qy+mz)\}$, where p and q are two independent eigenvalue parameters ranging throughout the real plane and $m=(1-p^2-q^2)^{\frac{1}{2}}$. Following the linearity of the Helmholtz equation the general solutions is then expressed in the form:

$$v(\mathbf{r})=\int_{-\infty}^{\infty}\int A\,(p,q)\exp\{ik\,(px+qy+mz)\}dp\,dq. \tag{1}$$

The function $A(p, q)$ is called the angular spectrum and is the Fourier transform of the boundary field $v(\xi, \eta)$;

$$A(p,q)=(k/2\pi)^2\int_{-\infty}^{\infty}\int v(\xi,\eta)\exp\{-ik(p\xi+q\eta)\}\,d\xi\,d\eta, \tag{2}$$

where ξ and η are the cartesian coordinates in the plane $z=0$.

The origin of the evanescent waves lies in the fact that for p and q such that $p^2+q^2>1$, m is imaginary and the eigenfunction represents a surface wave decaying exponentially along positive z-direction. In the language of the optical information processing, the eigenvalues (p, q) are related to the spatial frequencies (kp, kq). Thus, the evanescent waves are high frequency components in the spatial frequency decomposition of the optical field and carry information about the field variations over distances smaller than the wavelength.

We follow an analogous method for the cylindrical geometry. It is assumed that all the sources are confined within a cylinder of radius ρ_o and the boundary field is specified over the surface of this cylinder. The solution of the Helmholtz equation in the source free region $\rho>\rho_o$ is obtained by employing separation of variables in terms of cylindrical coordinates (ρ, θ, z). The procedure is straightforward, and one finds that the eigenfunctions in this case are (Stratton 1941):

$$H_m^{(1)}\,(kt\rho)e^{im\theta}\,e^{ikpz}, \tag{3}$$

where

$$t=(1-p^2)^{\frac{1}{2}}, \tag{4}$$

(p, m) are the two separation constants, and $H_m^{(1)}$ is the Hankel function (Watson 1966) of the first kind.* The continuity of the wavefield requires that m is restricted to integral values only. However p may take any value in the interval $(-\infty, \infty)$. It is to be noted that in contrast to the angular spectrum representation where both eigenvalue spectra (p, q) are continuous, in the present case one of the eigenvalue spectra m becomes discrete, the other still being continuous. The optical information is now being carried by discrete spatial frequencies. The general solution of the field $v(\mathbf{r})$ is given by

$$v(\mathbf{r})=\sum_m\int_{-\infty}^{\infty}A_m(p)H_m^{(1)}\,(kt\rho)\exp\{ikpz+\mathrm{i}m\theta\}\,dp, \tag{5}$$

*If we had chosen a negative sign in the definition of t [eq. (4)], we would find that the eigenfunction (3) will then involve Hankel function of the second kind.

where the cylindrical spectrum $A_m(p)$ is the Fourier transform of the boundary field $v(\rho_0, \theta_0, z_0)$:

$$A_m(p) = \frac{k}{4\pi^2}\{H_m^{(1)}(kt\rho_0)\}^{-1} \int_0^{2\pi} d\theta_0 \int_{-\infty}^{\infty} dz_0\, v(\rho_0, \theta_0, z_0) \exp\{-ikpz_0 - \mathrm{i}m\theta_0\}. \quad (6)$$

Equation (5) gives the normal mode expansion of the field. In the case of cartesian coordinates all the mode eigenfunctions were simple exponential functions, whereas in the present case, we also get the Hankel function. As such it is difficult, in general, to separate the field into outgoing and decaying waves analogous to the angular spectrum case. In the far field case ($\rho \to \infty$), however, we may characterise the field with two types of waves:

(i) *Homogeneous cylindrical waves:* For the values of p such that $|p| \leqslant 1$, t is real and the Hankel function $H_m^{(1)}(kt\rho)$ behaves asymptotically as an outgoing wave. The contribution to the integral (5) for these values of p may be referred to as homogeneous cylindrical waves.

(ii) *Evanescent cylindrical waves:* For the values of p such that $|p| > 1$, t, becomes imaginary and the Hankel function in this case is an exponentially decreasing function of ρ as is evident from the aymptotic expression (Watson 1966).

$$H_m^{(1)}(\alpha) \sim (2/\pi\alpha)^{1/2} \exp\{i(\alpha - \tfrac{1}{2}m\pi - \tfrac{1}{4}\pi)\}. \quad (7)$$

The waves $\exp\{i(kpz + m\theta)\}$ for these values of p will be confined to the surface of the cylinder and will be decaying exponentially with increasing ρ. In analogy with the angular spectrum case, we call these waves the 'evanescent cylindrical waves'.

It is worth pointing out that such a clear separation into homogeneous and evanescent waves has been valid only in the far field region ($k\rho \gg 1$). In the 'not so far-region' one finds (Berry 1975) that decaying waves may also occur when $|p| < 1$ (including the case when† $p = 0$). The waves decay as ρ increases away from the source until a certain value of ρ is reached when they change over to outgoing homogeneous waves which persist on to $\rho = \infty$.

We now briefly consider the case of the spherical geometry. Carrying out the separation of variables in the spherical coordinates ($\mathbf{r}, \theta, \phi$) the eigenfunction solution of the Helmholtz equation is given by (Stratton 1941).

$$v(\mathbf{r}) = \sum_m \sum_n A_{mn}\, h_n^{(1)}(k\mathbf{r})\, P_n^m(\cos\theta)\, e^{im\phi} \quad (8)$$

where $h_n^{(1)}$ is the spherical Hankel function of the first kind, P_n^m are the associated Legendre polynomials and (m, n) are the eigenvalues. The interesting point to be noted is that now the eigenvalue spectra for both m and n are discrete. We do not find any values of m and n such that the spherical Hankel function $h_n^{(1)}(k\mathbf{r})$ decays along $\mathbf{r}$ and consequently there will never be be evanescent waves on the surface of the sphere.

† This case must be handled with care, since the z-dependence is now not oscillatory, but is of the form $\alpha + \beta z$ where α and β are constants.

The absence of the evanescent waves in the spherical geometry is curious. Looking back we find that these waves can exist if and only if there is at least one eigenvalue whose spectrum is continuous. How is this related to the surface structure can be seen as follows: If the surface is open along a particular direction (i.e. it extends up to infinity) the spectrum is continuous while for the direction in which surface is bounded, the spectrum is necessarily discrete. An analogous situation occurs in quantum mechanics also where one obtains discrete or continuous eigenvalue spectrum depending on the potential well being finite or infinite.

We now consider the propagation problem. In cylindrical geometry eq. (5) together with (6) is the general solution. As an application consider the special case when the field distribution over the boundary is uniform so that $v(\rho_0, \theta_0, z_0)$ is a function of ρ_0 only. We then obtain

$$v(\mathbf{r}) = T(\rho, \rho_0)\, v(\rho_0), \tag{9}$$

where the transfer function $T(\rho, \rho_0) = H_0^{(1)}(k\rho)/H_0^{(1)}(k\rho_0)$ is a function of ρ only. In particular if a line source is situated along z-axis, the field on the boundary cylinder is given by the two-dimensional Green's function $v(\rho_0) \sim H_0^{(1)}(k\rho_0)$ and from (9) we find $v(\mathbf{r}) \sim H_0^{(1)}(k\rho)$. This merely indicates that a cylindrical wavefront remains cylindrical on propagation.

The general solution (5) gets little simplified in the usual far field approximation $k\rho \to \infty$. Using eq. (7) we rewrite (5) in this asymptotic approximation as

$$v(\mathbf{r}) = \sum_m \left(\frac{2}{\pi k\rho}\right)^{1/2} \exp\left\{im\left(\theta - \frac{\pi}{2}\right) - \frac{i\pi}{4}\right\} \int_{-\infty}^{\infty} (1-p^2)^{-1/4} A_m(p)$$
$$\times \exp\{ik\,\rho(1-p^2)^{1/2} + ikpz\}\, dp. \tag{10}$$

We now apply the method of stationary phase (Copson 1971) for evaluting the integral over p. We also assume $z \ll \rho$. We then find that

$$v(\mathbf{r}) = \frac{2}{k\rho}\, e^{i\psi} \sum_m A_m\left(\frac{z}{\rho}\right) e^{im(\theta-\pi/2)}, \tag{11}$$

where

$$\psi = k\rho + \tfrac{1}{2}(kz^2/\rho) - \tfrac{1}{2}\pi. \tag{12}$$

Also using eq. (6) we obtain

$$v(\mathbf{r}) = \frac{1}{2\pi^2\rho}\, e^{i\psi} \sum_m \{H_m^{(1)}(k\rho_0)\}^{-1} e^{im(\theta-\pi/2)}$$
$$\times \int_0^{2\pi} d\theta_0 \int_{-\infty}^{\infty} dz_0\, v(\rho_0, \theta_0, z_0)\, e^{-i(kz_0+m\theta_0)}. \tag{13}$$

In particular, if the source is circularly symmetric, i.e. if $v(\rho_0, \theta_0, z_0)$ does not depend on θ_0, only one term $m=0$ contributes in the summation over m. We then obtain from eq. (11)

$$v(\mathbf{r}) = \frac{2}{k\rho} e^{i\phi} A_0\left(\frac{z}{\rho}\right) \tag{14}$$

and from eq. (13)

$$v(\mathbf{r}) = \frac{1}{\pi\rho} e^{i\phi} \left\{ H_0^{(1)} (k\rho_0)\right\}^{-1} \int_{-\infty}^{\infty} v(\rho_0, z_0)\, e^{-ikzz_0/\rho}\, dz_0. \tag{15}$$

One may use this procedure also to consider the propagation of coherence function, etc. The mutual spectral density $W(\mathbf{r}_1, \mathbf{r}_2, k)$ which is the time-Fourier transform of the second order coherence function satisfies the two wave equations (Mehta and Wolf 1967)

$$(\nabla_i^2 + k^2)\, W(\mathbf{r}_1, \mathbf{r}_2, k) = 0;\; i = 1, 2. \tag{16}$$

We assume that $W(\mathbf{r}_1, \mathbf{r}_2, k)$ is given for the points $\mathbf{r}_1$ and $\mathbf{r}_2$ on a cylinder of radius ρ_0 and we are interested in obtaining the same for the points on a cylinder of radius ρ when $k\rho \gg 1$. In analogy with eq. (13), we thus find that

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi^4\rho^2} e^{i\phi} \sum_{m_1, m_2} \left\{ H_{m_1}^{(1)*} (k\rho_0)\, H_{m_2}^{(1)} (k\rho_0)\right\}^{-1}$$

$$\times \exp\left\{-im_1 (\theta_1 - \tfrac{1}{2}\pi) + im_2 (\theta_2 - \tfrac{1}{2}\pi)\right\} \int \ldots \int W(\vec{\rho}_{01}, \vec{\rho}_{02})$$

$$\exp\left\{ik(z_1 z_{01} - z_2 z_{02}) + i(m_1\theta_{01} - m_2\theta_{02})\right\} d\theta_{01}\, d\theta_{02}\, dz_{01}\, dz_{02}, \tag{17}$$

where

$$\phi = \tfrac{1}{2}k\, (z_1^2 - z_2^2)/\rho, \tag{18}$$

and we have suppressed the explicit k dependence of W.

We now consider two special cases when eq. (17) gets particularly simplified:

Cylindrical symmetry

In case when $W(\vec{\rho}_{01}, \vec{\rho}_{02})$ does not depend on θ_{01} and θ_{02}, we find that only $m_1 = m_2 = 0$ term contributes and we obtain from eq. (17), the relation,

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\pi^2\rho^2} e^{i\phi}\, |\, H_0^{(1)} (k\rho_0)\,|^{-2} \iint_{-\infty}^{\infty} dz_{01}, dz_{02}$$

$$\times\; W(\rho_0, z_{01}, z_{02}) \exp\left\{i(k/\rho)\, (z_1\, z_{01} - z_2\, z_{02})\right\}. \tag{19}$$

Further if the cylinder corresponds to an incoherent surface, i.e. if

$$W(\rho_0, z_{01}, z_{02}) \simeq \frac{2\pi}{k} I(\rho_0, z_0)\, \delta\, (z_{01} - z_{02}), \tag{20}$$

we obtain

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{2}{\pi k\rho^2} e^{i\phi}\, |\, H_0^{(1)} (k\rho_0)\,|^{-2} \int I(\rho_0, z_0)\, e^{i(k/\rho)(z_1 - z_2)\, z_0}\, dz_0. \tag{21}$$

Here $I(\rho_0, z_0)$ denotes the spectral density (the intensity per unit frequency interval) of the field. We thus find that cross-spectral density in the far field is proportional to the one dimensional Fourier transform of the intensity distribution (spectral density) across the source cylinder.

Boundary cylinder large ($k\rho_0 \gg 1$)

Equation (17) also gets simplified in the practically interesting case when $k\rho_0 \gg 1$. In this case we can use the asympotic expression (eq. 7) for $H_m^{(1)}(k\rho_0)$ and obtain,

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{k\rho_0}{2\pi\rho^2} e^{i\phi} \iint W(\rho_0, \theta_1, z_{01}, \rho_0, \theta_2, z_{02}) \times \exp\{i(k/\rho)(z_{01}z_1 - z_{02}z_2)\}\, dz_{01}\, dz_{02}. \tag{22}$$

In deriving eq. (22) we have used the relation

$$\sum_{m=-\infty}^{\infty} e^{im\theta} = 2\pi\delta(\theta). \tag{23}$$

Again if the boundary cylinder corresponds to an incoherent surface* so that

$$W(\rho_0, \theta_1, z_{01}, \rho_0, \theta_2, z_{02}) \simeq \frac{1}{k} I(\rho_0, \theta_1, z_{01})\, \delta(z_{01} - z_{02})\, \delta(\theta_1 - \theta_2), \tag{24}$$

we obtain the following relation for $W(\mathbf{r}_1, \mathbf{r}_2)$,

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{\rho_0}{2\pi\rho^2} e^{i\phi}\, \delta(\theta_1 - \theta_2) \int I(\rho_0, \theta_1, z_0) \exp\{i(k/\rho)(z_1 - z_2) z_0\}\, dz_0. \tag{25}$$

Equations (21) and (25) are the analogues of the van Cittert Zernike theorem in cylindrical geometry.

One can solve the propagation problem for a spherical source as well in an analogues way. In particular we find that the transfer function analogues to eq. (9) in this case is

$$T(\mathbf{r}, \mathbf{r}_0) = h_0^{(1)}(k\mathbf{r})/h_0^{(1)}(k\,\mathbf{r}_0), \tag{26}$$

where $h_0^{(1)}(k\mathbf{r})$ is the spherical Hankel function. We refer to a paper by Marathay for the construction of the Green's function in this geometry.

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*It may be observed that this condition is different from the assumption that $W(\rho_{01}, \rho_{02})$ is independent of θ_{01} and θ_{02} as in the earlier case [cf. eq. (19)].

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