Localized Coherent Structures of Ishimori Equation I through Hirota's Bilinearization method:Time dependent/Stationary boundaries

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Abstract. Ishimori equation is a $(2+1)$ dimensional generalization of the $(1+1)$ dimensional integrable classical continuous Heisenberg ferromagnetic spin equation. The richness of the coherent structures admitted by Ishimori equation I such as dromion, lump and rationally- exponentially localized solutions, have been demonstrated in the literature through $\bar{\partial}$ technique and binary Darboux transformation method. To our knowledge Hirota's method had been adopted to construct only the vortex solutions of Ishimori equation II. For the first time, the various types of localized coherent structures mentioned above have been constructed in this paper for the Ishimori equation I using the Hirota's direct method. In particular we have brought out the significance of boundaries and arbitrary functions in generating all these types of localized structures and proved that the absence of such boundaries leads only to line soliton solutions.

1. Introduction

A (2 + 1) dimensional integrable generalization of the (1+1) dimensional integrable Heisenberg ferromagnetic spin equation (isotropic Landau-Lifshitz equation) $\vec{S}_t(x,t) = \vec{S}\wedge\vec{S}_{xx}$ [1] was introduced by Ishimori in 1984 [2] to explain the dynamics of the classical spin system on a plane. Its form is

$$\vec{S}_t(x,y,t) = \vec{S}\wedge(\vec{S}_{xx}+\sigma^2\vec{S}_{yy}) + \phi_y\vec{S}_x + \phi_x\vec{S}_y, \tag{1.1a}$$

$$\phi_{xx} - \sigma^2\phi_{yy} = -2\sigma^2\vec{S}\cdot\vec{S}_x\wedge\vec{S}_y, \tag{1.1b}$$

where $\vec{S} = (S_1, S_2, S_3)$ is the three dimensional spin unit vector ($\vec{S}^2 = 1$), $\phi(x,y,t)$ is a scalar field and $\sigma^2 = \pm 1$. If $\sigma^2 = +1$, eq. (1.1) is referred to as Ishimori equation I(IE I) and if $\sigma^2 = -1$, it is referred to as Ishimori equation II(IE II). An important feature associated with eq. (1.1) is the existence of nontrivial topological invariant known as topological charge defined as

$$Q = \frac{1}{4\pi}\int\int \vec{S}\cdot\vec{S}_x\wedge\vec{S}_y dxdy \tag{1.2}$$

and the solutions of eq. (1.1) are classified in terms of the integer values of Q. Just like the (1+1) dimensional integrable spin system is geometrically equivalent [1] to the nonlinear Schrödinger equation through a moving space curve formalism, eq. (1.1) is geometrically equivalent to the Davey-Stewartson equation through a moving surface formalism [3].

The initial value problems of both IE I and IE II have been analysed by the $\bar{\partial}$ and nonlocal Riemann-Hilbert problem methods, respectively, in [4] by Konopelchenko and Matkarimov. For stationary boundaries, the initial boundary value problem for the IE I has been studied in [5] and three different types of localized solutions (soliton-soliton(ss), soliton-breather(sb), breather-breather(bb)) have been presented. The line solitons admitted by both IE I and IE II have been presented in [6]. The localized coherent structures for the IE I have been analysed in [7] for time dependent boundaries and the solutions such as rationally localized soliton, exponentially localized soliton and rationally-exponentially localized soliton have been reported using inverse scattering transform (IST) method. IE I has also been analysed through binary Darboux transformation method and different types of solutions have been constructed in terms of grammian determinants in [8]. Curiously, the Hirota's bilinearization method, which is one of the celebrated direct methods and applicable to almost all integrable soliton equations in (1+1) and (2+1) dimensions, has not been applied so far (as far as our knowledge goes) to obtain localized solutions for IE I, though it has been used by Ishimori himself [2] to obtain vortex solutions to IE II. The difficulty probably lies in introducing the boundaries in the bilinearized form appropriately for IE I. In this paper, we have successfully obtained all types of localized structures of IE I reported in [5–7] through the Hirota's bilinearization technique and presented the different types of localized structures admitted by it for the case of both time dependent and stationary boundaries.

The paper is organised as follows: In section 2, eq. (1.1) is bilinearized in laboratory coordinates through stereographic projection and Painlevé property and multiline soliton solutions for both IE I and IE II are presented for completeness. In section 3, the bilinearized version of IE I in terms of light cone coordinates is presented and the general form of the solution is given. In particular, we point out how boundaries can be introduced explicitly into the bilinearized version of eq. (1.1). The role of linear equations of modified Kadomtsev-Petviashvili in obtaining solutions of the bilinear equations is also pointed out. The different types of localized coherent structures driven by time-dependent boundaries are presented in section 4. In section 5, the behaviour of IE I in the background of stationary boundaries is analysed. Importance of boundaries in generating the various types of localized structures is also discussed. Finally, the results are summarized in section 6, where the importance of arbitrary function in expressing solutions is pointed out.

2. Bilinearization of IE

By making a stereographic projection of the spin of unit sphere on a complex plane, the spin components can be written in terms of the stereographic variable ω [9] as

$$S^+ = S_1 + iS_2 = \frac{2\omega}{1+|\omega|^2}, \quad S_3 = \frac{1-|\omega|^2}{1+|\omega|^2} \tag{2.1}$$

and eq. (1.1) takes the form

$$i\omega_t + \omega_{xx} + \sigma^2\omega_{yy} - \frac{2\omega^*}{1+|\omega|^2}(\omega_x^2 + \sigma^2\omega_y^2) - i\phi_y\omega_x - i\phi_x\omega_y = 0, \tag{2.2a}$$

$$\phi_{xx} - \sigma^2\phi_{yy} = \frac{4i\sigma^2}{(1+|\omega|^2)^2}(\omega_x^*\omega_y - \omega_x\omega_y^*). \tag{2.2b}$$

We find that this form is more convenient for further analysis as discussed below.

2.1. Painlevé singularity structure analysis

We can confirm the integrability nature of eq. (2.2), by performing a Painlevé analysis of it. Denoting ω and ω^* by F and G respectively, we rewrite eq. (2.2) as

$$(1+FG)[iF_t + F_{xx} + \sigma^2F_{yy} - i(\phi_yF_x + \phi_xF_y)] - 2G(F_x^2 + \sigma^2F_y^2) = 0, \tag{2.3a}$$

$$(1+FG)[-iG_t + G_{xx} + \sigma^2G_{yy} - i(\phi_yG_x + \phi_xG_y)] - 2F(G_x^2 + \sigma^2G_y^2) = 0, \tag{2.3b}$$

$$(1+FG)^2(\phi_{xx} - \sigma^2\phi_{yy}) = 4i\sigma^2(F_yG_x - F_xG_y), \tag{2.3c}$$

where eq. (2.3b) is the complex conjugate of eq. (2.3a). In order to carry out a singularity structure analysis of eq. (2.3), we effect the following local Laurent expansion for each dependent variable in the neighbourhood of a noncharacteristic singular manifold $\psi(x,y,t) = 0$:

$$F = \psi^m \sum_{j=0}^{\infty} F_j(x,y,t)\psi^j,\; G = \psi^n \sum_{j=0}^{\infty} G_j(x,y,t)\psi^j,\; \phi = \psi^p \sum_{j=0}^{\infty} \phi_j(x,y,t)\psi^j. \tag{2.4}$$

We now substitute eq. (2.4) into eq. (2.3) and look at the leading order behaviour of ψ. Here we come across two different branches, one at $m=0, n=-1, p=0$ and another one at $m=-1, n=0, p=0$. In both the cases, F_0, G_0 and ϕ_0 are found to be arbitrary functions of x, y and t and the resonances are found to occur at $j=-1,0,0,0,1,1$. Further analysis confirms that two arbitrary functions occur at the resonance values of $j=1$, without the introduction of any movable critical manifold, while the remaining coefficients can be expressed in terms of the earlier ones. Hence eq. (2.2) passes the Painlevé test and confirm its integrability nature.

2.2. Bilinearization in laboratory coordinates

To construct a formal Bäcklund transformation, we truncate the Laurent series at the constant level term, that is (for the case $m=-1$, $n=0$ and $p=0$)

$$F = F_0\psi^{-1} + F_1, \quad G = G_0, \quad \phi = \phi_0. \tag{2.5}$$

We can now construct the bilinear form of eq. (2.2), by considering

$$F_1 = 0.$$

Let $F_0 = g$ and $\psi = f$, then $\omega = \frac{g}{f}$. Under this transformation $\omega = \frac{g}{f}$,, where g and f are complex functions of x, y and t, eq. (2.2) can be written in terms of the Hirota's D-operators(which are defined as $D_x^i D_y^j D_t^k a(x,y,t)\cdot b(x,y,t) = (\partial_x - \partial_{x'})^i(\partial_y - \partial_{y'})^j$ $(\partial_t - \partial_{t'})^k a(x,y,t)b(x',y',t')|_{x=x',y=y',t=t'}$) as

$$(iD_t - D_x^2 - \sigma^2 D_y^2)(f^*\cdot g) = 0, \tag{2.6a}$$

$$(iD_t - D_x^2 - \sigma^2 D_y^2)(f^*\cdot f - g^*\cdot g) = 0, \tag{2.6b}$$

$$D_x(D_x(f^*\cdot f + g^*\cdot g))\cdot(f^*f + g^*g) = \sigma^2 D_y(D_y(f^*\cdot f + g^*\cdot g))\cdot(f^*f + g^*g), \tag{2.6c}$$

so that

$$\phi_x = -2i\sigma^2\frac{D_y(f^*\cdot f + g^*\cdot g)}{(f^*f + g^*g)}, \tag{2.6d}$$

$$\phi_y = -2i\frac{D_x(f^*\cdot f + g^*\cdot g)}{(f^*f + g^*g)}. \tag{2.6e}$$

Note that eq. (2.6c) arises due to the compatibility condition $\phi_{xy} = \phi_{yx}$ and it is biquadratic.

2.3. The line solitons

Now, the construction of the line soliton solutions to IE becomes standard and we briefly indicate their forms. One expands the functions g and f as power series in the arbitrary parameter ϵ as follows,

$$g = \sum_{n=0}^{\infty}\epsilon^{2n+1}g_{2n+1}, \quad f = 1 + \sum_{n=1}^{\infty}\epsilon^{2n}f_{2n}. \tag{2.7}$$

Substituting these expansions into eqs. (2.6a)-(2.6c) and equating the coefficients of various powers of ϵ, we get the respective following system of equations from (2.6a), (2.6b) and (2.6c):

$$\epsilon^{2n+1}:\ (i\partial_t+\partial_x^2+\sigma^2\partial_y^2)g_{2n+1}=-\sum_{k+m=n}D'(f_{2k}^*\cdot g_{2m+1}), \tag{2.8a}$$

$$\epsilon^{2n}:\ \ i\partial_t(f_{2n}^*-f_{2n})-(\partial_x^2+\sigma^2\partial_y^2)(f_{2n}^*+f_{2n})=$$
$$D'\left(\sum_{n_1+n_2=n-1}(g_{2n_1+1}^*\cdot g_{2n_2+1})-\sum_{m_1+m_2=n}(f_{2m_1}^*\cdot f_{2m_2})\right), \tag{2.8b}$$

$$(\partial_x^2-\sigma^2\partial_y^2)(f_{2n}^*-f_{2n})+D''\left(\sum_{n_1+n_2=n-1}(f_{2n_1}^*\cdot f_{2n_2})+\sum_{m_1+m_2=n-1}(g_{2m_1}^*\cdot g_{2m_2})\right)+$$
$$\Bigg\{D_x(f_{(2n-2)x}^*-f_{(2n-2)x}+\sum_{n_1+n_2=n-2}D_xg_{2n_1+1}^*\cdot g_{2n_2+1})-\sigma^2D_y(f_{(2n-2)y}^*-f_{(2n-2)y}+$$
$$\sum_{n_1+n_2=n-2}D_yg_{2n_1+1}^*\cdot g_{2n_2+1})\Bigg\}\cdot(f_{2n-2}^*-f_{2n-2}+\sum_{n_1+n_2=n-2}g_{2n_1+1}^*\cdot g_{2n_2+1})=0 \tag{2.8c}$$

with $D'=iD_t-D_x^2-\sigma^2D_y^2$, $D''=D_x^2-\sigma^2D_y^2$ and $f_0=0$.

2.3.1. 1-soliton solution In order to construct the exact N-soliton solutions (N-SS) of eq.(1.1), we make the ansatz

$$g_1=\sum_{j=1}^{N}\exp\chi_j,\quad \chi_j=l_jx+m_jy+n_jt, \tag{2.9}$$

where l_j, m_j and n_j are complex constants. As an example, we write the forms of g and f_2 with the help of eq. (2.8) for $N=1$ as

$$g_1=M\exp\chi_1,\quad \chi_1=l_1x+m_1y+n_1t,\quad f_2=\exp 2(\chi_{1R}+\psi), \tag{2.10}$$

where
$\chi_1=\chi_{1R}+i\chi_{1I}$, $n_1=i(l_1^2+\sigma^2m_1^2)$ and $\exp 2\psi=\frac{\sigma^2m_1^2-l_1^2}{(l_1+l_1^*)^2-\sigma^2(m_1+m_1^*)^2}MM^*$ and M is an arbitrary complex constant. Now we distinguish the two cases $\sigma^2=+1$ and $\sigma^2=-1$
Case(i): $\sigma^2=+1$ (IE I)

With the choice

$$M=\frac{(l_1+l_1^*)+i(m_1+m_1^*)}{m_1^*-il_1^*},$$

the corresponding 1-SS of IE I takes the form

$$S^+=2E\frac{(l_{1R}^2m_{1R}+m_{1R}^2l_{1I}+L)\exp i\chi_{1I}\mathrm{sech}\chi_{1R}}{A+2B\tanh\chi_{1R}+C\tanh^2\chi_{1R}}, \tag{2.11a}$$

$$S_3=1-\frac{2(l_{1R}^2-m_{1R}^2)^3\mathrm{sech}^2\chi_{1R}}{A+2B\tanh\chi_{1R}+C\tanh^2\chi_{1R}}, \tag{2.11b}$$

where

$$E=(l_{1R}+im_{1R})(l_{1R}^2-m_{1R}^2),$$

$$L = i(m_{1I}m_{1R}^2 - l_{1R}^3) + (m_{1R}^3 + l_{1I}l_{1R}^2 + i(m_{1I}l_{1R}^2 - l_{1R}m_{1R}^2))\tanh\chi_{1R},$$

$$A = 2l_{1R}^6 - 2m_{1R}^2(l_{1R}^4 + l_{1R}^3 m_{1I}) + m_{1R}^4 l_{1I}^2 + 2l_{1R}^2 l_{1I} m_{1R}^3 + m_{1I}^2 m_{1R}^4 + 3l_{1R}^2 m_{1R}^4 - m_{1R}^6,$$

$$B = l_{1R}^2 m_{1R}^2(m_{1R}^2 + l_{1I}^2 + m_{1I}^2 + l_{1R}^2) + l_{1R}^3 l_{1I} m_{1R} + l_{1I}(m_{1R}^5 - l_{1R}^5) - l_{1R}m_{1I}m_{1R}^4$$

$$C = 2m_{1R}^6 + 2l_{1I}l_{1R}^2 m_{1R}^3 + l_{1R}^4 l_{1I}^2 + m_{1I}^2 l_{1R}^4 + 3m_{1R}^2 l_{1R}^4 - 2l_{1R}^2 m_{1R}^4 - 2m_{1I}l_{1R}^3 m_{1R}^2 - l_{1R}^6.$$

<u>Case(ii): $\sigma^2 = -1$ (IE II)</u>

Choosing

$$M = \frac{(l_1 + l_1^*) + i(m_1 + m_1^*)}{l_1^* - im_1^*},$$

the 1-SS of IE II takes the form

$$S^+ = -2\frac{(l_{1R} + im_{1R}) + (m_{1I} - il_{1I} + (l_{1R} + im_{1R})\tanh\chi_{1R})\exp i\chi_{1I}\,\mathrm{sech}\chi_{1R}}{l_{1R}^2 + l_{1I}^2 + 2(l_{1R}m_{1I} - l_{1I}m_{1R})\tanh\chi_{1R} + (m_{1R}^2 + m_{1I}^2)\tanh^2\chi_{1R}}, \quad (2.12a)$$

$$S_3 = 1 - \frac{2(l_{1R}^2 + m_{1R}^2)\mathrm{sech}^2\chi_{1R}}{l_{1R}^2 + l_{1I}^2 + 2(l_{1R}m_{1I} - l_{1I}m_{1R})\tanh\chi_{1R} + (m_{1R}^2 + m_{1I}^2)\tanh^2\chi_{1R}}. \quad (2.12b)$$

2.3.2. 2-soliton solution To construct 2-SS, we take $N = 2$ in eq. (2.9). Then, g_1 takes the form

$$g_1 = \exp\chi_1 + \exp\chi_2. \quad (2.13)$$

Substituting (2.13) in (2.8) and after some calculations we obtain

$$f_2 = M_{11}\exp(\chi_1 + \chi_1^*) + M_{12}\exp(\chi_2 + \chi_1^*) + M_{21}\exp(\chi_1 + \chi_2^*) + M_{22}\exp(\chi_2 + \chi_2^*), \quad (2.14a)$$

$$g_3 = L_{112}\exp(\chi_1 + \chi_1^* + \chi_2) + L_{122}\exp(\chi_1 + \chi_2 + \chi_2^*), \quad (2.14b)$$

$$f_4 = K\exp(\chi_1 + \chi_1^* + \chi_2 + \chi_2^*), \quad (2.14c)$$

where

$$M_{rs} = \frac{\sigma^2 m_s^2 - l_s^2}{(l_r^* + l_s)^2 - \sigma^2(m_r^* + m_s)^2}, \quad r, s = 1, 2,$$

$$L_{rst} = \frac{(\sigma^2 m_s^{*2} - l_s^{*2})((l_r - l_t)^2 - \sigma^2(m_r - m_t)^2)}{((l_r + l_s^*)^2 - \sigma^2(m_r + m_s^*)^2)((l_t + l_s^*)^2 - \sigma^2(m_t + m_s^*)^2)}, \quad t = 1, 2,$$

$$K = \frac{((l_1^* - l_2^*)^2 - \sigma^2(m_1^* - m_2^*)^2)((l_1 - l_2)^2 - \sigma^2(m_1 - m_2)^2)P}{((l_1 + l_1^*)^2 - \sigma^2(m_1 + m_1^*)^2)((l_2 + l_1^*)^2 - \sigma^2(m_2 + m_1^*)^2)},$$

$$P = \frac{\sigma^2 m_1^2 - l_1^2}{(l_1 + l_2^*)^2 - \sigma^2(m_1 + m_2^*)^2}\,\frac{\sigma^2 m_2^2 - l_2^2}{(l_2 + l_2^*)^2 - \sigma^2(m_2 + m_2^*)^2}.$$

Making use of the fact that now $\omega = \frac{g}{f} = \frac{\epsilon g_1 + \epsilon^3 g_3}{1 + \epsilon^2 f_2 + \epsilon^4 f_4}$, and the relations (2.1) for the spin variables in terms of ω, the spin two soliton solution can be written explicitly.

2.3.3. N-soliton solution Finally, by taking g_1 as in eq. (2.9) and extending the above procedure, one can obtain the N-SS as

$$g = \sum_{\mu_j=0,1}^{''} \exp\{\sum_{i<j}^{2N} \phi(i,j)\mu_i\mu_j + \sum_{i=1}^{2N} \mu_i[\chi_i + \psi(i)]\}, \tag{2.15a}$$

where

$$\psi(i) = \begin{cases} \log(\sigma^2 m_i^2 - l_i^2) & \text{for } i = N+1,\ldots,2N, \\ 0 & \text{otherwise,} \end{cases}$$

$$\sum_{\mu_i=0,1}^{''} \text{ means } \sum_{i=1}^{N} \mu_i = 1 + \sum_{i=1}^{N} \mu_{i+N},$$

and

$$f = \sum_{\mu_j=0,1}^{'} \exp\{\sum_{i<j}^{2N} \phi(i,j)\mu_i\mu_j + \sum_{i=1}^{2N} \mu_i[\chi_i + \psi'(i)]\}, \tag{2.15b}$$

where

$$\chi_i = l_i x + m_i y + n_i t + \chi_i(0),$$

$$n_i = i(l_i^2 + \sigma^2 m_i^2), \quad \chi_{i+N} = \chi_i^*, l_{i+N} = l_i^*, m_{i+N} = m_i^*, n_{i+N} = n_i^*,$$

$$\phi(i,j) = \begin{cases} -\log((l_i+l_j)^2 - \sigma^2(m_i+m_j)^2) & \text{for } i = 1,\ldots,N \text{ and } j = N+1,\ldots,2N, \\ \log((l_i-l_j)^2 - \sigma^2(m_i-m_j)^2) & \text{for } i = 1,\ldots,N \text{ and } j = 1,\ldots,N, \end{cases}$$

$$\psi'(i) = \begin{cases} \log(\sigma^2 m_i^2 - l_i^2) & \text{for } i = 1,\ldots,N, \\ 0 & \text{otherwise,} \end{cases}$$

$$\sum_{\mu_i=0,1}^{'} \text{ means } \sum_{i=1}^{N} \mu_i = \sum_{i=1}^{N} \mu_{i+N}.$$

3. Bilinearization and solution of IE I in terms of light cone coordinates: The role of boundaries

In the case of most of the (2+1) dimensional integrable nonlinear evolution equations like Davey-Stewartson I, modified Kadomtsev-Petviashvili, and Nizhnik-Veselov-Novikov equations [10–13], the bilinearized forms are first transformed into systems of linear partial differential equations(pdes) while using the Hirota's direct method. By solving these linear pdes, one can construct the line solitons(as we have done in section 2) which are localized everywhere except along particular lines. Looking at the nature of line solitons, the presence of two nonparallel ghost solitons [10] (solitons which are visible only in the absence of the physical field) are identified. As a dromion is the two soliton solution made out of two nonparallel ghost solitons, they can be embedded in the two soliton solution to generate a (1,1) dromion. It implies that the solution consists of one bound state each in the x and y directions. This can be directly extended to generate multidromions. If we follow the same procedure for IE, we see from eq. (2.11) or eq. (2.12) that the two nonparallel ghost solitons are absent here. Hence we cannot construct dromions from the bilinearized form (2.6).

In the following, we analyse the IE I in a different frame of reference consisting of light cone coordinates ξ and η, which are defined as

$$\xi = \frac{1}{2}(y+x), \quad \eta = \frac{1}{2}(y-x). \tag{3.1}$$

Correspondingly, eq. (1.1) takes the form (after rescaling $-\frac{t}{2} \longrightarrow t'; t' \longrightarrow t$)

$$\vec{S}_t(\xi,\eta,t) = \vec{S} \wedge (\vec{S}_{\xi\xi} + \vec{S}_{\eta\eta}) + \phi_\xi \vec{S}_\xi - \phi_\eta \vec{S}_\eta, \tag{3.2a}$$

$$\phi_{\xi\eta} = \vec{S} \cdot \vec{S}_\xi \wedge \vec{S}_\eta. \tag{3.2b}$$

The form of eq. (3.2b) suggests that one can redefine the scalar field variable ϕ as

$$\phi(\xi,\eta,t) = \Phi(\xi,\eta,t) + \int m_1(\xi,t)d\xi + \int m_2(\eta,t)d\eta, \tag{3.3}$$

where $m_1(\xi,t)$ and $m_2(\eta,t)$ are the boundaries (arbitrary functions in the indicated variables). Eq. (3.2) can therefore be rewritten as

$$\vec{S}_t(\xi,\eta,t) = \vec{S} \wedge (\vec{S}_{\xi\xi} + \vec{S}_{\eta\eta}) + (\Phi_\xi + m_1(\xi,t))\vec{S}_\xi - (\Phi_\eta + m_2(\eta,t))\vec{S}_\eta, \tag{3.4a}$$

$$\Phi_{\xi\eta} = \vec{S} \cdot \vec{S}_\xi \wedge \vec{S}_\eta. \tag{3.4b}$$

In terms of the stereographic variable ω given by eq. (2.1), eq. (3.4) takes the form

$$i\omega_t + \omega_{\xi\xi} + \omega_{\eta\eta} - \frac{2\omega^*(\omega_\xi^2 + \omega_\eta^2)}{1+|\omega|^2} - i(\Phi_\xi + m_1(\xi,t))\omega_\xi - i(\Phi_\eta + m_2(\eta,t))\omega_\eta = 0, \tag{3.5a}$$

$$\Phi_{\xi\eta} = \frac{4i}{(1+|\omega|^2)^2}(\omega_\xi^*\omega_\eta - \omega_\xi\omega_\eta^*). \tag{3.5b}$$

After introducing the transformation $\omega = \frac{g}{f}$, the bilinear representations of eq. (3.4) can be written as

$$(iD_t - D_\xi^2 - D_\eta^2 - im_1(\xi,t)D_\xi + im_2(\eta,t)D_\eta)(f^* \cdot g) = 0, \tag{3.6a}$$

$$(iD_t - D_\xi^2 - D_\eta^2 - im_1(\xi,t)D_\xi + im_2(\eta,t)D_\eta)(f^* \cdot f - g^* \cdot g) = 0, \tag{3.6b}$$

$$D_\xi(D_\eta(f^* \cdot f + g^* \cdot g)) \cdot (f^*f + g^*g) = D_\eta(D_\xi(f^* \cdot f + g^* \cdot g)) \cdot (f^*f + g^*g), \tag{3.6c}$$

so that

$$\Phi_\xi = -2i\frac{D_\xi(f^* \cdot f + g^* \cdot g)}{(f^*f + g^*g)}, \tag{3.6d}$$

$$\Phi_\eta = -2i\frac{D_\eta(f^* \cdot f + g^* \cdot g)}{(f^*f + g^*g)}. \tag{3.6e}$$

One observes now the explicit introduction of the boundaries $m_1(\xi,t)$ and $m_2(\eta,t)$ into the bilinearized form (3.6), which turns out to be crucial to obtain localized solutions. Also eq. (3.6c) is a consequence of the compatibility between eqs. (3.6d) and (3.6e). It may also be noted that a similar introduction of arbitrary functions in the bilinear form was necessiated for the (2+1) dimensional sine-Gordan equation in order to obtain localized solutions [13].

Expanding now the functions g and f as power series,

$$g = \epsilon g_1 + \epsilon^3 g_3 + \epsilon^5 g_5 + \ldots, \tag{3.7a}$$

$$f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \epsilon^6 f_6 + \ldots \tag{3.7b}$$

and substituting them into eqs. (3.6a)-(3.6c), we will obtain the following set of linear pdes by equating the various powers of ϵ:

$$\epsilon : (iD_t - D_\xi^2 - D_\eta^2 - im_1(\xi,t)D_\xi + im_2(\eta,t)D_\eta)(1 \cdot g_1) = 0, \tag{3.8a}$$

$$\epsilon^2 : (iD_t - D_\xi^2 - D_\eta^2 - im_1(\xi,t)D_\xi + im_2(\eta,t)D_\eta)(1 \cdot f_2 + f_2^* \cdot 1 - g_1^* \cdot g_1) = 0, \tag{3.8b}$$

$$D_\xi(D_\eta(1 \cdot f_2 + f_2^* \cdot 1 - g_1^* \cdot g_1) \cdot 1) + D_\eta(D_\xi(1 \cdot f_2 + f_2^* \cdot 1 - g_1^* \cdot g_1) \cdot 1 = 0 \tag{3.8c}$$

and so on. Let us solve the above linear equations to obtain the solutions to IE I.

Eq. (3.8a) can be rewritten as

$$ig_{1t} + g_{1\xi\xi} + g_{1\eta\eta} - im_1(\xi,t)g_{1\xi} + im_2(\eta,t)g_{1\eta} = 0. \tag{3.9}$$

Let us try a variable separation, by postulating

$$g_1 = p(\xi,t)q(\eta,t), \tag{3.10}$$

where p and q are complex functions of the indicated arguments. Now, eq. (3.9) becomes

$$q(ip_t + p_{\xi\xi} - im_1(\xi,t)p_\xi) + p(iq_t + q_{\eta\eta} + im_2(\eta,t)q_\eta) = 0. \tag{3.11}$$

The above equation suggests that we should have

$$ip_t + p_{\xi\xi} - im_1(\xi,t)p_\xi = k_1 p, \tag{3.12a}$$

$$iq_t + q_{\eta\eta} + im_2(\eta,t)q_\eta = -k_1 q, \tag{3.12b}$$

where k_1 is a constant. Redefining $p = \hat{p}\exp -ik_1 t$ and $q = \hat{q}\exp ik_1 t$ and dropping the hats, eq. (3.12) becomes

$$ip_t + p_{\xi\xi} - im_1(\xi,t)p_\xi = 0, \tag{3.13a}$$

$$iq_t + q_{\eta\eta} + im_2(\eta,t)q_\eta = 0. \tag{3.13b}$$

Simplifying now eq. (3.8c), we will get

$$f_{2\xi\eta}^* - f_{2\xi\eta} = g_1^* g_{1\xi\eta} - g_1 g_{1\xi\eta}^*. \tag{3.14}$$

Let us write $f_2 = f_{2R} + if_{2I}$, where f_{2R} and f_{2I} are the real and imaginary parts of the function f_2, respectively. Eq. (3.14) now becomes

$$f_{2I\xi\eta} = \frac{1}{2i}(g_1 g_{1\xi\eta}^* - g_1^* g_{1\xi\eta}).$$

On carrying out integrations with respect to ξ and η, we get

$$f_{2I} = \frac{1}{2i}\int\int (g_1 g_{1\xi\eta}^* - g_1^* g_{1\xi\eta})d\xi d\eta + h(\xi,t) + r(\eta,t), \tag{3.15}$$

where $h(\xi,t)$ and $r(\eta,t)$ are arbitrary functions in the indicated arguments. Substitution of g_1 and f_2 in eq.(3.8b) gives rise to

$$\nabla^2 f_{2R} \equiv f_{2R\xi\xi} + f_{2R\eta\eta} = f_{2It} - m_1(\xi,t)f_{2I\xi} + m_2(\eta,t)f_{2I\eta} - (p_\xi p_\xi^* qq^* + pp^* q_\eta q_\eta^*), \tag{3.16}$$

which is nothing but the Poisson's equation in two dimensions for f_{2R}. For the given boundaries $m_1(\xi,t)$ and $m_2(\eta,t)$, once the eqs. (3.13a) and (3.13b) are solved for p and q respectively, the right hand sides of eqs. (3.15) and (3.16) can be found out. Hence they can be solved to obtain f_{2I} and f_{2R}. At this point, we may point out that exactly the same kind of linear equations (3.12a) and (3.12b) have appeared both in the case of IST method [7] and binary Darboux transformation method [8]. Now these equations have appeared from a different perspective, namely from the point of view of bilinearization method.

Now the general solution for ω can be written as

$$\omega = \frac{g}{f} = \frac{p(\xi,t)q(\eta,t)}{1+f_{2R}+if_{2I}}, \tag{3.17}$$

where the functions p, q, f_{2I} and f_{2R} are to be determined from equations (3.13a), (3.13b), (3.15) and (3.16), respectively. Here the main task lies in solving eq. (3.13) which is also associated with the linear problem of the following modified Kadomtsev-Petviashvili equation(mKP) [14]:

$$u_t + u_{xxx} + 6u^2u_x - 12\partial_x^{-1}u_{yy} + 12u_x\partial_x^{-1}u_y = 0. \tag{3.18}$$

The mKP eq. (3.15) is equivalent to the compatibility condition for the linear system,

$$(2i\partial_y + \partial_x^2 + 2iu\partial_x)\psi = 0, \tag{3.19a}$$

$$(i\partial_t + 4\partial_x^3 + 12iu\partial_x^2 + (6iu_x + 12\partial_x^{-1}u_y - 6u^2)\partial_x + a)\psi = 0, \tag{3.19b}$$

where a is an arbitrary constant. By comparing eq. (3.13) and eq. (3.19), it is seen that the proper solutions of eq. (3.13) are obtained by dropping the time dependence in the various types of solutions such as line solitons, line lumps and line breathers of eq. (3.19) and changing $y \rightarrow t$. Since we have two independent problems in eq. (3.13), large number of classes of exact solutions of IE I, which include rationally localized, rationally-exponentially localized and fully exponentially localized structures, are possible. We present the details of some of them in the next section.

4. Localized coherent structures of IE I: Time dependent boundaries

4.1. Lump-lump boundaries: Algebraically decaying structures

In this section, we choose specific forms of the boundaries $m_1(\xi,t)$ and $m_2(\eta,t)$ in eq. (3.8) to obtain localized solutions. For the algebraically decaying boundaries of the form (for easy comparison, we follow the notations of ref. [7])

$$m_1(\xi,t) = -\frac{2\alpha}{(\xi - \frac{2t}{\alpha} + c_1)^2 + \frac{\alpha^2}{4}}, \tag{4.1a}$$

$$m_2(\eta,t) = \frac{2\beta}{(\eta - \frac{2t}{\beta} + c_2)^2 + \frac{\beta^2}{4}}, \tag{4.1b}$$

where α, β, c_1 and c_2 are real constants, the solutions to eqs. (3.13) are obtained as

$$p(\xi,t) = \frac{\exp i(\frac{\xi}{\alpha} - \frac{t}{\alpha^2})}{\xi - \frac{2t}{\alpha} + c_1 - \frac{i\alpha}{2}}, \tag{4.2a}$$

$$q(\eta,t) = \frac{\exp i(\frac{\xi}{\beta} - \frac{t}{\beta^2})}{\eta - \frac{2t}{\beta} + c_2 - \frac{i\beta}{2}}. \tag{4.2b}$$

Substitution of eq. (4.2) in (3.15) leads to

$$f_{2I} = \frac{1}{2\alpha\beta}\frac{\beta(\xi - \frac{2t}{\alpha} + c_1) + \alpha(\eta - \frac{2t}{\beta} + c_2)}{((\xi - \frac{2t}{\alpha} + c_1)^2 + \frac{\alpha^2}{4})((\eta - \frac{2t}{\beta} + c_2)^2 + \frac{\beta^2}{4})}, \tag{4.3}$$

wherein $h(\xi,t) = 0$ and $r(\eta,t) = 0$ and substituting the values of p, q and f_{2I} in eq. (3.16) and solving the resultant Poisson's equation for f_{2R}, we get

$$f_{2R} = \frac{1}{\alpha\beta}\frac{(\xi - \frac{2t}{\alpha} + c_1) + (\eta - \frac{2t}{\beta} + c_2) - \frac{\alpha\beta}{4}}{((\xi - \frac{2t}{\alpha} + c_1)^2 + \frac{\alpha^2}{4})((\eta - \frac{2t}{\beta} + c_2)^2 + \frac{\beta^2}{4})}. \tag{4.4}$$

Therefore,

$$f_2 = f_{2R} + if_{2I} = \frac{1}{\alpha\beta(\xi - \frac{2t}{\alpha} + c_1 - \frac{i\alpha}{2})(\eta - \frac{2t}{\beta} + c_2 - \frac{i\beta}{2})}. \tag{4.5}$$

Hence, from eq. (3.17), we have

$$\omega = \frac{\exp i(\frac{\xi}{\alpha} - \frac{t}{\alpha^2} + \frac{\xi}{\beta} - \frac{t}{\beta^2})}{\frac{1}{\alpha\beta} + (\xi - \frac{2t}{\alpha} + c_1 - \frac{i\alpha}{2})(\eta - \frac{2t}{\beta} + c_2 - \frac{i\beta}{2})}. \tag{4.6}$$

Using eq. (2.1), the spin components corresponding to this form of ω are given below:

$$S^+ = 2\frac{(\frac{1}{\alpha\beta} + (\xi - \frac{2t}{\alpha} + c_1 - \frac{i\alpha}{2})(\eta - \frac{2t}{\beta} + c_2 - \frac{i\beta}{2}))\exp i(\frac{\xi}{\alpha} - \frac{t}{\alpha^2} + \frac{\xi}{\beta} - \frac{t}{\beta^2})}{A^2 + B^2}, \tag{4.7a}$$

$$S_3 = 1 - \frac{2}{A^2 + B^2}, \tag{4.7b}$$

where

$$A = \frac{1}{\alpha\beta} + (\xi - \frac{2t}{\alpha} + c_1)(\eta - \frac{2t}{\beta} + c_2) + \frac{\alpha\beta}{4}$$

and

$$B = \frac{\alpha}{2}(\eta - \frac{2t}{\beta} + c_2) - \frac{\beta}{2}(\xi - \frac{2t}{\alpha} + c_1).$$

The solution (4.7) decays as $\frac{1}{\xi\eta}$ as $\xi^2 + \eta^2 \longrightarrow \infty$ and moves with the velocity $V = (V_\xi, V_\eta) = (\frac{2}{\alpha}, \frac{2}{\beta})$. A snap shot of S_3 is plotted in fig. (1)

4.2. Soliton-soliton boundaries: Exponentially decaying dromions

Let us now choose the boundaries $m_1(\xi,t)$ and $m_2(\eta,t)$ as the following line solitons(again using the notations of [7]):

$$m_1(\xi,t) = \frac{\frac{8\mu_I}{|\mu|^2}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi}}{(1 + \frac{\mu_R}{\mu_I}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi})^2 + \exp\frac{4\mu_I}{|\mu|^2}\hat{\xi}}, \quad (\mu = \mu_R + i\mu_I) \tag{4.8a}$$

$$m_2(\eta,t) = -\frac{\frac{8\lambda_I}{|\lambda|^2}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}}{(1 + \frac{\lambda_R}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta})^2 + \exp\frac{4\lambda_I}{|\lambda|^2}\hat{\eta}}, \; (\lambda = \lambda_R + i\lambda_I) \tag{4.8b}$$

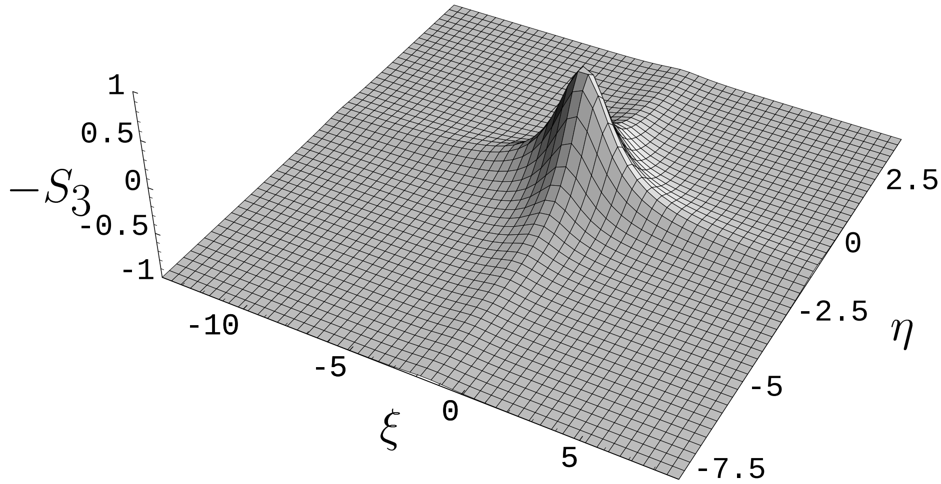

Figure 1. Algebraically decaying lump structure

where $\hat{\xi} = \xi - \frac{2\mu_R}{|\mu|^2}t + \xi_0$, $\hat{\eta} = \eta - \frac{2\lambda_R}{|\lambda|^2}t + \eta_0$ and μ_R, μ_I, λ_R, λ_I, ξ_0 and η_0 are real constants. The suffices R and I are used to denote the real and imaginary parts respectively. On solving eqs. (3.13), (3.15) and (3.16), we get the following expressions for $p(\xi, t)$, $q(\eta, t)$ and $f(\xi, \eta, t)$:

$$p(\xi,t) = \frac{\exp i(\frac{\xi}{\mu} - \frac{t}{\mu^2})}{(1 + \frac{\mu^*}{\mu_I}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi})}, \tag{4.9a}$$

$$q(\eta,t) = \frac{\exp i(\frac{\eta}{\lambda} - \frac{t}{\lambda^2})}{(1 + \frac{\lambda^*}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta})}, \tag{4.9b}$$

$$f_{2I} = \frac{1}{4}\frac{(1 + \frac{\lambda_R}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta})\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi} + (1 + \frac{\mu_R}{\mu_I}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi})\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}}{|1 + \frac{\mu}{\mu_I}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi}|^2 \quad |1 + \frac{\lambda}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}|^2}, \tag{4.9c}$$

$$f_{2R} = \frac{(1 + \frac{\mu_R}{\mu_I}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi})(1 + \frac{\mu_R}{\mu_I}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi}) - \exp\frac{2\mu_I}{|\mu|^2}\hat{\xi} + \frac{2\lambda_I}{|\lambda|^2}\hat{\eta}}{4|1 + \frac{\mu}{\mu_I}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi}|^2 \quad |1 + \frac{\lambda}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}|^2}. \tag{4.9d}$$

Consequently, we have

$$\omega = \frac{\exp i(\frac{\xi}{\mu} - \frac{t}{\mu^2} + \frac{\eta}{\lambda} - \frac{t}{\lambda^2})}{(1 + \frac{\mu^*}{\mu_I}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi})(1 + \frac{\lambda^*}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}) + \frac{1}{4}}. \tag{4.10}$$

The spin components corresponding to this form of ω are given below:

$$S^+ = \frac{2((1 + \frac{\mu^*}{\mu_I}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi})(1 + \frac{\lambda^*}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}) + \frac{1}{4})\exp i(\frac{\hat{\xi}}{\mu} - \frac{t}{\mu^2} + \frac{\hat{\eta}}{\lambda} - \frac{t}{\lambda^2})}{C^2 + D^2 + \exp(\frac{2\mu_I}{|\mu|^2}\hat{\xi} + \frac{2\lambda_I}{|\lambda|^2}\hat{\eta})}, \tag{4.11a}$$

$$S_3 = 1 - \frac{2\exp(\frac{2\mu_I}{|\mu|^2}\hat{\xi} + \frac{2\lambda_I}{|\lambda|^2}\hat{\eta})}{C^2 + D^2 + \exp(\frac{2\mu_I}{|\mu|^2}\hat{\xi} + \frac{2\lambda_I}{|\lambda|^2}\hat{\eta})}, \tag{4.11b}$$

where

$$C = 1 + \frac{1}{4} + \frac{\mu_R}{\mu_I}\exp\frac{2\mu_I}{|\mu|^2}\hat{\xi} + \frac{\lambda_R}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta} + \frac{1}{\lambda_I\mu_I}(\lambda_R\mu_R - \lambda_I\mu_I)\exp(\frac{2\mu_I}{|\mu|^2}\hat{\xi} + \frac{2\lambda_I}{|\lambda|^2}\hat{\eta}),$$

$$D = \exp\frac{2\mu_I}{|\mu|^2}\hat{\xi} + \exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta} + (\frac{\lambda_R}{\lambda_I} + \frac{\mu_R}{\mu_I})\exp(\frac{2\mu_I}{|\mu|^2}\hat{\xi} + \frac{2\lambda_I}{|\lambda|^2}\hat{\eta}).$$

The solution (4.11) decays exponentially (see fig. (2)) in all directions on the plane ξ, η and moves with the velocity $V = (V_\xi, V_\eta) = (\frac{2\mu_R}{|\mu|^2}, \frac{2\lambda_R}{|\lambda|^2})$.

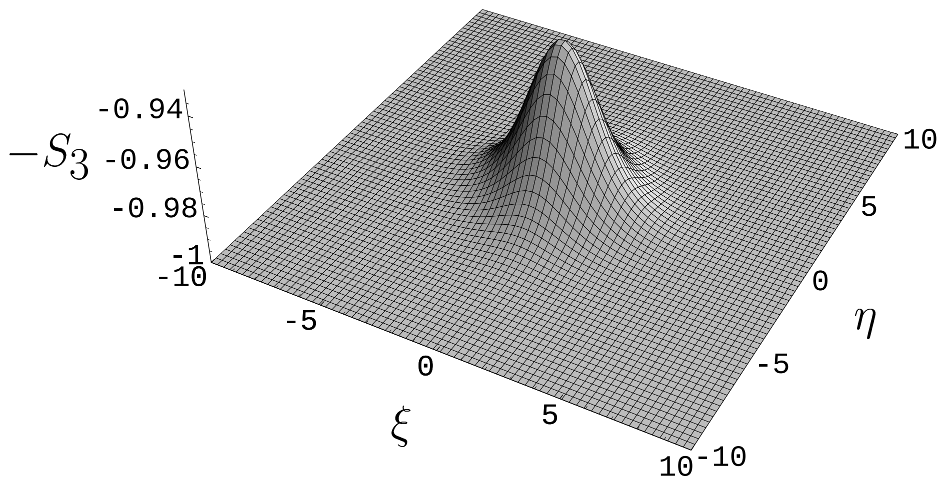

Figure 2. A dromion

4.3. Lump-line soliton boundaries: Rationally-exponentially decaying nature

Taking the choice of the rational lump as the boundary $m_1(\xi, t)$ and of the line soliton as the boundary $m_2(\eta, t)$, that is

$$m_1(\xi, t) = -\frac{2\alpha}{(\xi - \frac{2t}{\alpha} + c_1)^2 + \frac{\alpha^2}{4}}, \tag{4.12a}$$

$$m_2(\eta, t) = -\frac{\frac{8\lambda_I}{|\lambda|^2}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}}{(1 + \frac{\lambda_R}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta})^2 + \exp\frac{4\lambda_I}{|\lambda|^2}\hat{\eta}}, \tag{4.12b}$$

and proceeding as before, we can find the solution of ω as

$$\omega = \frac{\exp i(\frac{\xi}{\alpha} - \frac{t}{\alpha^2} + \frac{\eta}{\lambda} - \frac{t}{\lambda^2})}{\frac{1}{2\alpha} + (\xi - \frac{2t}{\alpha} + c_1 - \frac{i\alpha}{2})(1 + \frac{\lambda^*}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta})}. \tag{4.13}$$

The spin components in this case are

$$S^+ = \frac{2(\frac{1}{2\alpha} + (\xi - \frac{2t}{\alpha} + c_1 + \frac{i\alpha}{2}) + (1 + \frac{\lambda}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}))\exp i(\frac{\xi}{\alpha} - \frac{t}{\alpha^2} + \frac{\hat{\eta}}{\lambda} - \frac{t}{\lambda^2})}{E^2 + F^2 + \exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}}, \qquad (4.14a)$$

$$S_3 = 1 - \frac{2\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}}{E^2 + F^2 + \exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}}, \qquad (4.14b)$$

where

$$E = (\xi - \frac{2t}{\alpha} + c_1) - \frac{\alpha}{2}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta} + \frac{\lambda_R}{\lambda_I}(\xi - \frac{2t}{\alpha} + c_1)\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta} + \frac{1}{2\alpha}$$

$$F = \frac{\alpha}{2}(1 + \frac{\lambda_R}{\lambda_I}\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}) + (\xi - \frac{2t}{\alpha} + c_1)\exp\frac{2\lambda_I}{|\lambda|^2}\hat{\eta}.$$

The above solution (4.14) decays rationally in the ξ direction and exponentially in the η direction on the plane ξ, η and it moves with the velocity $V = (V_\xi, V_\eta) = (\frac{2}{\alpha}, \frac{2\lambda_R}{|\lambda|^2})$ (see fig. (3)).

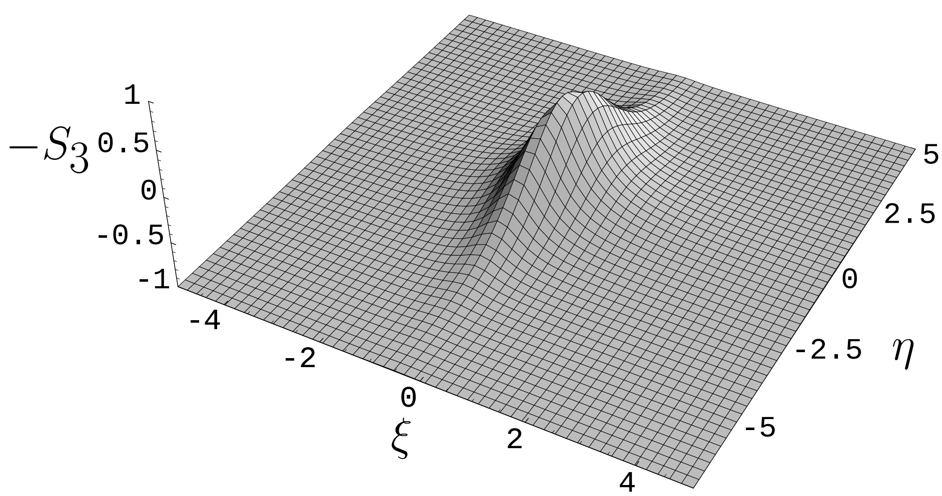

Figure 3. Rationally-exponentially deacying structure

We can take the other choice also, that is $m_1(\xi, t)$ as the plane soliton boundary and $m_2(\eta, t)$ as the rational boundary. Replacing ξ by η, α by β, $\hat{\eta}$ by $\hat{\xi}$ and λ by μ in (4.12) and (4.13), we will get the solution corresponding to this case.

5. Stationary boundaries

In the case of stationary boundaries, that is $m_1(\xi, t) = m_1(\xi)$ and $m_2(\eta, t) = m_2(\eta)$, the function g_1 can be written as

$$g_1 = p'(\xi)q'(\eta)T(t) \qquad (5.1)$$

where p', q' and T are complex functions of their indicated arguments. Substitution of this expression for g_1 in eq. (3.9) leads to

$$ip'q'T_t + q'T(p'_{\xi\xi} - im_1(\xi)p'_\xi) + p'T(q'_{\eta\eta} + im_2(\eta)q'_\eta) = 0. \tag{5.2}$$

Eq. (5.2) suggests that

$$p'_{\xi\xi} - im_1(\xi)p'_\xi = -\lambda^2 p' \tag{5.3a}$$

$$q'_{\eta\eta} + im_2(\eta)q'_\eta) = -\lambda^{'2} q^{'} \tag{5.3b}$$

$$T_t + i(\lambda^2 + \lambda^{'2})T = 0, \tag{5.3c}$$

where λ and λ' are some complex arbitrary parameters. Eq. (5.3c) can be solved immediately and it gives

$$T = \exp -i(\lambda^2 + \lambda^{'2})t. \tag{5.4}$$

Rescaling $p'(\xi) = \hat{p}(\xi) \exp \frac{i}{2} \int m_1(\xi) d\xi$ and $q'(\eta) = \hat{q}(\eta) \exp \frac{-i}{2} \int m_2(\eta) d\eta$ and then dropping the hats, eqs. (5.3a) and (5.3b) take the form

$$p'_{\xi\xi} + (\lambda^2 + \frac{m_1^2}{4} + i\frac{m_{1\xi}}{2})p' = 0, \tag{5.5a}$$

$$q'_{\eta\eta} + (\lambda'^2 + \frac{m_2^2}{4} - i\frac{m_{2\eta}}{2})q' = 0. \tag{5.5b}$$

The forms of f_{2I} and f_{2R} remain the same as in eqs. (3.15) and (3.16), respectively, except that here we have to replace p by p' and q by q' and the solution to ω is also given by (3.17). Again the above linear equations are exactly the same as the ones obtained in ref. [7] through IST formalism, which are however now obtained from the bilinear form of IE I.

It is seen that the problems (5.3a) and (5.3b) are gauge equivalent to the spectral problems of the Schrödinger type

$$\phi_{xx} + (\lambda^2 + u^2(x) \pm iu_x(x))\phi(x, \lambda) = 0 \tag{5.6}$$

with the very special potential [14]

$$V(x) = -(u^2(x) \pm iu_x(x)), \tag{5.7}$$

where $u(x)$ is a real valued function. In this case also, the class of coherent structures admitted by (5.6) and hence by IE I are very rich.

5.1. Constant boundaries:Lump solution

If the boundaries are constant, that is $m_1(\xi, t) = m_1$ and $m_2(\eta, t) = m_2$, then going through the various steps indicated above, the function ω is found out to be

$$\omega = -\frac{\exp i((m_1 + \frac{1}{\alpha})(\xi - \frac{t}{\alpha}) + (-m_2 + \frac{1}{\beta})(\eta - \frac{t}{\beta}))}{1 - \left(\frac{1}{(m_1 + \frac{1}{\alpha})^2} + \frac{1}{(m_2 - \frac{1}{\beta})^2}\right)\frac{\xi^2 + \eta^2}{2}}. \tag{5.8}$$

The corresponding spin components are

$$S^+ = \frac{(-2 + \frac{\xi^2+\eta^2}{(m_1+\frac{1}{\alpha})^2} + \frac{\xi^2+\eta^2}{(m_2-\frac{1}{\beta})^2}) \exp i((m_1 + \frac{1}{\alpha})(\xi - \frac{t}{\alpha}) + (-m_2 + \frac{1}{\beta})(\eta - \frac{t}{\beta}))}{(1 - \left(\frac{1}{(m_1+\frac{1}{\alpha})^2} + \frac{1}{(m_2-\frac{1}{\beta})^2}\right) \frac{\xi^2+\eta^2}{2})^2 + 1}, \qquad (5.9a)$$

$$S_3 = 1 - \frac{2}{(1 - \left(\frac{1}{(m_1+\frac{1}{\alpha})^2} + \frac{1}{(m_2-\frac{1}{\beta})^2}\right) \frac{\xi^2+\eta^2}{2})^2 + 1}. \qquad (5.9b)$$

In this case, the solution decays algebraically as $(\frac{\xi^2+\eta^2}{2})^{-1}$ as $\xi^2 + \eta^2 \longrightarrow \infty$.

5.2. Boundaries are absent:Line solitons

If $m_1(\xi, t) = m_2(\eta, t) = 0$, then on solving equations (3.13), (3.15)-(3.17), we will get the following results:

$$p = A \exp(l\xi + il^2t), \qquad (5.10a)$$

$$q = B \exp(m\eta + im^2t), \qquad (5.10b)$$

$$f_{2I} = k_1 \exp 2(l_R\xi + m_R\eta - 2(l_Rl_I + m_Rm_I));\ k_1 = -AA^*BB^* \left(\frac{l_Rm_I + l_Im_R}{4l_Rm_R}\right), \qquad (5.10c)$$

$$f_{2R} = k_2 \exp 2(l_R\xi + m_R\eta - 2(l_Rl_I + m_Rm_I));\ k_2 = AA^*BB^* \left(\frac{l_Im_I - l_Rm_R}{4l_Rm_R}\right), \qquad (5.10d)$$

$$\omega = \frac{AB \exp(l\xi + m\eta + i(l^2 + m^2)t)}{1 + k_3 \exp 2(l_R\xi + m_R\eta - 2(l_Rl_I + m_Rm_I))};\ k_3 = -\frac{lm}{4l_Rm_R}AA^*BB^*, \qquad (5.10e)$$

which is nothing but the line soliton solution similar to the line solitons obtained in section 2 except for the difference in constant values. Here A and B are complex constants. So, if the boundaries are absent to start with in the IE I, eq. (3.2) or its bilinearized version (3.6), we obtain line soliton solutions only. From this, one can appreciate the important role of boundaries to generate localized coherent structures in the bilinearized version of IE I.

6. Conclusions and Discussions

If we look at the eqs. (3.13), (3.15), (3.16), it is seen that we are having four equations for six unknowns viz., $p(\xi, t), q(\eta, t), m_1(\xi, t), m_2(\eta, t), f_{2R}(\xi, \eta, t)$ and $f_{2I}(\xi, \eta, t)$. Hence any two of them can be treated as arbitrary functions. In this case, one can manipulate these arbitrary functions to get a large number of solutions similar to the solutions generated by Lou and his coworkers for equations such as Davey-Stewartson model, Nizhnik-Novikov-Veselov system and dispersive long wave equation [15]. For example, if we choose $p(\xi, t) = \exp \chi_1$ and $q(\eta, t) = \exp \chi_2$, where $\chi_1 = lx + \omega_1 t$ and $\chi_2 = my + \omega_2 t$, we will get the line solitons. It should be noted that we can interpret our analysis on localized coherent structures done in section 4 entirely in a different way. That is, choose the values of $p(\xi, t)$, and $q(\eta, t)$ given in eqs. (4.2a) and (4.2b), then solve the eqs. (3.13a), (3.13b), (3.15) and (3.16) to obtain the values of $m_1(\xi, t), m_2(\eta, t), f_{2I}(\xi, \eta, t)$ and

$f_{2R}(\xi,\eta,t)$ respectively. Once again we arrive at the same algebraically decaying structure of IE I given in eq. (4.7). Similar interpretation holds good for generating localized structures of other types also.

On the otherhand, if we take the more general solution of mKP I equation and by solving eqs. (3.13),(3.15) and (3.16), we can generate the multi-soliton (multilump/ multidromion) solutions for the IE I. For example, we take

$$m_1(\xi,t) = -\frac{2\alpha_1 X_2^2 + 2\alpha_2 X_1^2 + \frac{\alpha_1\alpha_2(\alpha_1+\alpha_2)^2}{2(\alpha_1-\alpha_2)^2}}{(X_1X_2 - \frac{\alpha_1\alpha_2(\alpha_1+\alpha_2)^2}{4(\alpha_1-\alpha_2)^2})^2 + \frac{1}{4}(\alpha_1 X_2 + \alpha_2 X_1)^2}, \qquad (6.1)$$

where $X_i = \xi - \frac{2t}{\alpha_i} + \delta_i,\ i = 1,2$, which describes the scattering of two lumps of the form given in eq. (4.1a) and $m_2(\eta,t)$ given in eq. (4.1b). On solving eqs. (3.13), (3.15) and (3.16), we can generate a (2,1) lump solution for IE I. Hence by appropriately choosing the boundaries $m_1(\xi,t)$ and $m_2(\eta,t)$, one can generate multidromions or multilumps and in general multisoliton structures.

To conclude, in the case of IE also the Hirota method has proved to be a straightforward but powerful tool to obtain different kinds of localized and other solutions. This has become feasible by appropriate bilinearization procedure in the light cone coordinates. We have also reported how the explicit forms of various localized solutions can be deduced.

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