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Nonlinear dynamics: Challenges and perspectives

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Abstract. The study of nonlinear dynamics has been an active area of research since 1960s, after certain path-breaking discoveries, leading to the concepts of solitons, integrability, bifurcations, chaos and spatio-temporal patterns, to name a few. Several new techniques and methods have been developed to understand nonlinear systems at different levels. Along with these, a multitude of potential applications of nonlinear dynamics have also been enunciated. In spite of these developments, several challenges, some of them fundamental and others on the efficacy of these methods in developing cutting edge technologies, remain to be tackled. In this article, a brief personal perspective of these issues is presented.

Keywords. Nonlinear dynamics; integrability; chaos; patterns; applications.

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1. Introduction

Though the field of nonlinear dynamics is of recent origin, only three or four decades old, the genesis of the basic ideas can be traced to the ingenious works of the giants of the past. To name a few of them, the following may be cited (see for example [1,2]:

- John Scott-Russel's observation of solitary wave and its remarkable stability properties in the year 1837.
- Sophia Kovalevksaya's analysis of rigid body problem based on singularity structure analysis in 1888 and the subsequent classification of nonlinear ordinary differential equations by Painlevé and co-workers.
- Poincaré's analysis on the sensitive dependence on initial conditions of nonlinear systems in the beginning of the 20th century.
- van der Pol's observations of chaotic oscillations in an electrical circuit in 1927.
- Einstein's insistence on the importance of nonlinear field theories in understanding the structure of matter.
- Fisher's formulation of gene propagation in terms of a nonlinear diffusion equation.

In spite of these important early developments, it is generally accepted that the modern beginning of the field of nonlinear dynamics can be traced to the far-reaching and completely unexpected results on the energy sharing among the various normal modes of an anharmonic lattice in the famous Fermi–Pasta–Ulam (FPU) numerical experiments of 1955 [3]. Attempts to explain the unexpected results of FPU experiments, and the ensuing interest in the analysis of nonlinear systems, led eventually to the notions of solitons and chaos during the 1960s: Zabusky and Kruskal's numerical experiments [4] on the collision properties of solitary waves of Korteweg–de Vries equations leading to the concept of solitons and the eventual development of the inverse scattering transform method (IST) [5] to identify completely integrable infinite dimensional nonlinear dynamical systems [6], Hénon–Heiles analysis of two-dimensional nonlinear Hamiltonian systems and the path-breaking investigations of Lorenz on a simplified version of atmospheric dynamics ultimately leading to the concepts of highly sensitive dependence of dynamics on initial conditions and chaos [7–9].

The period since 1970 has seen an explosive growth of investigations [9] of the different aspects of integrable and chaotic systems, including spatio-temporal patterns, leading to very many advances in nonlinear dynamics. These studies have led to considerable advances in our understanding of nonlinear systems both from the mathematical and physical points of view as well as in identifying potential applications. These include

- Identification and characterization of completely integrable nonlinear dynamical systems and their solution structures.
- Mathematical theory of solitons in $(1+1)$ dimensions and certain generalizations to $(2+1)$ dimensions and quantum integrable systems.
- Identification of various bifurcations and routes to chaos, characterization of chaos in low and higher dimensions.
- Different types of instabilities and spatio-temporal patterns.
- Controlling and synchronization of chaos.
- Quantum integrable systems and quantum chaotic systems.
- Applications toward real world technology like optical communications, computing, secure communication, cryptography, etc.

In spite of these multifaceted progress in understanding nonlinear systems, one very quickly recognizes the fact that what has been achieved is minimal when compared to the more challenging fundamental problems one faces while considering the totality of nonlinear systems. For example, when does a given system is integrable and when does it become non-integrable and then chaotic are highly intricate questions to answer. How does the nature of the solution distinguish between integrable, non-integrable and chaotic behaviour of a given nonlinear dynamical system? What is the nature of the solution and basic excitations in higher spatial dimensions? How to understand the various spatio-temporal patterns from the point of view of solution structures? Can nonlinear systems lead to technologically advanced concepts which can be considered to be distinctly superior to the existing technologies so as to claim that nonlinear effects do make a crucial implication in real-world life? Many such challenging and fascinating questions are to be answered in order to realize the full potentials of nonlinear effects in physical and other natural systems.

In this article, we present a brief personal perspective on the various challenging problems that require detailed investigations in the near future. To begin with, we present a brief overview of the various types of dynamical systems modeled by nonlinear ordinary and partial differential equations, differential-difference and difference equations (maps), nonlinear integral and integro-differential equations among others in §2. Integrable, non-integrable and chaotic dynamical behaviours exhibited by these systems are classified. In §3 we identify some of the most pressing issues and challenges faced in the study of nonlinear dynamics. In particular we concentrate on the notions of integrability, non-integrability and chaos, coherent and chaotic structures in higher spatial dimensions, spatio-temporal patterns and evolving technologies. Finally, we present a brief overview and outlook in §4.

2. A brief classification of nonlinear dynamical systems

Considering various nonlinear dynamical systems, it can be easily stated that these systems are modeled by nonlinear equations which arise in a wide variety of forms. These occur in all branches of science, engineering and technology as well as in other fields such as economics, social dynamics and so on [6–10]. These nonlinear equations include differential (ordinary/partial/delay), difference, differential-difference, integro-differential and integral equations.

It is well-known that with the advent of differential calculus in the 17th century, differential equations have started playing a pivotal role in scientific investigations. This is particularly so for evolutionary problems as differential equations are obvious candidates for the description of natural phenomena. Since most of the natural processes are nonlinear, it is no wonder that nonlinear differential equations arise abundantly in theoretical descriptions of physical, chemical, biological and engineering problems as well as population dynamics, economics, social dynamics, etc. [11].

2.1 *Nonlinear ordinary differential equations (ODEs) as nonlinear dynamical systems*

Nonlinear ODEs occur in numerous forms depending on the nature of the problem. These problems could be either Hamiltonian (conservative) or dissipative. The solution properties of the underlying nonlinear differential equations like integrability, non-integrability, chaos, localization, patterns, etc. depend upon the order of the highest derivative, number of dependent variables (representing the physical variables), degree of nonlinearity (which corresponds to the nature of the forces), whether homogeneous or inhomogeneous (whether time dependent forces are present or not) and the numerical values of the controlling parameters as well as on the initial conditions. A short list of standard nonlinear differential equations arising in various physical situations and their significance are indicated in table 1. For more detailed description of such systems, see refs [6–11].

Table 1. Some important nonlinear ordinary differential equations ($\cdot = \frac{\mathrm{d}}{\mathrm{d}t}$). Here standard notations for parameters are used.

Name	Equation	Significance
Logistic equation	$\dot{x} = ax - bx^2$	Population growth model
Bernoulli equation	$\dot{x} + P(t)x = Q(t)x^n$	Linearizable
Riccati equation	$\dot{x} + P(t) + Q(t)x + R(t)x^2 = 0$	Admits nonlinear superposition principle
Lotka–Volterra equation	$\dot{x} = ax - xy$, $\dot{y} = xy - by$	Population dynamics; exhibits limit cycle
Anharmonic oscillator	$\ddot{x} + \omega_0^2 x + \beta x^3 = 0$	Integrable by Jacobian elliptic function(s)
Pendulum equation	$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$	Integrable by Jacobian elliptic function(s)
Painlevé II equation	$\ddot{x} = 2x^3 + tx + \alpha$	Satisfies Painlevé property
Duffing oscillator	$\ddot{x} + p\dot{x} + \omega_0^2 x + \beta x^3 = f\cos\omega t$	Exhibits chaotic dynamics
Damped driven pendulum	$\ddot{\theta} + \alpha\dot{\theta} + \omega_0^2\sin\theta = \gamma\cos\omega t$	Exhibits chaotic dynamics
Lorenz equation	$\dot{x} = \sigma(y - x)$, $\dot{y} = -xz + rx - y$, $\dot{z} = xy - bz$	Prototypical example of chaotic motion
Hénon–Heiles system	$\ddot{x} + x + 2xy = 0$, $\ddot{y} + y + x^2 - y^2 = 0$	Hamiltonian chaos
Mackey–Glass equation	$\dot{x} + \frac{ax_\tau}{(1+x_\tau^{10})} + bx = 0$, $x_\tau = x(t-\tau), \tau$: constant	Delay differential system

2.2 *Nonlinear dynamical systems represented by partial differential equations (PDEs)*

Nonlinear physical processes which vary continuously in space such as various wave phenomena (optical/magnetic/mechanical/acoustic/electrical, etc.) are described often by nonlinear partial differential equations of different types [6–10]. They also have a long history dating back to the early days of differential calculus. For example, the basic equations of fluid dynamics, namely, the Euler equation, the Navier–Stokes equation, etc. are highly nonlinear. Many of the equations describing nontrivial surfaces in (differential) geometry are essentially nonlinear PDEs. In modern times many interesting nonlinear partial differential equations such as Korteweg–de Vries equation (describing the Scott–Russel phenomenon), sine-Gordon equation, nonlinear Schrödinger equation, Burger's equation, FitzHugh–Nagumo equation, Ginzburg–Landau equation, etc. have been identified to describe various physical phenomena. Each one of these equations possesses a fascinating history, solution properties, methods of analysis and applications. They can be further classified as integrable and non-integrable, dispersive/non-dispersive/diffusive and so on. The properties of the solutions and the corresponding physical behaviour of the nonlinear dynamical system depend heavily on the number of independent and dependent

Table 2. Some important nonlinear partial differential equations (suffix denotes partial derivative with respect to that variable).

Name	Equation	Significance
(i) *Dispersive equations*		
Kortweg–de Vries equation	$u_t - 6uu_x + u_{xxx} = 0$	Integrable soliton eqn. in (1+1) dimensions
Sine-Gordon equation	$u_{xt} = \sin u$	Integrable soliton eqn. in (1+1) dimensions
Nonlinear Schrődinger equation	$iu_t + u_{xx} + 2\|u\|^2 u = 0$	Integrable soliton eqn. in (1+1) dimensions
Heisenberg ferromagnetic spin system	$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \ \ \mathbf{S} = (S_1, S_2, S_3)$, $\mathbf{S}^2 = 1$	Integrable soliton eqn. in (1+1) dimensions
Kadomtsev–Petviashvili equation	$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0$, $\sigma^2 = \pm 1$	Integrable soliton eqn. in (2+1) dimensions
Davey–Stewartson equation	$iq_t + \frac{1}{2}(q_{xx} + q_{yy}) + \alpha\|q\|^2 q + q\phi = 0$, $\phi_{xx} - \phi_{yy} + 2\alpha(\|q\|^2)_{xx} = 0, \alpha = \pm 1$	Integrable soliton eqn. in (2+1) dimensions
ϕ^4 equation	$u_{tt} - u_{xx} + u - u^3 = 0$	Non-integrable equation
(ii) *Diffusive equations*		
Burgers equation	$u_t + uu_x - u_{xx} = 0$	Linearizable through Cole–Hopf transformation
FitzHugh–Nagumo equation	$u_t = u_{xx} + u - u^3 - v$, $v_t = a(u - b)$	Nerve impulse propagation model
Kuramoto–Sivashinsky equation	$u_t = -u - u_{xx} - u_{xxxx} - uu_x$	Spatio-temporal patterns and chaos
Ginzburg–Landau equation	$u_t = (a + ib)\nabla^2 u + (c + id)(\|u\|^2)u$	Spatio-temporal patterns and chaos

variables, the order, the nature of nonlinearity, initial and boundary conditions, nature of external forcing, strength of parameters, etc. A select set of such equations and their significance is indicated in table 2 in which standard notations are used.

2.3 *Nonlinear difference equations/maps*

Yet another class of nonlinear dynamical systems, which is of considerable physical and mathematical relevance, is the so-called nonlinear difference equations/recurrence equations/iterated maps [12] of the form

$$\vec{x}_{n+1} = \vec{F}(\vec{x}_n, n), \tag{1}$$

where the independent/time variable n takes discrete values $n = 0, 1, 2, \ldots$, and $\vec{x}_n$ stands for m dependent variables, while $\vec{F}$ is an m-dimensional nonlinear function. Equations of the form (1) often correspond to various finite-difference schemes of nonlinear ODEs/PDEs; explicit Euler, implicit midpoint, forward difference, etc. For example, the logistic differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bx^2, \tag{2}$$

where a and b are parameters, can be approximated by Euler's forward difference scheme as

$$x_{n+1} - x_n = h(ax_n - bx_n^2), \tag{3}$$

where $h > 0$. However, nonlinear difference equations can arise as dynamical systems on their own merit as in population dynamics, radioactive decay, etc., where it is more appropriate to treat time as a discrete independent variable. Also such nonlinear maps play a fundamental role as prototypical examples of chaotic dynamical systems, through which the bifurcation routes and mechanisms of onset of chaos can be analyzed more exhaustively.

Of course, the most famous and ubiquitous example of a nonlinear map is the logistic map [9]

$$x_{n+1} = \lambda x_n(1 - x_n), \quad 0 \le x \le 1, \ \ 0 \le \lambda \le 4, \tag{4}$$

which is a typical dynamical system exhibiting period doubling bifurcation route to chaos. Note that the general solution to eq. (4) cannot be written down in closed form for all values of the control parameter λ, except for the special choice $\lambda = 4$ (see eq. (14)). There are many other interesting one-dimensional maps such as the tent map, exponential map, etc. which are all well-studied in the literature.

Important two-dimensional nonlinear maps of considerable physical significance, exhibiting various bifurcations and routes to chaos, self-similar and fractal behaviour, etc. include the Hénon map, the standard map, the circle map and so on. For example, the Hénon map has the form

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + y_n, \\ y_{n+1} &= bx_n, \quad a, b > 0. \end{aligned} \tag{5}$$

Chaos synchronization and spatio-temporal patterns of very many types have been identified in coupled logistic map lattices of the form

$$x_{n+1}^j = f(x_n^j) + \varepsilon[f(x_n^{j-1}) + f(x_n^{j+1}) - 2f(x_n^j)], \quad j = 0, 1, 2, \ldots, L-1, \tag{6}$$

where $f(x) = \lambda x(1 - x)$, $0 \le x \le 1$, $0 \le \lambda \le 4$, and ε is a small parameter.

In addition, there also exist several families of integrable maps (see for example, [13]). A simple example is the McMillan map:

$$x_{n+1} + x_{n-1} = \frac{2\mu x_n}{(1 - x_n^2)}, \tag{7}$$

where μ is a parameter. Other examples include various integrable discrete Painlevé equations, Quispel–Robert–Thompson (QRT) map, etc. Several integrable nonlinear difference–difference and differential-difference equations also exist in the literature. For example, the Toda lattice equation

$$\ddot{x}_n = \exp(-(x_n - x_{n-1})) - \exp(-(x_{n+1} - x_n)), \quad n = 0, 1, 2, \ldots, N \quad (8)$$

is an integrable soliton system of great physical significance. For more details, see [6].

2.4 *Nonlinear integro-differential equations*

Apart from nonlinear differential and difference equations there exist several situations where nonlinear integro-differential equations arise naturally as dynamical systems. In fact, historically, nonlinear integro-differential equations arose with the work of Vitto Volterra (see [14]) when he introduced a hereditary factor encountered in the past as a modification to the logistic equation as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - by^2 + y\int_c^t k(t-s)y(s)\mathrm{d}s, \quad (9)$$

where a, b and c are parameters and $k(t)$ is a given function. One of the most important nonlinear integro-differential equations in physics is the Boltzmann transport equation [15] for the evolution of the distribution function $f(\vec{r}, \vec{v}, t)$ of a dilute gas,

$$\left(\frac{\partial}{\partial t} + \vec{v}_1 \cdot \vec{\nabla}_r + \frac{\vec{F}}{m} \cdot \vec{\nabla}_{v_r}\right) f(\vec{r}, \vec{v}_1, t) = \int \mathrm{d}^3 v_2 \int \mathrm{d}\Omega\sigma(\Omega)|\vec{v}_2 - \vec{v}_1|(f_1' f_2' - f_1 f_2), \quad (10)$$

where $f_i' = f(\vec{r}_i, \vec{v}_i', t), \ f_i = f(\vec{r}_i, \vec{v}_i, t), \ i = 1, 2$. In the integrable soliton case, there exist several interesting integro-differential equations. An important example is the Benjamin-One equation [6],

$$u_t + 2uu_x + Hu_{xx} = 0, \quad (11)$$

where H is the Hilbert transform. Solving and analysing such nonlinear integro-differential equations is a much more complex problem than analysing differential equations.

3. Challenging problems and perspectives

So far, we have provided a rather brief summary of the types of nonlinear dynamical systems which have arisen in the recent literature and the notions which have been developed through their analysis. The basic concepts which arise out of the study of these systems [6–10] are essentially (1) integrability, (2) soliton behaviour, (3) near integrable systems, (4) bifurcations and chaos and (5) spatio-temporal patterns, among others. Occurrence of any of these properties and the associated physical behaviour essentially depend upon the number of dynamical variables and spatial dimensions, the nature of nonlinearity and external forces, the range of control parameters, boundary and initial conditions, etc. All these aspects have to be essentially reflected in the nature of the general solution of the underlying nonlinear dynamical systems. But the point is that very rarely one is able to solve the evolution equations in any general sense. And there lies the crux of the problem in analysing nonlinear dynamical equations.

Though integrable nonlinear dynamical systems are rare to find, increasing number of them are being found in finite degrees of freedom dynamical systems, difference equations and in partial differential equations, mostly in (1+1) dimensions and occasionally in higher spatial dimensions [6–9]. Often such systems exhibit periodic or solitonic behaviour. However, non-integrability and chaos are predominant even in systems with few degrees of freedom, which are mostly identified by numerical analysis. Instabilities and spatio-temporal patterns dominate non-integrable extended nonlinear dynamical systems. However, it is obvious that one has touched only the tip of the iceberg as far as nonlinear systems are concerned and our present understanding is confined to a narrow range of them. To understand the nature of excitations in physically relevant higher dimensional systems, the transition from integrable regular systems to non-integrable systems and then to chaotic systems through different bifurcation routes and the formation of various spatio-temporal patterns on perturbation of soltion and other structures, and to understand them from the solution structure point of view, remain to be some of the most challenging problems for the future. Also the definition of integrability itself, from the point of view of exact solutions, integrals of motion, behaviour in the complex plane of the independent variables, its relation to real-time behaviour and existence and uniqueness of solutions is one of the most important fundamental notions which has to be understood and extended to non-integrable and chaotic situations. Many of the new technologies which are in the process of unfolding as a result of the various applications of the notions of solitons and chaos remain to be harnessed to their full potential, in such areas as optical soliton-based communication, optical computing, magnetoelectronics, secure communications, chaotic cryptography, etc. In this section, we will give a brief view of some of these problems.

3.1 *Integrability and chaos*

All soliton systems (which are of nonlinear dispersive type) are completely integrable infinite dimensional nonlinear dynamical systems of Hamiltonian type [6]. They also satisfy the Painlevé property in the sense that the solutions are meromorphic and are free from movable critical singular manifolds of the independent variables. From a mathematical point of view existence and uniqueness of solutions in these cases can also be established. On the other hand, certain nonlinear diffusive equations are also considered to be integrable: they are linearizable, admit infinite number of generalized symmetries, satisfy Painlevé property, etc. The examples include Burger's equation, Fokas–Yortsos equation, generalized Fisher equation, etc. [9].

In the case of finite degrees of freedom nonlinear dynamical systems also, there exist both integrable Hamiltonian systems and integrable dissipative systems satisfying the Painlevé property, existence of sufficient integrals, generalized symmetries, etc.

Example 1: *The generalized Emden-type equation* [16]

$$\ddot{x} + k\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x = 0 \tag{12}$$

is integrable in the sense that the Painlevé property is satisfied, sufficient integrals exist and explicit general solutions can be constructed. The system is dissipative for $\lambda_1 \leq 0$ and Hamiltonian for $\lambda > 0$ (see [16]).

Example 2: *Coupled quartic anharmonic oscillator with the potential* [17]

$$V = Ax^2 + By^2 + \alpha x^4 + \beta y^4 + \delta x^2 y^2 \tag{13}$$

is integrable only for four parametric choices where the Painlevé property is satisfied and sufficient integrals of motion exist. For other values of the parameters, the system is non-integrable, shows complicated structure of singularities in the complex t-plane showing high multi-valuedness, non-integrability and chaotic behaviour.

Example 3: *Lorenz system* [18]

Only for very specific parameter values the system is integrable and satisfies the Painlevé property. For other values, the singularity structure is very complicated and the dynamics shows regular and chaotic behaviour depending on the parameters.

Example 4: *The logistic map* [9]

The solution of the map (4) is in general cannot be expressed in terms of explicit functions except for special solutions. However, for $\lambda = 4$, the fully chaotic case, one has the explicit solution

$$x_n = \sin^2 2\pi\theta_n, \quad \theta_{n+1} = 2^n\theta_0 (\mathrm{mod}\, 1), \tag{14}$$

exhibiting sensitive dependence on initial conditions.

Example 5: *The Murali–Lakshmanan–Chua circuit equations* [19]

$$\dot{x} = y - h(x) \tag{15}$$

$$\dot{y} = -\beta(1+\gamma)y - \beta x + f\sin\omega t \tag{16}$$

with the piece-wise linear function

$$h(x) = \begin{cases} bx + a - b & x \geq 1 \\ ax & |x| \leq 1 \\ bx - a + b & x < -1, \end{cases} \tag{17}$$

where a, b, β, γ, f and ω are parameters, admits explicit solutions in different regions of x. However, the system exhibits all kinds of complex nonlinear dynamical behaviour, bifurcations and chaos.

So what is integrability? When does it arise? When is a given system non-integrable? What distinguishes near-integrable systems with chaotic systems and so on? How does the solution changes its structure at different bifurcations? Why should complex time/space singularities affect the behaviour of the system in real time/space? What is the exact relation between the type of multivaluedness in the complex plane/manifold of the independent variables and the behaviour of the system in real time-space including various bifurcations, chaos and spatio-temporal patterns? These are some of the most challenging questions at a fundamental level. Progress in these directions will substantially enhance our understanding of physical systems modeled by nonlinear interactions.

Yet another intriguing feature is the relationship between integrability, non-integrability, bifurcations and chaos with the occurrence of explicit solutions. As in the case of integrable systems, even chaotic systems do admit exact solutions (see Examples 4 and 5), yet they show all the properties of non-integrability. This shows that occurrence of explicit solutions need not be the hall mark of integrability and it will be a real intellectual adventure to identify ways and means to trace analytical solutions of such nonlinear dynamical systems to identify all dynamical behaviour in a rigorous sense.

3.2 *Nonlinear excitations in higher spatial dimensions*

While $(1+1)$-dimensional nonlinear dynamical systems exhibit both localized and coherent structures such as solitons as well as spatio-temporal chaotic structures, one would like to know whether these excitations survive in higher dimensions too and/or whether totally new kinds of structures exist in higher spatial dimensions. In fact, in the case of natural physical extensions of $(1+1)$-dimensional soliton equations such as (2+1)-dimensional versions of sine-Gordon, nonlinear Schrödinger and Heisenberg ferromagnetic spin equations of the form

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2 \sin\phi = 0, \tag{18}$$

$$iq_t + \nabla^2 q + \frac{1}{2}|q|^2 q = 0, \tag{19}$$

$$\vec{\mathbf{S}}_t = \vec{\mathbf{S}} \times \nabla^2\vec{\mathbf{S}}, \;\; \vec{\mathbf{S}}^2 = 1, \tag{20}$$

respectively, where ∇^2 is the two-dimensional or three-dimensional Laplacian, do not seem to possess soliton-like exponentially localized structures. In fact, it has been shown in recent times that the solutions of many of these equations in $(2+1)$ dimensions develop wave collapse or singularities at finite time and the solution blows up [20]. However, under special geometries, these equations may possess interesting classes of particular solutions, like time-independent spherically symmetric, axially symmetric, instanton, vortex, monopole, hedgehog and other types of solutions. But the nature of their general solutions is simply not known. This is another important problem that requires considerable analysis in the near future.

On the other hand, there are several possible ways of taming the tendency for blow up:

1. Increasing the number of physical variables corresponding to multicomponent nonlinear evolution equations.

2. Introducing coupling to additional scalar mean field variables or by the introduction of suitable non-local interactions.
3. Introducing higher order dispersive effects.
4. Introducing additional saturating-type nonlinearity or additional confining potentials, and so on.

Such modifications of physically relevant nonlinear evolution equations which give rise to desirable localized structures raise the question as to which of the models give the correct description of physical phenomena and what are the underlying dynamical processes. Typical generalized equations in (2+1) dimensions include the Davey–Stewartson equation, Ishimori equation, Novikov–Nizhnik–Veslov equations, which are the generalization of nonlinear Schrödinger, Heisenberg and Korteweg–de Vries equations, respectively. All these modified equations admit exponentially localized solutions.

In particular, the following problems in higher spatial dimensions need urgent attention.

1. What are all the possible stable structures (special solutions) in $(2+1)$ and $(3+1)$ dimensions? How stable are they? What are their collision properties? If unstable, are they metastable? If not, do they give rise to new spatio-temporal patterns? What is the effect of external forces including damping?
2. Can one develop techniques to solve the Cauchy initial value problem of physically important $(2+1)$-dimensional extensions of $(1+1)$-dimensional solution equations? Or can one obtain enough information about the nature of general excitations through numerical analysis? How can the numerics be simplified to tackle such problems?
3. Is it possible to perceive something similar to FPU experiments in $(2+1)$ and $(3+1)$ dimensions? What new phenomena are in store here? Can actual analog simulations be made with suitable miniaturization of electronic circuits so that these $(2+1)$- and $(3+1)$-dimensional systems can be analysed systematically?
4. Is it possible to extend the inverse scattering formulation to $(3+1)$-dimensional systems also as in the case of $(2+1)$-dimensional evolution equations? The main difficulty seems to arise in the inverse analysis due to certain non-uniqueness arising from constraints on the scattering data (see ref. [6]).

It is very certain that the future of nonlinear physics will be much concentrated around such higher dimensional nonlinear systems, where new understandings and applications will arise in large numbers. A long-term sustained numerical and theoretical analysis of $(2+1)$- and $(3+1)$-dimensional nonlinear evolution equations both for finite and continuous degrees of freedom will be one of the major tasks for several decades to come which can throw open many new nonlinear phenomena. Also, one might consider discretization and analog simulation of these systems, to which nonlinear electronics community can contribute much.

3.3 *Nonintegrable systems, spatio-temporal patterns and chaos*

Integrable nonlinear dynamical systems are relatively rare and fewer in number. Most natural systems are non-integrable. Yet many of them may be considered as perturbations of integrable nonlinear systems. Examples include many of the important condensed matter systems such as magnetic, electronic and lattice systems, optoelectronic systems, hydrodynamical systems and so on. Thus it is imperative to study the effect of these various additional forces with reference to the basic nonlinear excitations of integrable systems and extend the analysis to far from integrable regimes.

Such an analysis needs to consider the different length scales of the perturbation (both space and time) with respect to the nonlinear excitations of the unperturbed cases. Depending on such scales, the original entities with suitable deformations may undergo chaotic or complex motions/deformations giving rise to spatio-temporal patterns. Several preliminary studies on soliton perturbations in $(1 + 1)$ dimensions do exist in the literature (see [21,22]). Detailed classification of the types of perturbations and the resulting coherent and chaotic structures and spatio-temporal patterns can be used as dictionaries to explain varied physical situations in condensed matter, fluid dynamics, plasma physics, magnetism, atmospheric physics, biological and chemical systems, etc. Further, such studies in $(2 + 1)$-dimensional systems, wherein any stable entity when perturbed by additional weak forces can lead to exciting new structures corresponding to realistic world description, are of considerable importance and pose significant challenges. A large body of studies on instabilities and pattern formations, including various Turing patterns, spatio-temporal chaos, etc. already exist in [9]. A concerted effort through analytical and numerical investigations using integrable structures can be expected to provide further rich dividends in the future.

3.4 *Nonlinear dynamics and emerging technologies*

The growth of the field of nonlinear dynamics, which is essentially conceived as an interdisciplinary subject, depends not only on the novel and fundamental concepts that are being/will be unraveled to explain very many natural phenomena but also on the fact that these concepts can lead to new cutting edge engineering and technology applications that can replace the existing technologies. Such developments then can clearly show why the study of the effects of nonlinearities under varied circumstances is an essential ingredient of scientific enquiry. In this connection several potential applications have been envisaged at different times in recent years. These include [9] (1) nonlinear optics, (2) optical soliton-based communication in fibers, (3) optical soliton-based computing, (4) micromagnetics and magnetoelectronics, (5) synchronization of chaos and secure communication, (6) chaotic cryptography, (7) communicating by chaos, (8) computing using chaos, (9) chaos and financial markets, and so on. Each of these potential applications offer exciting engineering, technological and commercial applications. Already some of these aspects have found their way into real-world applications such as second harmonic generation, photonic switching, optical soliton communication, etc. However, the intriguing point is that while the importance and significance of nonlinear effects are indisputable, one starts wondering how far nonlinear dynamical concepts have lead to really path-breaking technological advancements at the forefront of human endeavours that can take over the existing technologies based on linear concepts. In my opinion, this is still a weak area in nonlinear dynamics in which much attention needs to be paid.

One may cite several examples to emphasize the above point.

1. Soliton-based communication in optical fibers [23,24] is a technological reality, which has been demonstrated impressively in a practical sense, where the optical soliton pulses are used as the information carrying bits in optical fibers. In fact, optical communication is considered to be an excellent example of how mathematical advances in nonlinear dynamics can lead to technological advances of considerable significance in the form of development of the nonlinear optical transmission line.

   In recent times considerable advances have been made in increasing the capacity of data transmission in soliton communication systems overcoming various inhibitory effects such as fiber loss, chirping and so on and new techniques such as dispersion management method have been developed. Multinational organizations have experimentally demonstrated the viability of soliton communication system in different ways. Even then much effort needs to be done in order to demonstrate that from a commercial point of view the soliton transmission system can replace the existing ones. One hopes for continued progress here which will clearly demonstrate the significance of nonlinear dynamical concepts. For details, see the special focus issue, Optical Solitons: Perspectives and Applications, *Chaos* **10(3)**, (2000).
2. In recent times, there has been considerable discussion on the possibility of developing all-optical computers in homogeneous bulk media such as photorefractive crystals using non-trivial optical soliton collisions pointed out by Radhakrishnan *et al* [25] which can occur during incoherent beam propagation in these materials. The corresponding system is represented by coupled nonlinear Schrödinger equations. The shape changing collisions can be represented by linear fractional transformations [26], which in turn leads to the identification of various logic gates, including the universal NAND gate, represented by multisoliton solutions. Thus in a theoretical sense one can envisage the possibility of an all-optical computer without interconnecting discrete components and information transfer is purely through light–light collisions. But the question is how far these theoretical developments can be advanced to actual physical implementation.
3. Similarly, nonlinear effect in ferromagnetic films can lead to very many practical applications in magnetoresistive recording, magneto-optical recording and development of possible magnetoelectronics. The underlying nonlinear evolution equation is the Landau–Lifshitz–Gilbert equation [27]

$$\frac{\partial \vec{S}(\vec{r},t)}{\partial t} = \vec{S} \times \vec{F}_{\text{eff}} + \lambda \vec{S} \times (\vec{S} \times \vec{F}_{\text{eff}}), \;\; \vec{S}^2 = 1, \; \vec{S} = (S_x, S_y, S_z), \quad (21)$$

   where λ is the damping constant and $\vec{F}_{\text{eff}}$ is the effective magnetic field representing exchange interaction, anisotropic interaction, external electromagnetic field interaction, etc. Much needs to be done on theoretical analysis of the above system and from an application point of view it is very important to demonstrate that nonlinear effects do play a predominant role in realizing the full potential of micromagnetics.

4. There had been considerable excitement and expectation for sometime that the notion of controlling and synchronization [19,28] of chaos can be effectively used for effective secure communication by making use of sensitive dependence of chaotic dynamical systems on initial conditions. Further, many suggestions have been given to develop effective chaos-based cryptographic systems, which find immense practical applications not only in military applications but also in such areas as Internet protocols, e-commerce, electronic voting, Internet banking, etc. All these possibilities depend critically on the fact that the chaotic attractor on which information signal is superimposed or masked cannot be reconstructed easily so that an eavesdropper will have very little chance of breaking security. However, this hope has not been realized fully, because of the definite possibility of reconstruction of chaotic attractor as demonstrated by Pérez and Cedeira [29] for the case of Lorenz system. Several modifications have been suggested in recent times but all of them have been shown to have one or more lacunae and cannot compete with existing methods. One needs to find more foolproof and easily implementable algorithms/methodologies which will demonstrate the technological applications of the notion of chaos.

Similar comments can be made about all other possible applications of nonlinear dynamical concepts. To put it in a nutshell, the cutting edge technological applications of nonlinear dynamical concepts are still lagging far behind theoretical developments. This is an area in which much progress can be made and has to be made in order to keep the momentum in the field at a satisfactory level.

4. Summary and outlook

The subject of nonlinear dynamics has come a long way from a position of insignificance to the central stage in physics and even in science as a whole. While the pace of such a development was rather slow in early days, an eventful golden era which ensued during the period 1950–1970 saw the stream-rolling of the field into an interdisciplinary topic of great relevance of scientific endeavor. I have tried to present here a rather personal perspective of some of these developments and the outstanding tasks urgently needed to be carried out in the near future.

Particularly, we have tried to stress that many new developments can come out by (i) clearly understanding the concepts of integrability and non-integrability from a unified point of view, (ii) by analysing nonlinear structures in $(2+1)$- and $(3+1)$-dimensional cases, which are more realistic, and (iii) through in-depth analysis of the effect of perturbation on integrable nonlinear systems and analysis of other non-integrable systems and classifying the types of novel spatio-temporal structures which might arise. We have also tried to point out some of the tasks and potentialities in certain emerging technology oriented topics such as magneto-electronics, optical soliton-based communications, chaotic cryptography and so on, which are the off-shoots of progress at the fundamental level.

There are numerous important topics which we have not touched upon or discussed their future development here, including such topics as nonlinearities in plasma physics, acoustics, biological physics, many areas of condensed matter physics, astrophysics, gravitational theory, detailed quantum aspects and so on. Similarly, the quest towards the ultimate theory of matter in particle physics, whichever form it may take, will ultimately be a nonlinear one. There is no doubt that nonlinearity will rule the world for many more years to come and there is scope for everybody to try his hand in the field for a better understanding of Nature.

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