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Squeezed vacuum as an eigenstate of two-photon annihilation operator

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Abstract. We introduce the inverse annihilation and creation operators $\hat{a}^{-1}$ and $\hat{a}^{\dagger -1}$ by their actions on the number states. We show that the squeezed vacuum $\exp(\frac{1}{2}\xi\hat{a}^{\dagger 2}]|0\rangle$ and squeezed first number state $\exp[\frac{1}{2}\xi\hat{a}^{\dagger 2}]|n=1\rangle$ are respectively the eigenstates of the operators $(\hat{a}^{\dagger -1}\hat{a})$ and $(\hat{a}\hat{a}^{\dagger -1})$ with the eigenvalue ξ.

Keywords. Coherent states; squeezed states; squeezed vacuum; two-photon annihilation operators.

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1. Introduction

There are several representations of the harmonic oscillator states. The extensively studied states are the number states, the coherent states [1] and the squeezed states [2–6]. The squeezed states are particularly important due to their potential application in noise reduction.

The number states are the eigenstates of the number operator $\hat{a}^\dagger\hat{a}$:

$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

whereas the coherent states are the eigenstates of the annihilation operator $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.2}$$

Here $\hat{a}$ and $\hat{a}^\dagger$ are the boson annihilation and creation operators satisfying the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ and α is an arbitrary complex number.

It is interesting to note that the ground state $|0\rangle$ corresponding to $n = 0$ is also a coherent state with $\alpha = 0$. This state, also called the vacuum state, is an eigenstate of the annihilation operator $\hat{a}$ with eigenvalue zero. A unitarily transformed vacuum $\hat{U}|0\rangle$ will then be an eigenstate of the unitarily transformed operator $\hat{U}\hat{a}\hat{U}^\dagger$ with zero eigenvalue:

$$\hat{U}\hat{a}\hat{U}^\dagger(\hat{U}|0\rangle) = 0. \tag{1.3}$$

Here we have used the unitarity relation of the operator $\hat{U}$:

$$\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = I. \tag{1.4}$$

If we choose $\hat{U}$ to be the displacement operator

$$\hat{D}(\alpha)=\exp(\alpha\hat{a}^{\dagger}-\alpha^{*}\hat{a}), \tag{1.5}$$

we get the coherent state eigenvalue equation

$$(\hat{a}-\alpha)|\alpha\rangle=0. \tag{1.6}$$

Similarly, if we choose $\hat{U}$ to be the squeeze operator

$$\hat{S}(\sigma)=\exp[\tfrac{1}{2}(\sigma\hat{a}^{\dagger 2}-\sigma^{*}\hat{a}^{2})], \tag{1.7}$$

we obtain

$$(\hat{a}\cosh r-\hat{a}^{\dagger}e^{i\phi}\sinh r)|\sigma,0\rangle=0, \tag{1.8}$$

where we have written $|\sigma,0\rangle$ for squeezed vacuum, i.e.,

$$|\sigma,0\rangle=\hat{S}(\sigma)|0\rangle \tag{1.9}$$

and

$$\sigma=re^{i\phi}. \tag{1.10}$$

While (1.6) can be expressed as the eigenvalue equation $\hat{a}|\alpha\rangle=\alpha|\alpha\rangle$, we note that such a separation is not possible for (1.8), where the operator could be made independent of the parameter σ.

In this paper we aim to obtain a new eigenvalue equation for the squeezed vacuum which does not have the above mentioned limitation. Our motivation is, therefore, to look for an operator independent of σ of which the state $|\sigma,0\rangle$ is an eigenstate. In fact, we consider a slight generalization: In place of the usual squeeze operator (1.7) we let the exponential of a general quadratic $[\tfrac{1}{2}(\sigma\hat{a}^{\dagger 2}-\sigma^{*}\hat{a}^{2})+i\beta(\hat{a}^{\dagger}\hat{a}+\hat{a}\hat{a}^{\dagger})]$ act on to the vacuum. By making use of the normal ordered form of this exponential operator, we may express the state

$$\exp[\tfrac{1}{2}(\sigma\hat{a}^{\dagger 2}-\sigma^{*}\hat{a}^{2})+i\beta(\hat{a}^{\dagger}\hat{a}+\hat{a}\hat{a}^{\dagger})]|0\rangle \tag{1.11}$$

in the form [7]

$$(1-|\xi|^{2})^{1/4}\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})|0\rangle, \tag{1.12}$$

with

$$\xi=\frac{\sigma}{(\lambda\coth\lambda-2i\beta)}, \tag{1.13}$$

where $\lambda^{2}=(|\sigma|^{2}-4\beta^{2})$ and β is real.

In §2 we introduce the inverse operators $\hat{a}^{-1}$ and $\hat{a}^{\dagger^{-1}}$ and discuss some of the commutation relations obeyed by them. In §3 the desired eigenvalue equation is derived and we show that the state (1.12) is an eigenstate of the operator $(\hat{a}^{\dagger^{-1}}\hat{a})$. Also, we show that the squeezed first number state $\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})|n=1\rangle$ is the eigenstate of another operator $(\hat{a}\hat{a}^{\dagger^{-1}})$.

2. Inverse operators

We introduce inverse of the operators $\hat{a}$ and $\hat{a}^{\dagger}$ by their actions on the number states $|n\rangle$. These operators $\hat{a}^{-1}$ and $\hat{a}^{\dagger^{-1}}$ are defined as [8]

$$\hat{a}^{-1}|n\rangle=\frac{1}{(n+1)^{1/2}}|n+1\rangle,\qquad n=0,1,2,\ldots, \tag{2.1a}$$

$$\hat{a}^{\dagger^{-1}}|n\rangle = \frac{1}{\sqrt{n}}|n-1\rangle, \qquad n = 1, 2, \ldots, \tag{2.1b}$$

$$= 0, \qquad n = 0. \tag{2.1c}$$

From (2.1) one can see that $\hat{a}^{-1}$ behaves as a creation operator, while $\hat{a}^{\dagger^{-1}}$ behaves as an annihilation operator. The matrix elements of these operators are given by the following relations:

$$\langle m|\hat{a}^{-1}|n\rangle = \frac{1}{(n+1)^{1/2}}\delta_{m,n+1}, \tag{2.2a}$$

$$\langle m|\hat{a}^{\dagger^{-1}}|n\rangle = \frac{1}{\sqrt{n}}(1-\delta_{n,0})\delta_{m,n-1}. \tag{2.2b}$$

It is readily seen that $\hat{a}^{-1}$ is the right inverse of $\hat{a}$ and $\hat{a}^{\dagger^{-1}}$ is the left inverse of $\hat{a}^{\dagger}$, i.e.

$$\hat{a}\hat{a}^{-1} = \hat{a}^{\dagger^{-1}}\hat{a}^{\dagger} = I, \tag{2.3}$$

whereas

$$\hat{a}^{-1}\hat{a} = \hat{a}^{\dagger}\hat{a}^{\dagger^{-1}} = I - |0\rangle\langle 0|. \tag{2.4}$$

Here $|0\rangle\langle 0|$ is the projection operator on to the vacuum. We find that these operators satisfy the following commutation relations:

$$[\hat{a}, \hat{a}^{-1}] = [\hat{a}^{\dagger^{-1}}, \hat{a}^{\dagger}] = |0\rangle\langle 0|, \tag{2.5}$$

$$[\hat{a}, \hat{a}^{\dagger k}] = k\hat{a}^{\dagger(k-1)}, \tag{2.6}$$

$$[\hat{a}^{\dagger}, \hat{a}^{k}] = -k\hat{a}^{(k-1)}, \tag{2.7}$$

where k is any positive or negative integer.

3. The eigenvalue equation

As mentioned earlier, the squeezed vacuum [eq. (1.11)] may be expressed in the form:

$$|\xi, 0\rangle = (1-|\xi|^2)^{1/4}\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})|0\rangle. \tag{3.1}$$

We are, therefore, looking for an operator of which the state (3.1) is an eigenstate. Since the generation of a squeezed vacuum is essentially a two photon process, we consider the two photon annihilation operator $(\hat{a}^{\dagger^{-1}}\hat{a})$. From eq. (2.3) and the relation

$$[\hat{a}, \exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})] = \xi\hat{a}^{\dagger}\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2}), \tag{3.2}$$

we find that

$$[\hat{a}^{\dagger^{-1}}\hat{a}, \exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})] = \xi\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2}) + [\hat{a}^{\dagger^{-1}}, \exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})]\hat{a}. \tag{3.3}$$

If we let (3.3) act on the vacuum, we obtain

$$[\hat{a}^{\dagger^{-1}}\hat{a}, \exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})]|0\rangle = \xi\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})|0\rangle. \tag{3.4}$$

Equation (3.4) is of interest since it can be expressed in the form

$$\hat{a}^{\dagger^{-1}}\hat{a}[\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})|0\rangle] = \xi[\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})|0\rangle], \tag{3.5}$$

which is an eigenvalue equation. Hence, we conclude that the squeezed vaccum (3.1), i.e., $\exp[\frac{1}{2}\xi\hat{a}^{\dagger 2}]|0\rangle$ is an eigenstate of the operator $(\hat{a}^{\dagger^{-1}}\hat{a})$ with an eigenvalue ξ.

If we consider another two photon annihilation operator $(\hat{a}\hat{a}^{\dagger^{-1}})$, we find that

$$[\hat{a}\hat{a}^{\dagger^{-1}}, \exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})] = \xi\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})\hat{a}^{\dagger}\hat{a}^{\dagger^{-1}} + \hat{a}[\hat{a}^{\dagger^{-1}}, \exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})]. \tag{3.6}$$

On letting (3.6) operate on the first number state $|n=1\rangle$ we find that

$$[\hat{a}\hat{a}^{\dagger^{-1}}, \exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})]|n=1\rangle = \xi\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})|n=1\rangle. \tag{3.7}$$

One can write immediately from this equation that

$$\hat{a}\hat{a}^{\dagger^{-1}}[\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})|n=1\rangle] = \xi[\exp(\tfrac{1}{2}\xi\hat{a}^{\dagger 2})|n=1\rangle]. \tag{3.8}$$

Hence, we conclude that the squeezed first number state, i.e., $\exp[\frac{1}{2}\xi\hat{a}^{\dagger 2}]|n=1\rangle$, is an eigenstate of the operator $(\hat{a}\hat{a}^{\dagger^{-1}})$ with an eigenvalue ξ.

References

[1] R J Glauber, *Phys. Rev.*, **131**, 2766 (1963)
[2] D F Walls, *Nature (London)* **306**, 141 (1983) and references therein
[3] R E Slusher, L W Hollberg, B Yurke, J C Mertz and J F Valley, *Phys. Rev. Lett.*, **55**, 2409 (1985)
[4] R M Shelby, M D Levenson, S H Perlmutter, R G Devoe and D F Walls, *Phys. Rev. Lett.*, **57**, 691 (1986)
[5] L A Wu, H J Kimble, J L Hall and H Wu, *Phys. Rev. Lett.*, **57**, 2520 (1986)
[6] R M Shelby, M D Levenson, D F Walls, A Aspect and G J Milburn, *Phys. Rev.* **A33**, 4008 (1986)
[7] Anil K Roy and C L Mehta, *J. Mod. Opt.*, **39**, 1619 (1992)
[8] C L Mehta, Anil K Roy and G M Saxena, *Phys. Rev.* **A46**, 1565 (1992)