

Nonlinear Physics: Integrability, Chaos and Beyond

M. Lakshmanan
Centre for Nonlinear Dynamics
Department of Physics
Bharathidasan University
Tiruchirapalli 620 024
India

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Abstract

Integrability and chaos are two of the main concepts associated with nonlinear physical systems which have revolutionized our understanding of them. Highly stable exponentially localized solitons are often associated with many of the important integrable nonlinear systems while motions which are sensitively dependent on initial conditions are associated with chaotic systems. Besides dramatically raising our perception of many natural phenomena, these concepts are opening up new vistas of applications and unfolding technologies: Optical soliton based information technology, magnetoelectronics, controlling and synchronization of chaos and secure communications, to name a few. These developments have raised further new interesting questions and potentialities. We present a particular view of some of the challenging problems and payoffs ahead in the next few decades by tracing the early historical events, summarizing the revolutionary era of 1950-70 when many important new ideas including solitons and chaos were realized and reviewing the current status. Important open problems both at the basic and applied levels are discussed.

1 Introduction

Fifty years is a long period in modern science and it is very hazardous to visualize how a given subject will grow and blossom in a period of five decades. Particularly this is so for an area like Nonlinear Science, whose growth is quite recent and phenomenal, and even the most immediate ramifications of its exciting developments have not yet been fully understood and utilized. Even then it will be fruitful and relevant to digress the past history, evaluate the present status and visualize the future course of directions and developments.

Of course if one asks the question how was nonlinear science 50 years ago, the answer is simply that no such field existed then and probably not many had envisaged its emergence and potentialities. It was the linear science that generally ruled the scientific world, whether it was quantum mechanics or field theory, fluid mechanics or solid state physics or electronics or name any other field barring exceptions. Linear physics is associated with beautifully structured linear mathematics, including spectral theory, integral transforms, linear vector spaces, linear differential equations and so on. There were only very few instances where nonlinear systems in physics were considered to be important or relevant or tractable. One thought nonlinearities are essentially perturbations to exactly solvable linear systems, their effects could be analysed through well developed perturbation methods, statistical analysis and so on. Of course there were isolated studies of nonlinear systems starting from the pendulum motion solvable by elliptic functions to Kepler problem and rigid body problem in classical mechanics, classification of nonlinear ordinary differential equations, geometrical theory of partial differential equations, and isolated problems in fluid mechanics and field theory. Also gravitation theory is a patently nonlinear theory, where essentially special solutions were sought and obtained. Thus except for such isolated studies, little systematic analysis of nonlinear systems on its own merit was carried out in general.

Yet there were enough warnings and forbodings by men of great foresight and vision during the past 100 years or so that a lot of new phenomena and insight are in store and are associated with nonlinear systems: John Scott-Russel's observation of solitary wave and its remarkable stability properties, Sophia Kovalevskaya's analysis of rigid body problem based on singularity structure analysis, Poincare's analysis on the sensitive dependence on initial conditions of nonlinear systems, van der Pol's observations of chaotic oscillations in electrical circuits, Einstein's insistence on the importance of nonlinear field theories and so on. Each of these gems of ideas took a long time of gestation period before their implications could be understood. One might say that essentially these ideas lie at the root of the modern development of nonlinear science.

The modern renaissance of nonlinear science in general and nonlinear physics in particular, of course, starts from the famous Fermi-Pasta-Ulam(FPU) experiments on the energy sharing phenomenon in nonlinear lattices in 1955. The ensuing many faceted investigations on the one hand by the remarkable analysis of Zabusky and Kruskal in 1965 led to the concept of solitons and ultimately to the development of integrable systems, while on the other hand led to the study of Hamiltonian chaos. Parallel path breaking studies of Lorenz in 1963 on the thermal convection equations ultimately led to the full power of chaotic dynamics. During the same period the remarkable foresight of Skyrme has led to the revival of interest in nonlinear field theories in particle physics. In retrospect one might say that the period 1950-70 is the golden age of nonlinear physics, revolutionizing our concept and outlook.

Some people even call it a third revolution in physics in this century, besides relativity and quantum mechanics.

The period 1970-96 is then a period of consolidation and enjoying the initial fruits of the earlier two era. Much understanding in both the fields of integrable and chaotic systems have been achieved and their ramifications and applications have touched almost all fields of physics during this period.

Though these developments are substantial, still many fundamental questions remain to be answered. For example, when a given system is integrable and when it is nonintegrable and then chaotic is a tricky question which needs to be answered rigorously. What are the natural excitations in higher dimensional systems? Are there other types structures besides solitonic and chaotic structures? These are some of the crucial questions which must be answered soon. How are nonintegrable systems to be understood from the point of view of integrable systems and how can one obtain effective spatio temporal patterns? What are the phenomena that are lying dormant to be exploited for technological developments? What kind of technical aspects one can develop in application oriented topics such as magneto-electronics, optoelectronics, controlling and synchronization of chaos and so on? These and other potential future problems in nonlinear physics are also of considerable importance to be considered.

This article will start with a brief account of the historical perspectives. Then it will discuss how based on these ideas the modern development of nonlinear science and the various topics in nonlinear physics it has led to developed. Then the current status of each of these fields will be briefly reviewed. In the last phase of the article, some of the outstanding problems in each of these areas will be discussed and finally the future perspective will be analysed.

2 Early Visions of Nonlinear Science and their Physical Ramifications

In this section, let us consider briefly some of the sailent features of the remarkable foresightedness of a few visionary scientists in realizing the importance of nonlinear phenomena. Their discoveries or vision, though not fully appreciated or even drew derisive comments by contemporary scientists, withstood the test of time and each one formed a cornerstone of the modern concepts of nonlinear science and in particular nonlinear physics.

2.1 Scott-Russel and his great wave of translation

John Scott-Russel, the Victorian entrepreneur and naval engineer for East India Company, was making investigations on the size of the ship-hull, weight, speed etc. in the Union canal connecting the cities of Edinburgh and Glasgow. It is now a scientific folklore [Bullough, 1988] that during the course of these investigations he was careful enough to make the first scientific observation of a patently nonlinear phenomenon, namely the formation of a solitary wave which can propagate without change of speed and form and whose velocity is dependent on its amplitude. This observation in the own words of Scott-Russel as he reported in the British Association Reports (1844) runs as follows.

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion: it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost in the windings of the channel. Such, in the month of August 1834, was my first chance interview with the singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now generally bears: which I have since found to be an important element in almost every case of fluid resistance, and ascertained to be of the type of that great moving elevation of the sea, which, with the regularity of a planet, ascends our rivers and rolls along our shores.

To study minutely this phenomenon with a view to determining accurately its nature and laws, I have adopted other more convenient modes of products it than that which I just described, and have employed various methods of observation. A description of these will probably assist me in conveying just conception of the nature of this wave.

Genesis of the Wave of the First Order

Scott-Russel immediately realized the importance of his observations, went back to his laboratory and carried out a series of experiments during 1834-40 and confirmed the permanent nature of the solitary wave. Scott-Russel also deduced a phenomenological relation between the velocity c and amplitude η ,

$$c = \sqrt{g(h + \eta)}, \quad (1)$$

where g is the acceleration due to gravity, h is the depth of the undisturbed water in the canal. It is also remarkable that Russel had also realized even the collision properties of the solitary waves [Bullough,1988]. John Scott-Russel was obviously far ahead of his time in realizing the importance of such solitary waves of permanence in that his contemporaries such as Airy, Stokes and others refused to believe Scott-Russel’s observations and explanations. In fact Scott-Russel died as a disappointed man in that during his lifetime he could not make his fellow scientists to accept his findings.

It took many more years before Boussinesq in 1872 and later on Korteweg and de Vries in 1895 could put Scott-Russel’s observations in the proper perspective rigorously (For details see Bullough [1988]). Starting from the basic equations of hydrodynamics and the fact that there are two small parameters available in the problem, namely the ratio of the height(amplitude) of the water wave in the canal to the depth of the channel and the depth of the channel to the length of the canal (or solitary wave), and using fixed and free (non-linear) boundary conditions, and by a systematic (asymptotic) expansions (see for example, Ablowitz and Clarkson, [1991]), Korteweg and de Vries showed that the Scott-Russel wave phenomenon can be described by a nonlinear wave equation for the amplitude $\eta(x, t)$. Its modern version is the ubiquitous form

$$u_t + 6uu_x + u_{xxx} = 0, \quad (2)$$

where $u(x, t)$ is related to $\eta(x, t)$ linearly. Eq.(2) admits solitary wave solutions of the form

$$u(x, t) = 2k^2 \operatorname{sech}^2 \left[k \left(x - 4k^2 t - \delta \right) \right], \quad k, \delta : \text{constants}. \quad (3)$$

One can easily check that the empirical formula of Scott-Russel given in eq.(1) also naturally follows from eq.(3), showing the correctness of Russel's observations.

Unfortunately even after the Korteweg-de Vries analysis, which comprehensively showed the presence of an entirely new phenomenon, namely solitary waves of permanence being supported by nonlinear partial differential equations, little further interest seems to have been shown by the scientific community in such a patently nonlinear phenomenon for another 70 years or so until Zabusky and Kruskal came across exactly at the same K-dV equation albeit in an entirely new physical situation, namely the propagation of waves in a nonlinear lattice-the famous Fermi-Pasta-Ulam(FPU) lattice. It is now part of the scientific history that how the Zabusky-Kruskal work had led to the concept of soliton (and Russel's solitary wave is indeed a soliton) and how this concept is playing a paradigmic role in many branches of physics ranging from astronomy to particle physics, condensed matter, fluid dynamics, ferromagnetism, optical physics and so on to biological physics.

2.2 S.Kovalevskaya's work on integrability of dynamical systems and singularity structure analysis

Sophia Kovalevskaya who migrated from Russia, and studied under Weisstrass, considered the problem of integrating the equations of motion of nonlinear dynamical systems. Particularly she took up the problem of analysing for what parametric values the equations of motion of a rigid body (top) rotating about a fixed point [Kovalevskaya,1889] is completely integrable and analytic integrals of motion can be obtained. This was a problem posed by the Paris Academy of Sciences for the Bordin Prize of 1888 and S.Kovalevskaya approached this problem in an entirely novel way, whose full ramifications are only realized in recent times with great potential applications for the future.

Considering the dynamics of a rigid body with one fixed point under the influence of gravitation the equation of motion can be written as

$$A \frac{d\Omega_1}{dt} = (B - C)\Omega_2\Omega_3 - \beta x_0 + \gamma y_0, \quad \frac{d\alpha}{dt} = \beta\Omega_3 - \gamma\Omega_2, \quad (4a)$$

$$B \frac{d\Omega_2}{dt} = (C - A)\Omega_1\Omega_2 - \gamma x_0 + \alpha z_0, \quad \frac{d\beta}{dt} = \gamma\Omega_1 - \alpha\Omega_3, \quad (4b)$$

$$C \frac{d\Omega_3}{dt} = (A - B)\Omega_1\Omega_2 - \alpha y_0 + \beta x_0, \quad \frac{d\gamma}{dt} = \alpha\Omega_2 - \beta\Omega_1, \quad (4c)$$

for the components of the angular velocity vector $\vec{\Omega}$ and angular momentum \vec{I}

$$\vec{\Omega} = \sum_{i=1}^3 \Omega_i \vec{e}_i, \quad \vec{I} = A\Omega_1 \vec{e}_1 + B\Omega_2 \vec{e}_2 + C\Omega_3 \vec{e}_3, \quad (4d)$$

with respect to the moving trihedral $\vec{e}_i, i = 1, 2, 3$ fixed on the body. Here the vertical unit vector \vec{e} and the centre of mass \vec{r}_0 are given by

$$\vec{e} = \alpha \vec{e}_1 + \beta \vec{e}_2 + \gamma \vec{e}_3, \quad \vec{r}_0 = x_0 \vec{e}_1 + y_0 \vec{e}_2 + z_0 \vec{e}_3. \quad (4e)$$

In order to identify the parametric choices for which the nonlinear dynamical system (4) becomes completely integrable Kovalevskaya used the novel idea that the solutions of integrable cases will be meromorphic (that is solutions will be free from movable critical points, namely movable branch points and essential singularities) in the complex time plane, just as in the case of equations satisfied by elliptic functions. S.Kovalevskaya was far ahead of her time in realizing the connection between meromorphicity and integrability. It took almost a century before mathematical physicists could appreciate such an approach and even now the implications are not clearly understood.

S.Kovalevskaya was probably motivated by the works of R.Fuchs, who isolated that class of odes whose solutions are meromorphic from out of all first order differential equations of the form

$$\frac{dy}{dx} = F(x, y), \quad (5)$$

where F is analytic in x and algebraic in y . It was shown that only the solution of Riccati equation

$$\frac{dy}{dx} + P_1(x)y + P_2(x)y^2 = P_3(x), \quad (6)$$

is free from movable critical points. Further the fact that elliptic functions are meromorphic might have lead Kovalevskaya to seek solutions of the form

$$x_i(t) = \sum_{n=0}^{\infty} a_{i,n}(t - t_0)^{n - p_i}, \quad (7)$$

for the dynamical variables in eq.(4) where p_i is an integer, which is to be determined along with the coefficients $a_{i,n}$.

Kovalevskaya identified essentially four nontrivial parametric choices from the above analysis for which eqs.(4) are integrable, out of which three were already known:

- i) $A = B = C$ (well known)
- ii) $x_0 = y_0 = z_0$ (due to Euler)
- iii) $x_0 = y_0 = 0, A = B$ (due to Lagrange)
- iv) $y_0 = z_0 = 0, A = B = 2C$ (new)

For each of the four integrable cases, four independent involutive integrals of motion exist [Lakshmanan and Sahadevan, 1993]. For the first three cases, the solutions are expressible in terms of elliptic functions which are meromorphic, while for the Kovalevskaya's fourth case, the solutions are given by hyperelliptic functions, which are in this particular case again meromorphic [Kruskal and Clarkson, 1991]. Even after one century, these results stand the test of time and no further new integrable cases have been found, thereby proving Kovalevskaya's farsighted intuition.

Unfortunately the method was not pursued further by dynamical systems community until late 1970s, except for the mathematicians P.Painlevé and his coworkers, Gambier, Garnier and so on(during 1900-10). The latter authors isolated those second order differential equations, whose solutions are free from movable critical points, of the form

$$\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}, y, x\right), \quad (8)$$

where F is a rational function of $\frac{dy}{dx}$ and y , and analytic in x . Painlevé and coworkers showed that out of all possible equations (8), there are only fifty or so canonical types which

have the property of their solutions having no movable critical points, out of which 44 are solvable by elementary functions including elliptic functions and the remaining required new transcendental functions, the so called Painlevé transcendental functions. For fuller details see Ablowitz and Clarkson [1991], Lakshmanan and Sahadevan [1993].

Again unfortunately, these important developments have not received much attention among the scientific community and very little work was done on the classification of higher order ordinary differential equations or partial differential equations, except for some isolated studies on third order odes by Bureau, Chazy and so on (see for example, Ablowitz and Clarkson, 1991). The full implications of these investigations have to again wait for many more decades until 1980s when the connection to soliton equations was established.

2.3 Poincaré's work on sensitive dependence on initial conditions

Henri Poincaré (1854-1912), the pioneering mathematician, physicist and philosopher of the early part of this century, in his famous works on celestial mechanics was involved with the problem of stability of motion of dynamical systems, like the three body gravitational problem, and the problem of finding precise mathematical formulas for the dynamical history of complex systems. In the course of these studies he was led to the notion of (Poincaré) surface of section and the concept of sensitive dependence of motions on initial conditions.

Poincaré concluded in his essay on Science and Method "It may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible". Thus Poincaré conceived the notion of deterministic chaos and the associated motion which is sensitively dependent on initial conditions in nonlinear systems even as early as the beginning of this century (for more details see for example Brillouin[1964] and Holmes[1990]). However it took several decades before Poincaré's ideas could be understood for their full ramifications.

2.4 van der Pol's investigations on coexisting multiperiodic solutions in forced nonlinear oscillators

The coexistence of several multistable periodic solutions and non-periodic (highly unstable) solutions in forced nonlinear oscillators was realized as early as 1927 in their remarkable experimental study by van der Pol and van der Mark [1927]. They analysed essentially the circuit given in Fig.1 [Jackson, 1991]. It has a neon glow lamp Ne, a battery $E(\approx 200V)$, a resistor R(of several megaohms), and an applied emf $E_0(\approx 10V)$. In the absence of the emf the period of the system increases with increasing capacitance C.

There are essentially three important discoveries associated with the experiments of van der Pol and van der Mark.

- 1) Presence of more than 40 subharmonics (Ω/n) of the applied frequency, Ω .
- 2) These subharmonics were found to be entrained over a limited range of C. As C was further varied, the frequency changed discontinuously to another subharmonic(Fig.2).
- 3) Observation of bands of 'noise' in the regions of many transitions of the frequency, which was regarded as a 'subsidiary phenomena'. Also their figure clearly showed a hysteresis effect,

hence a dynamical bistability in the system.

Thus van der Pol-van der Mark's observations were clearly the precursors to the to modern theory of chaotic nonlinear oscillators. Again it took more than five or six decades to fully understand the implications of these far reaching discoveries. However, one must also note that the experiments of van der Pol and van der Mark had historically lead to a number of important theoretical works to understand nonlinear phenomena in the underlying oscillator systems, notable among which is the work of Cartwright and Littlewood[1945] and of Levinson[1949]. The latter authors works, though abstract, were concerned with the analysis of the underlying dynamical equation to describe the current/voltage in Fig.1 in the form

$$\ddot{x} + k(\dot{x}^2 - 1)\dot{x} + x = f \cos \Omega t. \quad (. = \frac{d}{dt}) \quad (9)$$

They noted the existence of the limit cycle solution to (6). Ultimately all these works got perfected in the recent works in nonlinear dynamics(see for example, Lakshmanan and Murali, 1996).

2.5 Einstein's observations

Starting from the days of electromagnetism there had been attempts to describe elementary particles in terms of localized solutions of nonlinear field equations. Typical examples in this direction are Mie's theory of point particles and nonlinear electrodynamics of Born and Infeld (see for example, Schiff [1962]). It is interesting to note Einstein's view [Einstein, 1965] regarding classical nonlinear field theories. He remarks: "Is it conceivable that a classical field theory permits one to understand the atomistic and quantum structure of reality? ... I believe that at present time nobody knows anything reliable about it ... We do not possess any method at all to derive systematically solutions that are free of singularities. Approximate methods are of no avail since one never knows whether or not there exists to a particular approximate solution an exact solution free of singularities. Only a significant progress in the mathematical methods can help here ..."

The above remark very clearly makes profound prediction of the relevance of nonlinear dynamics.

3 The Revolution: Integrability and Chaos

The period 1950-70 can be considered as the golden age of nonlinear physics when revolutionary discoveries were made on integrable and chaotic systems leading to the present advances. Among these developments, the celebrated Fermi-Pasta-Ulam(FPU) experiments and the associated paradox [Fermi, Pasta & Ulam, 1955] may be rightly considered as the harbinger of a new era in physics. The various attempts to explain the FPU paradox ultimately resulted in the discovery of 'solitons' in the KdV equation by Martin Kruskal and Norman Zabusky in 1965 [Zabusky and Kruskal, 1965], simultaneously giving an integrable approximation resolution of the FPU paradox.

Interestingly, the concept of chaos was also found to be lurking behind the FPU experiments. In fact the simplest FPU lattice can be mapped onto the celebrated Henon-Heiles

system discovered in 1964 [Henon and Heiles, 1964] exhibiting the notion of chaos, namely sensitive dependence on initial conditions, when the nonlinearity is sufficiently large. In the same period, the atmospheric scientist Lorenz discovered that a grossly reduced set of convection equations in the form of a set of three coupled first order ordinary nonlinear differential equations, namely Lorenz equations [Lorenz, 1963], also show sensitive dependence on initial conditions leading to dissipative chaos, thereby heralding the era of chaotic dynamics.

The above first mentioned works on soliton systems were followed by the further discoveries of inverse scattering method and other soliton generating techniques for a large class of nonlinear dispersive systems in (1+1) dimensions and confirmed the fact that the KdV solitons are not fortuitous entities but form one of the most important basic coherent structures of nonlinear dynamics. This has led to the stage for the application of solitons in diverse fields of physics. Similarly the path breaking discoveries of Henon and Heiles, and of Lorenz that certain nonlinear systems can exhibit sensitive dependence on initial conditions culminated in the development of possible routes to and characterization of chaos in the 1970s, leading to revolutionary implications in physics.

We will concisely discuss these developments in the following sections.

3.1 The FPU paradox: A harbinger of revolutionary era of non-linear physics

In the early 1950s, Enrico Fermi, Stan Ulm and John Pasta were set to make use of the MANIAC-I at Los Alamos Laboratory on important problems in physics (for a detailed account see for example Ford, [1992]). In particular, Fermi felt that it will be highly instructive to integrate the equations of motion for judiciously chosen, one dimensional, harmonic chain of mass points, weakly perturbed by nonlinear forces. The expectation was that the state of the chain as it evolves could not be accurately predicted after a finite time and it could form a simple model to test the various sophisticated questions related to irreversible statistical mechanics. To begin with they intended to test the simplest and most widely believed assertions of equilibrium statistical mechanics such as equipartition of energy, ergodicity and the like.

The FPU model is essentially the one-dimensional chain of (N-1) moving mass points having the Hamiltonian

$$H = \sum_{i=1}^{N-1} \frac{P_i^2}{2} + \frac{1}{2} \sum_{i=0}^{N-1} (Q_{i+1} - Q_i)^2 + \frac{\alpha}{3} \sum_{i=0}^{N-1} (Q_{i+1} - Q_i)^3, \quad (10)$$

where $Q_0 = Q_N = 0$ and Q_i and P_i are the coordinate and momentum of the i th particle, and α is a small nonlinearity parameter. FPU had in addition considered quartic and broken linear couplings.

Now looking through a normal mode decomposition

$$A_l = \sqrt{\frac{2}{N}} \sum_{k=1}^{N-1} Q_k \sin\left(\frac{kl\pi}{N}\right), \quad (11)$$

one has essentially a system of independent harmonic oscillators weakly coupled by terms cubic in the normal mode positions, given by the Hamiltonian

$$H = \frac{1}{2} \sum (\dot{A}_k^2 + \omega_k^2 A_k^2) + \alpha \sum C_{klm} A_k A_l A_m, \quad (12)$$

where $\omega_k = 2 \sin(\frac{k\pi}{2N})$ is the frequency of the k th normal mode and C_{klm} are constants. Then $E_k = \frac{1}{2}(\dot{A}_k^2 + \omega_k^2 A_k^2)$ is the energy of the k th normal mode and to a first approximation $H = \sum E_k$, when α is small.

How does E_k change as a function of time when the weak nonlinear forces are present? The results of FPU for the lattice (12) (or (10)) with $N=32$ and $\alpha = \frac{1}{4}$ are given in Fig.3, with an initial shape at $t = 0$ in the form of a half-sine wave given by $Q_k = \sin(\frac{k\pi}{32})$ so that only the fundamental harmonic mode was excited with an amplitude $A_1 = 4$ and energy $E_1 = 0.077 \dots$. During the time interval $0 \leq t \leq 16$ in Fig.3, where t is measured in periods of the fundamental mode, modes 2,3,4, etc. sequentially begin to absorb energy from the initially dominant first mode as one would expect from a standard analysis. After this, the pattern of energy sharing undergoes a dramatic change. Energy is now exchanged primarily only among modes 1 through 6 with all higher modes getting very little energy. In fact the motion of the anharmonic lattice is almost periodic and even perhaps quasiperiodic, with a recurrence period (FPU recurrence) at about $t = 157$ fundamental periods. The energy in the fundamental mode returns to within 3% of its value at $t = 0$.

FPU immediately realized that the above results are simply astounding. First, they appear to violate the canons of statistical mechanics, which assert that the above type of nonlinear system should exhibit an approach to equilibrium with energy being shared equally among degrees of freedom. But even more astonishing, they seem to invalidate Fermi's theorem regarding ergodicity in nonlinear systems. Indeed, Fermi is said to have remarked that these results might be one of the most significant discoveries of his career.

Though the FPU results are truly path-breaking, it is curious to know that it took almost ten years for the matter to reach open literature that too just as part of Fermi's collected works [see Ford, 1992]. Originally a preprint was available in November 1955 as Los Alamos preprint LA-1940(7 November 1955), but then unfortunately Fermi died and the paper was never published. This had a rather inhibitory effect as many people took the view that the results are too preliminary and it did not undergo peer review and perhaps does not warrant full attention. Nevertheless, the results became familiar through word of mouth and personal discussions and many serious efforts started being made to explain the FPU paradox.

3.2 Integrable approximation: The Zabusky-Kruskal discovery of soliton

Kruskal and Zabusky, through an asymptotic analysis, had sought a continuum approximation to FPU(see for example, Ablowitz and Clarkson, [1991]). As the equation of motion of the anharmonic FPU lattice (10) can be written as

$$\ddot{Q}_k = (Q_{k+1} - 2Q_k + Q_{k-1})[1 + \alpha(Q_{k+1} - Q_{k-1})], \quad (13)$$

in the lowest order continuous limit it takes the form

$$Q_{tt} = Q_{xx} + \varepsilon Q_x Q_{xx} = (1 + \varepsilon Q_x) Q_{xx}. \quad (14)$$

When $\varepsilon = 0$, eq.(14) is just the wave equation and when $\varepsilon \neq 0$, eq.(14) is hyperbolic and can develop shock. Going over to the next order correction, under suitable asymptotic limit, Kruskal and Zabusky showed that the shock formation can be avoided with the addition of suitable dispersion so that a useful approximation to the FPU lattice results. Its specific form read

$$Q_{tt} = Q_{xx} + \varepsilon Q_x Q_{xx} + \beta Q_{xxx}. \quad (15)$$

Restricting attention to unidirectional waves, with the replacement of x by $\sigma = x - t$, t by $\tau = \varepsilon t$, Q_x by $U = \frac{1}{2}Q_x = \frac{1}{2}Q_\sigma$ in eq.(15), and neglecting the terms proportional to ε^2 , they obtained the celebrated Korteweg- de Vries equation

$$U_\tau + UU_\sigma + \delta^2 U_{\sigma\sigma\sigma} = 0, \quad (16)$$

which under rescaling can be recast in the standard form

$$u_t + 6uu_x + u_{xxx} = 0. \quad (17)$$

Eq.(17) is nothing but the Korteweg-de Vries equation (1) derived to represent the Scott-Russel phenomenon as described in Sec.2.1, but now appearing in an entirely different physical context.

Of course eq.(17) admits solitary waves of the form (2), but Zabusky and Kruskal[1965] went on further to numerically integrate eq.(16)(and so (17)) using periodic boundary conditions and one cycle of a cosine as initial condition. Much to their surprise, the initial cosine shape evolved into a finite number of relatively short pulses(Fig.4) that moved at distinct speeds about their periodic paths like runners on a track. Upon collision, the pulses would exhibit a nonlinear superposition during overlap and then would emerge unchanged in shape or speed.

The almost-periodic behaviour of the FPU systems could now be understood at an especially clear and intuitive level. The first full recurrence of the FPU motion occurs when all the pulses approximately overlap, generating a near return to the initial cosine shape. The recurrence period calculated by Zabusky and Kruskal closely approximates the actual FPU recurrence period, showing that KdV is a suitable long wavelength approximation of FPU system. Thus the Zabusky-Kruskal analysis provides an intuitively delightful interpretation of the FPU phenomenon, wherein the power of nonlinearity is made transparent.

But more interestingly the Zabusky-Kruskal experiments paved the way to provide insight into a much larger class of problems, turning the FPU paradox into a real discovery in physics (and mathematics), namely the invention of solitons, which are ubiquitous in nature. The KdV equation itself has become a paradigm for an expanding class of completely integrable nonlinear differential equations of dispersive type admitting soliton solutions, and they possess Lax pair and solvable by inverse scattering transform procedure. In particular the solitary waves of these systems under collision retain their shape and speed except for a phase shift, as demonstrated by Zabusky and Kruskal[1965] for the KdV equation. Such solitary waves have been termed as solitons due to their particle-like properties by Zabusky and Kruskal. A typical two soliton scattering property (for the KdV) is illustrated in Fig.5.

3.3 Lax pair and inverse scattering formulation of KdV

The remarkable stability properties of the soliton solutions of the KdV equation and the asymptotic form of the solutions in the form of N-number of solitons in the background of small amplitude dispersive waves had motivated Kruskal and coworkers to search for analytic methods to solve the initial value problem of the KdV equation. In particular Gardner, Greene, Kruskal and Miura[1967] realized that the KdV equation is linearizable in the sense that it can be associated with two linear differential operators L and B, the so called Lax pairs. Considering the time-independent Schrödinger spectral problem

$$L\psi = \lambda\psi, \quad L = -\frac{\partial^2}{\partial x^2} + u(x, t), \quad (18)$$

where λ is the eigenvalue parameter and u is the unknown potential in which t is a parameter and the associated time evolution equation for $\psi(x, t)$,

$$\psi_t = B\psi, \quad B = -4\frac{\partial^3}{\partial x^3} - 6u\frac{\partial}{\partial x} - 3u_x, \quad (19)$$

then the compatibility of eqs.(18) and (19) with the condition that λ is a constant in time leads to the Lax equation

$$L_t = [L, B] \iff KdV. \quad (20)$$

Thus given the initial condition $u(x, 0)$ and analysing the linear equations (18) and (19), the initial value problem can be solved and all the numerical results of Zabusky and Kruskal discussed in the earlier sections can be obtained exactly. For example the two soliton solution of the KdV can be obtained in the form

$$u(x, t) = 2(k_2^2 - k_1^2) \frac{k_2^2 \operatorname{cosech}^2 \gamma_2 + k_1^2 \operatorname{sech}^2 \gamma_1}{(k_2 \coth \gamma_2 - k_1 \tanh \gamma_1)^2}, \quad (21)$$

$$\gamma_1 = k_1 x - 4k_1^3 t + \delta_1, \quad \gamma_2 = k_2 x - 4k_2^3 t + \delta_2,$$

where $k_1, k_2, \delta_1, \delta_2$ are constants, whose structure is exactly the same as given in Fig.5. Similarly the explicit form of N-soliton solutions can also be obtained (see for example, Ablowitz and Clarkson, [1991]). KdV equation has also the remarkable property that it is a completely integrable infinite dimensional dynamical system in the sense that it possesses infinite number of involutive, functionally independent integrals of motion, which can be directly related to its linearizability property (see Sec.4. below).

3.4 Other soliton equations in (1+1) dimensions

The KdV equation is ubiquitous in the sense it occurs in a large number of physical problems in areas as disparate as fluid dynamics, condensed matter physics, quantum field theory and astrophysics and so on. Interestingly it is not only the KdV equation that admits solitons - there exists a large number of equally ubiquitous nonlinear dispersive wave equations of great physical importance which also admit soliton solutions, the major classes of which have been found during the decade following the work on KdV by Kruskal and his coworkers and the

list is ever growing (for details, see for example Ablowitz and Clarkson, [1991]). Informations about some of the well known equations in (1+1) dimensions are given in Table I.

The most crucial aspects of soliton equations is that it is not only the KdV equation but also a large class of equally important systems such as the one given in Table I which become linearizable in terms of different Lax pairs or linear spectral problems. For example, the Zakharov-Shabat(Z-S) and Ablowitz-Kaup-Newell-Segur(AKNS) matrix spectral problem

$$V_x = MV, \quad (22)$$

with the corresponding time evolution

$$V_t = NV, \quad (23)$$

gives rise to the compatibility equation

$$M_t - N_x + [M, N] = 0, \quad (24)$$

which is equivalent to the Lax equation (20). Different choices of M and N lead to different nonlinear evolution equations. The modified KdV, the nonlinear Schrödinger, the sine-Gordon and the Heisenberg ferromagnetic spin equations are some of the important evolution equations which are linearizable in the above sense. The linearized forms are also given in Table I.

Once the linearization is effected in the above sense that for a given nonlinear system $u_t = k(u)$, where $k(u)$ is a nonlinear functional of u and its spatial derivatives, the Cauchy initial value problem corresponding to the boundary condition $u \rightarrow 0$ as $x \rightarrow \pm\infty$ can be solved by a three step process shown schematically in Fig.6. This process which is known as the Inverse Scattering Transform(IST) procedure may be considered as the nonlinear analogue of the Fourier transform method applicable to linear dispersive systems. The method consists of the following steps:

(i) Direct scattering analysis: An analysis of the linear eigenvalue problem with the initial condition $u(x, 0)$ as the potential is carried out to obtain the scattering data $S(0)$. For example for the KdV,

$$S(0) = \{k_n(0), n = 1, 2, \dots, N, C_n(0), R(k, 0), -\infty < k < \infty\}. \quad (24)$$

Here N is the number of bound state eigenvalues k_n , $C_n(0)$ are the normalization constants of the bound state eigenfunctions, and $R(k, 0)$ is the reflection coefficient for the scattering states.

(ii) Time Evolution: Using the asymptotic form of the time evolution equation for the eigenfunctions, the time evolution of the scattering data $S(t)$ is determined.

(iii) Inverse Scattering: The set of Gelfand-Levitan- Marchenko linear integral equations corresponding to the scattering data $S(t)$ is constructed and solved. The resulting solution consists typically of N number of localized, exponentially decaying(soliton) solutions asymptotically ($t \rightarrow \pm\infty$).

Thus once a given nonlinear evolution equation is fitted into the Lax pair and inverse scattering formalism, its Cauchy initial value problem can be solved, soliton solution obtained and completely integrability proved. For a more general list of such integrable equations, we refer the reader to Ablowitz & Clarkson [1991].

3.5 The FPU lattice, Henon-Heiles system and transition to chaos

The fact that the FPU lattice given by eq.(10) did not give rise to the expected statistical behaviour and that the integrable KdV approximation gives satisfactory explanation of the recurrence behaviour in the long wavelength limit made many people to wonder whether the FPU lattice itself is a completely integrable dynamical system. However this doubt was soon dispelled by the following fact[Ford, 1992]. Consider a three-particle FPU system having periodic boundary conditions which is governed by the Hamiltonian

$$H = \sum_{k=1}^3 \frac{P_k^2}{2} + \frac{1}{2} \sum_{k=1}^3 (Q_{k+1} - Q_k)^2 + \frac{\alpha}{3} \sum_{k=1}^3 (Q_{k+1} - Q_k)^3, \quad (25)$$

where $Q_4 = Q_1$. After introduction of a canonical change of variables to harmonic normal mode coordinates (ξ_k, η_k) , the Hamiltonian (25) becomes

$$H = \frac{1}{2}(\eta_1^2 + \eta_2^2 + \eta_3^2) + \frac{3}{2}(\xi_2^2 + \xi_3^2) + \frac{3\alpha}{\sqrt{2}}(\xi_2\xi_3^2 - \frac{1}{3}\xi_2^3). \quad (26)$$

Now transforming to the centre of mass frame and setting $t = \frac{\tau}{\sqrt{3}}$, $\eta_2 = \frac{\sqrt{2}}{\alpha}q_2$, and $\eta_3 = \frac{\sqrt{2}}{\alpha}q_1$, the Hamiltonian (26) can be rewritten as

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2q_2 - \frac{1}{3}q_3^2. \quad (27)$$

But the Hamiltonian (27), canonically equivalent to the three-particle FPU lattice, is the celebrated Henon-Heiles system originally introduced by astronomers Henon and Heiles [1964] during the same revolutionary era which shows the other facet of nonlinear dynamical systems, namely chaos in deterministic systems.

Henon and Heiles were interested in determining whether a third integral existed which constrained the motion of a star in a galaxy which had an axis of symmetry. Such a system has three degrees of freedom and two known isolating integrals of motion which are the energy and one component of angular momentum. It was long thought that such systems do not have a third isolating integral because none had been found analytically. However, the nonexistence of a third integral implies that the dispersion of velocities of stellar objects in the direction of the galactic center is the same as that perpendicular to the galactic plane. What was observed, however was a 2:1 ratio in these dispersions. Henon and Heiles [1964] constructed exactly the Hamiltonian (27) to model the essential features of the problem and studied it numerically by solving the equation of motion,

$$\frac{dq_i}{dt} = p_i, \quad i = 1, 2, \quad (28a)$$

$$\frac{dp_1}{dt} = -q_1 - q_1q_2, \quad \frac{dp_2}{dt} = -q_2 - 2q_1^2 - 2q_2^2. \quad (28b)$$

A sketch of their results is shown in Fig.7 in the form of a Poincaré surface of section. At low energy (Fig.7a) there appears to be a third integral, at least to the accuracy of these

figures (enlargement of the region around the hyperbolic fixed points would show a scatter of points). As the energy is increased (equivalent to the increase of the effect of the nonlinear terms as may be seen from a scaling argument) the third integral appears to be destroyed in the neighborhood of the hyperbolic fixed points (Fig.7b). At still higher energies (Fig.7c), the second isolating integral appears to have been totally destroyed. The scattered points in the Henon-Heiles plots correspond to a single trajectory which is chaotic. Such trajectories are chaotic in that they have "sensitive dependence on initial conditions". This is an entirely new kind of structure, originally envisaged by Poincaré (cf sec.2.3) but which is transparent in a simple Hamiltonian system now. Additionally, this shows that the original FPU system is not integrable and it can develop complex motion if the strength of the nonlinearity in eq. (10) is increased. (For fuller details see for example Reichl [1992]). And with this the era of chaos in Hamiltonian system has started in right earnest in physics!

3.6 The Lorenz system and dissipative chaos

Edward Lorenz, an atmospheric scientist and meteorologist, who was interested in the long term behaviour of the atmospheric weather, reduced [Lorenz, 1963] the bare essentials of the underlying dynamical equations to a system of three first order nonlinear coupled differential equations

$$\begin{aligned}\dot{x} &= -\sigma(x - y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy.\end{aligned}\tag{29}$$

The above equations essentially represent a model of two-dimensional convection in a horizontal layer of fluid heated from below. In eq.(29) x represents the velocity and y, z the temperature of the fluid at each instant, and r, σ, b are positive parameters determined by the heating of the layer of the fluid, the physical properties of the fluid, and the height of the layer.

The general understanding in dynamics, particularly in weather prediction, is strongly influenced by Laplace dictum of complete predictability of solving Newton's equations of motion. This means small deviations in the input or at any stage of the calculation will lead to small uncertainties only in the output also. When Lorenz numerically integrated the equations of motion (29), which constitutes a dissipative system as the phase space volume shrinks,

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(\sigma + b + 1) < 0,\tag{30}$$

he discovered the now ubiquitous chaotic behaviour, the so called butterfly effect: small deviations can grow exponentially fast in a finite time - in the popular language as small as the effect of a butterfly fluttering its wings somewhere in the Amazons can lead to a tornados in Texas in a few days time.

The physical consequence is that the long term weather prediction becomes almost impracticable and geometrically in the phase space dynamical systems can admit limiting motions which are extremely sensitively dependent on initial conditions (strange attractors).

Combined with Henon-Heiles invention of similar effect in Hamiltonian systems, the concept of chaos has now come to stay in dynamics.

Of course it is now well known that the Lorenz system admits several kinds of dynamical motions: equilibrium points, limit cycles of different periods, chaotic motions, strange attractors and so on. The known results for a typical choice of the parameters to illustrate them and the form of chaotic attractors are given in Figs.8.

3.7 The Skyrme's model

During the same revolutionary era, the distinguished English physicist T. H. R. Skyrme proposed a model of baryons as topological solitons in a series of papers during the period 1955-63 (for details see for example Chadso et al, [1993]). Unfortunately the model was mostly ignored by particle physics community until 1980s, when it was realized that the Skyrme model could be considered as the possible low energy limit of quantum chromodynamics. Since then the model has received enormous attention and reverence.

In the Skyrme model, the nucleus is considered to be a classical, electrically neutral incompressible (mesonic fluid), which occupies a region with radius R . The nucleons are immersed into this mesonic fluid, which saturates them, while freely moving inside it. Further, the mesonic fields could take their values on S^3 as the field manifold so that conserved topological charge, which can be interpreted as the baryon number, is associated with it. The Lagrangian suggested by Skyrme was of the form

$$L = -\frac{1}{4\lambda^2}Tr\vec{L}_\mu^2 + \frac{\epsilon^2}{16}Tr [\vec{L}_\mu, \vec{L}_\nu]^2, \quad (31)$$

where the chiral currents \vec{L}_ν are vector fields defined on the S^3 manifold with values on the $s(u, 2)$ algebra. Suggesting a hedgehog ansatz, Skyrme suggested [Skyrme, 1962] that the model could describe a stable extended particle with a unit topological charge and all finite dynamical characteristics, which should be quantized as fermions. Though the response was belated, the particle physics community has realized the importance of Skyrme's model as one of the most fundamental theories.

In the above discussions in the present section 3, we have tried to give a very personal bird's overview of the golden era of nonlinear dynamics when both the concepts of solitons and chaos were born and the role of the FPU paradox, highlighting the analysis of the KdV equation by Zabusky and Kruskal leading to the concept of solitons and the discovery of sensitive dependence on initial conditions in Henon-Heiles and Lorenz systems. These pioneering works will then obviously have far reaching consequences in physics. The remaining period of this century is then essentially a period of further understanding of these developments and consolidation of these results and application of these concepts in various physical systems. These are taken up in the next section.

4 The Modern Era of Nonlinear Physics: Coherent and Chaotic Structures and Their Applications

The period starting from 1970s has seen an exponential growth of research in nonlinear science, particularly in nonlinear physics and mathematics. Correspondingly novel areas of applications have been identified to utilize both the concepts of coherent and chaotic structures. The identification of more than a hundred completely integrable soliton systems in (1+1) dimensions and some possible extensions to (2+1) dimensions to identify exponentially localized solutions and the development of various analytic and algebraic techniques to isolate and investigate soliton systems are some of the important developments regarding integrable nonlinear systems in this period. Similarly, identification of various routes to chaos, characterization of chaotic attractors and identification of numerous chaotic dynamical systems in nature are few of the main progress in chaotic dynamics during the last three decades. There has been tremendous amount of applications of these ideas and potential technologies are unfolding due to these studies. We may mention a few of them: magnetoelectronics using nonlinear magnetic excitations, soliton propagation in optical fibres as a means of lossless propagation revolutionizing information technology and various opto-electronic devices in nonlinear optics, controlling aspects of chaos with its various ramifications including weather forecasting, synchronization of chaos and secure communications are some of the important payoffs realized during this current era. We will briefly discuss these developments in this section.

4.1 Soliton equations and techniques

As noted in the previous section, it is now almost 30 years since the soliton was invented by Zabusky and Kruskal in their numerical experiments on KdV, followed by the invention of the inverse scattering transform method by Gardner, Greene, Kruskal and Miura[1967] to solve the Cauchy initial value problem of it for vanishing boundary conditions (at infinity). This has ultimately led to the notion of complete integrability of infinite dimensional soliton systems, the list of which is ever expanding. Consequently the field has grown from strength to strength (see, for example, Ablowitz and Segur [1981], Ablowitz and Clarkson [1991], Fokas and Zakharov [1992], Lakshmanan [1988,1993,1995]). A great number of increasingly sophisticated mathematical concepts from linear operator theory, complex analysis, differential geometry, Lie algebra, graph theory, algebraic geometry, and so on are being ascribed to soliton phenomenon, while new physical, engineering and biological applications, where the soliton concept is found to play a crucial role, appear all the time (Lakshmanan [1995]).

There are two broad theoretical approaches available to deal with the soliton bearing nonlinear evolution equations, namely (i) analytic and (ii) algebraic methods, though overlapping and strong interconnections exist between them (Fig.9). Analytic approaches include the IST method its generalization (namely, the \bar{d} -bar approach) for solving the Cauchy initial value problem, and other soliton generating techniques such as the Hirota bilinearization method(Matsuno, [1984]), operator dressing method(Novikov et al, [1984]), Bäcklund transformation method (Rogers and Shadwick, [1982]), direct linearizing transform method (Ablowitz and Clarkson, [1991]), apart from Painlevé analysis(Weiss et al, [1983]) to test

integrability. On the other hand, complete integrability aspects of soliton systems including the existence of infinite number of integrals of motion can be associated with the generalized Lie-Bäcklund symmetries and the associated group theoretic, Lie algebraic, bihamiltonian structure and differential geometric properties (Asano and Kato, [1991]; Dickey, [1991]; Magnano and Magri, [1991]). Such complete integrability aspects can be further generalized to the area of quantum integrable systems, exactly solvable statistical models and so on (Wadati et al, [1989]) through Yang-Baxter relations and the quantum inverse scattering method. Also the study of perturbation of soliton systems, often leading to spatiotemporal complexity, is of great physical interest in condensed matter physics, fluid dynamics, nonlinear optics, liquid crystals and so on (Sanchez and Vazquez, [1991]; Hasegawa, [1989]; Lam and Prost, [1991]). Thus one finds the study of soliton-bearing systems is of fundamental importance in several branches of physics and natural sciences. We will now briefly mention some basic ideas of these various aspects.

4.2 Direct methods

The inverse scattering formalism described earlier in Sec.3.4 is quite sophisticated, although elegant. Often one would like to have simpler analytic methods to obtain explicit N-soliton solutions. Several invaluable direct methods have been developed in the literature for this purpose during the last three decades. Among them, the Hirota's bilinearization method and the Bäcklund transformation method have played very crucial roles in the development of the field as they help one to quickly obtain soliton solutions even when the IST formalism is not yet available for a particular evolution equation. Besides, they have deep algebraic and geometric set-ups, giving special significance to soliton systems. Apart from these methods, the prolongation structure method of Wahlquist & Estabrook [1975], the dressing method [Zakharov & Shabat, 1974], the Darboux method [Matveev & Salle, 1991] and the direct linearizing transform method [Santini et al., 1984] are some of the other important techniques available for soliton solutions. In this section we give the salient features of the Hirota method and the Bäcklund transformation procedures only. For the other methods, which have close connections with the above two and IST, the reader may refer to the references cited.

4.3 Hirota's bilinearization method

The essence of the method [Hirota, 1980; Ablowitz & Segur, 1981; Matsuno, 1984] is (i) to convert the given nonlinear evolution equation into what is known as bilinear forms, wherein each term is of degree 2 in the dependent variables, and (ii) to derive a formal power series solution which turns out to be the soliton solution for all soliton bearing NLEEs.

As an example, we again consider the KdV equation (17). Under the transformation

$$u = 2 \frac{\partial^2}{\partial x^2} \log F, \quad (32a)$$

Eq.(17) takes the bilinear form (after integration and setting the integration constant to zero)

$$F_{xt}F - F_xF_t + F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 = 0. \quad (32b)$$

It is advantageous to introduce the so-called Hirota's bilinear operator

$$D_x^m D_t^n (a.b) = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n a(x, t) b(x', t') \Big|_{\substack{x=x' \\ t=t'}}, \quad (33)$$

so that Eq.(32) takes the notationally simpler form

$$(D_x D_t + 4D_x^4) F.F = 0. \quad (34)$$

The properties of the bilinear operator can be easily worked out. Samples: $D_m^m a.1 = \partial_x^m a$, $D_m^m a.b = (-1)^m D_x^m b.a$, $D_x^m a.a = 0$, m odd, and so on. Using such properties the calculations can be simplified considerably.

Now expanding F in a formal power series in ε [Ablowitz & Segur, 1981],

$$F = 1 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \dots, \quad (35a)$$

where

$$f^{(1)} = \sum_{i=1}^N e^{\eta_i}, \eta_i = k_i x + \omega_i t + \eta_i^{(0)} \quad (35b)$$

and $k_i, \omega_i, \eta_i^{(0)}$ are constants, the N -solitons of KdV can be obtained. To see this, we substitute (34) in (33), equate each power of ε separately to zero and obtain to $0(\varepsilon^3)$

$$0(1) : 0 = 0, \quad (36a)$$

$$0(\varepsilon) : 2(\partial_x \partial_t + \partial_x^4) f^{(1)} = 0, \quad (36b)$$

$$0(\varepsilon^2) : 2(\partial_x \partial_t + \partial_x^4) f^{(2)} = -(D_x D_t + D_x^4) f^{(1)}.f^{(1)}, \quad (36c)$$

$$0(\varepsilon^3) : 2(\partial_x \partial_t + \partial_x^4) f^{(3)} = -2(D_x D_t + D_x^4) f^{(1)}.f^{(2)}, \quad (36d)$$

The procedure is then to use (35b) in (36b) and successively solve the remaining equations. In practice, one finds the solution for $N=1,2,3$ and then hypothesize it for arbitrary N which is to be proved by induction.

For example, for $N=1$, $f^{(1)} = e^{\eta_1}$ and from (36b), $\omega_1 = -k_1^3$ and $(\partial_x \partial_t + \partial_x^4) f^{(2)} = 0$, so that $f^{(2)} = 0$ and $f^{(i)} = 0, i > 2$. Thus the solution of (34) can be written as

$$F_1 = 1 + e^{\eta_1}, \eta_1 = k_1 x - k_1^3 t + \eta_1^{(0)}. \quad (37)$$

Making use of (31), it is straightforward to obtain the 1-soliton solution (3) of the KdV equation.

Similarly for the case $N=2$,

$$f^{(1)} = e^{\eta_1} + e^{\eta_2}, \eta_i = k_i x - k_i^3 t + \eta_i^{(0)}, i = 1, 2$$

so that the solution of (34) becomes

$$F_2 = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}}, \quad (38)$$

where A_{12} is constant. Again this leads to the 2-soliton solution (21) of the KdV discussed in Sec.2 by using the transformation (31). In an analogous fashion, one can proceed to find the N -soliton solution also. We also note that with the solution of the Hirota equation in a

form such as (38), it is quite easy to understand the elastic nature of the soliton interaction discussed in Sec.2.

As noted above, all the other known soliton equations can also be bilinearized and the soliton solution obtained through the Hirota method. Even though the Hirota method appears to be merely a device to generate soliton solutions, recent investigations reveal deeper meaning to the role of bilinear theory; new, beautiful group-theoretic and geometric connections can be ascribed to this method and one finds such integrable equations live in a phase-space which is an infinite dimensional Kac-Moody Lie algebra [Date et al., 1983].

4.4 Bäcklund transformations

Bäcklund transformations (BTs) are essentially a set of relations involving the solutions of differential equations. They arose originally in the theories of differential geometry and differential equations as a generalization of contact transformations (see, for example, Rogers & Shadwick [1982]). A classical example is the Cauchy-Riemann conditions, $u_x = v_y$ and $v_x = -u_y$, so that they are a BT from the Laplace equation into itself, as both u and v satisfy the Laplace equation. A BT is essentially defined as a pair of partial differential equations involving two dependent variables and their derivatives which together imply that each one of the dependent variables satisfies separately a partial differential equation. Thus, for example, the transformation

$$\begin{aligned} v_x &= F(u, v, u_x, u_t, x, t), \\ v_t &= G(u, v, u_x, u_t, x, t), \end{aligned} \tag{39}$$

will imply that u and v satisfy pdes of the operational form

$$P(u) = 0, \quad Q(v) = 0. \tag{40}$$

If P and Q are of the same form, then (40) is an auto-BT.

Thus for the sine-Gordon equation the original transformation of Bäcklund derived in 1880 is

$$\begin{aligned} \left(\frac{u+v}{2}\right)_x &= k \sin\left(\frac{u-v}{2}\right), \\ \left(\frac{u-v}{2}\right)_t &= \frac{1}{k} \sin\left(\frac{u+v}{2}\right), \end{aligned} \tag{41}$$

where k is a parameter, so that u and v satisfy the sine-Gordon equation, $\phi_{xt} = \sin\phi$. Similarly for the KdV, one can write down the BT as

$$w_x + w'_x = 2k - \frac{1}{2}(w - w')^2, \tag{42a}$$

$$w_t + w'_t = (w - w')(w_{xx} - w'_{xx}) - 2(u^2 + uu' + u'^2), \tag{42b}$$

where $w_x = u$ and $w'_x = u'$, so that u and u' satisfy (17). Similarly BTs can be worked out for all other soliton evolution equations also. For the various methods available to obtain BTs, see for example Miura[1976], Rogers & Shadwick[1982].

BTs can in principle be integrated to generate higher-order solitons, though in practice they are difficult to solve in general. The usual way to get around this difficulty is to utilize the permutability property of the BT, which states that if we make two successive BTs from a given initial solution u_0 , we will end up with the same final solution no matter what sequential order of the two BTs we take. In other words, if k_1 and k_2 represent the parameters of the two BTs and if

$$u_0 \xrightarrow{k_1} u_1 \xrightarrow{k_2} u_{12}, \quad u_0 \xrightarrow{k_2} u_2 \xrightarrow{k_1} u_{21},$$

then $u_{12} = u_{21}$. From such permutability one can derive a nonlinear superposition formula, expressing u_{12} algebraically in terms of u_0 , u_1 and u_2 . For example for the sG equation, from (40) the following superposition law can be derived:

$$u_{12} = u_0 + 4 \tan^{-1} \left[\frac{k_2 + k_1}{k_2 - k_1} \tan \left(\frac{u_1 - u_2}{4} \right) \right]. \quad (43)$$

The superposition law can be repeatedly applied to construct soliton solutions of higher and higher order and one can make symbolic computation packages such as MAPLE effectively for this purpose.

Another effective method to derive explicit BTs is to use the so-called dressing method [Zakharov & Shabat, 1974]. This method requires solving a certain factorization problem in order to generate new solutions from a given input solution. In the case of (1+1) dimensions, this factorization problem takes the form of a Riemann problem in the complex eigenvalue plane of the associated linear system.

4.5 Complete integrability of soliton systems

The IST solvable soliton equations may be considered to be completely integrable infinite dimensional Hamiltonian systems and they are associated with infinite number of integrals of motion (see for example, Ablowitz & Clarkson, [1991]). One can associate with each of these soliton equations an infinite number of conservation laws. For example for the KdV the first three reads

$$u_t + (3u^2 + u_{xx})_x = 0, \quad (50a)$$

$$(u^2)_t + (4u^3 + 2uu_{xx} - u_x^2)_x = 0, \quad (50b)$$

$$(u^3 - \frac{1}{2}u_x^2)_t + \left(\frac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 - u_xu_{xxx} + \frac{1}{2}u_{xx}^2 \right)_x = 0. \quad (50c)$$

The rest of them can be obtained recursively.

Furthermore, the existence of these infinite number of conservation laws can be associated with the existence of infinite number of generalized symmetries, namely the so called Lie-Bäcklund symmetries, from which using Noether's theorem through suitable recursion operators the integrals can be obtained. For fuller details see for example, Bluman & Kumei [1989].

One of the fundamental concepts which underlies the IST method of the solution is the interpretation that nonlinear evolution equations which are solvable by the IST scheme are completely integrable infinite dimensional Hamiltonian systems and IST can be thought of as a nonlinear transformation from physical variables to an infinite set of action-angle variables.

Such a description was developed for the KdV equation by Zakharov and Faddeev [1971] followed by others to different soliton equations (see Ablowitz & Clarkson [1991]).

Considering the KdV equation, it can be written as

$$u_t = \{u, H\}, \quad (52)$$

where H is the Hamiltonian

$$H = - \int_{-\infty}^{\infty} (u^3 - \frac{1}{2}u_x^2) dx, \quad (53)$$

and the Poisson bracket between two functionals $A(\alpha)$ and $B(\beta)$ are defined by

$$\{A(\alpha), B(\beta)\} \equiv \int_{-\infty}^{\infty} \left\{ \frac{\delta A(\alpha)}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta B(\beta)}{\delta u(x)} \right\} dx. \quad (54)$$

Here $\frac{\delta}{\delta u}$ stands for functional derivative. Then defining the canonical coordinates

$$P_j = k_j^2, \quad Q_j = -2 \ln |C_j|, \quad j = 1, 2, \dots, N, \quad (55a)$$

$$P(k) = k\pi^{-1} \ln |a(k)|^2, \quad Q(k) = -\frac{i}{2} \ln \left[\frac{b(k)}{\bar{b}(k)} \right], \quad -\infty < k < \infty \quad (55b)$$

where k_j is the bound state eigenvalue of the associated Schrödinger spectral problem, C_j is the corresponding normalization constant and $a(k)$ and $b(k)$ are related to transmission and reflection coefficients, one finds

$$\{P_j, Q_j\} = \delta_{ij}, \quad \{P_i, P_j\} = \{Q_i, Q_j\} = 0 \quad (56)$$

$$\{P(k), Q(k')\} = \delta(k - k'), \quad \{P(k), P(k')\} = \{Q(k), Q(k')\} = 0. \quad (57)$$

Then the Hamiltonian H becomes

$$\hat{H} = -\frac{32}{5} \sum_{j=1}^N P_j^{5/2} + 8 \int_{-\infty}^{\infty} k^3 P(k) dk, \quad (58)$$

so that writing the equations of motion one finds that

$$P(k, t) = P(k, 0), \quad Q(k, t) = Q(k, 0) + 8k^3 t, \quad (59a)$$

$$P_j(t) = P_j(0), \quad Q_j(t) = Q_j(0) - 16P_j^{3/2} t. \quad (59b)$$

Thus P 's and Q 's constitute an infinite set of action-angle variables and in this sense the KdV is a completely integrable Hamiltonian system. Any other soliton system can be shown to have exactly similar description.

4.6 The Painlevé property

In the earlier sections, we have seen that several interesting nonlinear dispersive wave equations admit soliton solutions so that the Cauchy initial value problem can be solved and they can be considered completely integrable (in the Liouville sense). However the question arises whether given a nonlinear partial differential equation, one can conclude before hand that it is integrable and that linearization can be effected. It is now well-recognized that a systematic approach to determine whether a nonlinear pde is integrable or not is to investigate the singularity structure of the solutions, in particular, their so-called Painlevé property. This approach, originally suggested by Weiss, Tabor & Carnevale [1983] (WTC), aims to determine the presence or absence of movable noncharacteristic critical singular manifolds (of branching type, both algebraic and logarithmic, and essential singular type). When the system is free from movable critical manifolds, the Painlevé property holds, suggesting its integrability. Otherwise the system is nonintegrable in general.

The above development is a natural generalization of analyzing ordinary differential equations (odes) as per the movable critical singular points admitted by the solutions in the complex plane of the independent variable. Such a procedure was originally advocated by the mathematicians Fuchs, Painlevé, Gambier, Garnier, Chazy, Bureau and others and was applied successfully to the rigid body dynamics by S. Kovalavskaya as mentioned in Sec.2.2 [Kruskal & Clarkson, 1992]. A recent revival is due to the findings of Ablowitz, Ramani & Segur [1980], who conjectured that similarity reductions of integrable soliton equations always lead to odes free from movable critical singular points [Lakshmanan & Kaliappan, 1983]. In recent times several integrable dynamical systems have been identified to be free from movable critical points (see, for example Lakshmanan & Sahadevan [1992]), thereby giving a useful criterion for integrability.

4.6.1 Painlevé analysis

Let us consider a NLEE of the form

$$u_t + K(u) = 0, \quad (60)$$

where $K(u)$ is a nonlinear functional of $u(x_1, x_2, \dots, x_M, t) = u(\mathbf{x}, t)$ and its derivatives up to order N , so that Eq.(60) is an N th order nonlinear pde. Then one may say that (60) possesses the Painlevé or P-property if the following conditions are satisfied.

(A) The solutions of (60) must be single-valued about the noncharacteristic movable singular manifold. More precisely, if the singular manifold is determined by

$$\phi(\mathbf{x}, t) = 0, \quad \phi_{x_i}(\mathbf{x}, t) \neq 0, \quad \phi_t(\mathbf{x}, t) \neq 0, \quad i = 1, 2, \dots, M, \quad (61)$$

and $u(\mathbf{x}, t)$ is a solution of the pde (60), then we have the Laurent expansion

$$u(\mathbf{x}, t) = [\phi(\mathbf{x}, t)]^{-m} \sum_{j=0}^{\infty} u_j(\mathbf{x}, t) \phi^j(\mathbf{x}, t), \quad (62)$$

where $\phi(\mathbf{x}, t)$, $u_j(\mathbf{x}, t)$ are analytic functions of (\mathbf{x}, t) in a deleted neighborhood of the singular manifold (61), and m is an integer.

(B) By the Cauchy-Kovalevskaya theorem the solution (62) should contain N arbitrary functions, one of them being the singular manifold ϕ itself and the others coming from the u_j 's.

Then the WTC procedure to test the given pde for its P-property essentially consists of the following three steps [Weiss et al., 1983]:

- (i) Determination of leading-order behaviours,
- (ii) Identification of powers j (resonances) at which the arbitrary functions can enter into the Laurent series expansion (62), and
- (iii) Verifying that at the resonance values j a sufficient number of arbitrary functions exist without the introduction of movable critical singular manifold.

An important feature of the WTC formalism is that the generalized Laurent series expansion can not only reveal the singularity structure aspects of the solution and integrability nature of a given pde, but can also provide an effective algorithm which in most cases successfully captures all its properties, namely the linearization, symmetries and so on.

As a simple application, we illustrate the above aspects with KdV as an example. Any other soliton system can also be likewise analyzed. For the KdV equation (17), we substitute the formal Laurent series expansion (62) around the singularity manifold $\phi(x, t) = 0$ and equate equal powers of ϕ to zero. One finds that the exponent $m=2$ and that at $j = -1, 4, 6$ arbitrary functions can enter the power series (62). Identifying the arbitrariness of ϕ with $j = -1$, recursively one finds

$$j = 0 : u_0 = -2\phi_x^2, \quad (63a)$$

$$j = 1 : u_1 = 2\phi_{xx}, \quad (63b)$$

$$j = 2 : \phi_x \phi_t + 6u_2 \phi_x^2 + 4\phi_x \phi_{xxx} - 3\phi_{xx}^2 = 0, \quad (63c)$$

$$j = 3 : \phi_{xt} + 6u_2 \phi_{xx} - 2u_3 \phi_x^2 + \phi_{xxxx} = 0, \quad (63d)$$

$$j = 4 : \frac{\partial}{\partial x}(\phi_{xt} + 6u_2 \phi_{xx} - 2u_3 \phi_x^2 + \phi_{xxxx}) = 0. \quad (63e)$$

Now it is clear that by the condition (63d), (63e) is always satisfied so that $u_4(x, t)$ is arbitrary. Similarly one can derive the condition at $j = 5$ and prove that at $j = 6$, $u_6(x, t)$ is arbitrary. The KdV equation is of third order, the Laurent series admits three arbitrary functions (without the introduction of a movable critical manifold) and so the Painlevé property is satisfied.

Now, if the arbitrary functions u_4 and u_6 are chosen to be identically zero and, further, if we require $u_3 = 0$ then $u_j = 0, j \geq 3$, provided u_2 satisfies the KdV. Thus we obtain the BT for the KdV in the form

$$u = (\log \phi)_{xx} + u_2, \quad (64)$$

where u and u_2 solve the KdV and ϕ satisfies (63c-d) with $u_3 = 0$. By a set of transformations, it is possible to show that the defining equations for ϕ are indeed equivalent to the linearizing equations (19), as well as the bilinear equation (32). Thus the analytic properties associated with the KdV equation can also be obtained from the Painlevé procedure as well. The same procedure can be applied to any other NLEE to obtain its integrability property in any dimension.

4.7 Some applications of soliton concept

The remarkable stability of soliton excitations in nonlinear dispersive systems caught immediately the imagination of physicists working in different areas of physics, apart from the traditional areas of hydrodynamics and fluid mechanics. Innumerable applications of soliton concept in diverse areas of physics have been realized during the past three decades (see for example, Ablowitz and Clarkson [1991]; Lakshmanan [1995]). We will consider here only a select few of them to illustrate the vast potentialities of the concept. These include applications in magnetism, nonlinear optics, liquid crystals and elementary particle physics.

4.7.1 Solitons in ferromagnets

Spin excitations in ferromagnets are effectively expressed in terms of the Heisenberg's nearest neighbour spin-spin exchange interaction with additional anisotropy and external field dependent forces. For the simplest isotropic case, the Hamiltonian for quasi-one dimensional ferromagnets is given by

$$H = -J \sum_{\{i,j\}} \vec{S}_i \cdot \vec{S}_j \quad (65)$$

where the spin operator $\vec{S}_i = (S_i^x, S_i^y, S_i^z)$ and J is the exchange integral. The Heisenberg equation of motion

$$\frac{d\vec{S}_i}{dt} = [S_i, H], \quad (66)$$

in the long wavelength, low temperature (semiclassical $\hbar \rightarrow 0$) limit can be expressed in terms of classical unit vectors

$$\frac{d\vec{S}_i}{dt} = \{S_i, H\}_{PB}, \quad \vec{S}_i^2 = 1, \quad (67)$$

where the Poisson brackets between two functions of spin can be defined as

$$\{A, B\}_{PB} = \sum_i \epsilon_{\alpha\beta\gamma} \frac{\partial A}{\partial S_i^\alpha} \frac{\partial B}{\partial S_i^\beta} S_i^\gamma. \quad (68)$$

Correspondingly the equation of motion for the Hamiltonian (65) can be written as

$$\frac{d\vec{S}_i}{dt} = J \vec{S}_i \times \{\vec{S}_{i+1} + \vec{S}_{i-1}\}. \quad (69)$$

Additional interaction can also be included in the same way. For example with a uniaxial anisotropy and external magnetic field along the z-direction, the Hamiltonian is

$$H = -J \sum_{\{i,j\}} \vec{S}_i \cdot \vec{S}_j + A \sum_i (S_i^z)^2 - \mu \vec{B} \cdot \sum \vec{S}_i, \quad (70)$$

so that the equation of motion for the spins becomes

$$\frac{d\vec{S}_i}{dt} = \vec{S}_i \times \{J (\vec{S}_{i+1} + \vec{S}_{i-1}) - 2AS_i^z \vec{n} + \mu \vec{B}\} \quad (71)$$

where $\vec{B} = (0, 0, B)$ and $\vec{n} = (0, 0, 1)$.

In the continuum limit,

$$\vec{S}_i(t) \rightarrow \vec{S}(x, t), \quad \vec{S}_{i+1}(t) = \vec{S}(x, t) + a \frac{\partial \vec{S}}{\partial x} + \frac{a^2}{2} \frac{\partial^2 \vec{S}}{\partial x^2} + \dots, \quad (72)$$

where a is the lattice parameter, the equation of motion (69) in the limit $a \rightarrow 0$ and after a suitable rescaling becomes

$$\frac{\partial \vec{S}}{\partial t} = \vec{S} \times \frac{\partial^2 \vec{S}}{\partial x^2}. \quad (73)$$

Similarly eq.(71) in the continuum limit becomes

$$\frac{\partial \vec{S}}{\partial t} = \vec{S} \times \left(\frac{\partial^2 \vec{S}}{\partial x^2} + 2AS^z \vec{n} + \mu \vec{B} \right). \quad (74)$$

Spin equations of the type (73) or (74) are special cases of the so called Landau-Lifshitz equation, which were derived originally by Landau and Lifshitz[1935] from phenomenological arguments. It was not until 1977 that the complete integrable nature of many of these systems was realized. In fact, by mapping eq.(73) on a moving space curve with curvature (Lakshmanan [1977])

$$\kappa(x, t) = \left[\frac{\partial \vec{S}}{\partial x} \cdot \frac{\partial \vec{S}}{\partial x} \right]^{1/2}, \quad (75)$$

and torsion

$$\tau(x, t) = \kappa^{-2} \left(\vec{S} \cdot \frac{\partial \vec{S}}{\partial x} \times \frac{\partial^2 \vec{S}}{\partial x^2} \right), \quad (76)$$

which are respectively related to energy density and current density, eq.(73) gets mapped onto the ubiquitous nonlinear Schrödinger equation (see Table I) (Lakshmanan [1977])

$$iq_t + q_{xx} + 2|q|^2 q = 0, \quad (77)$$

where the complex variable

$$q(x, t) = \frac{\kappa(x, t)}{2} \exp i \int_{-\infty}^x \tau dx'. \quad (78)$$

Thus the spin equation (73) itself becomes a completely integrable soliton system with the one soliton solution for the energy density being given by

$$\left(\frac{\partial \vec{S}}{\partial x} \right)^2 = \kappa^2(x, t) = 16\eta^2 \operatorname{sech}(2\eta(x - x_0) + 8\eta\xi t) \quad (79)$$

and the spin component becomes

$$S^Z(x, t) = 1 - \frac{2\eta^2}{(\xi^2 + \eta^2)} \operatorname{sech}^2(2\eta[x - 2\xi t - \theta_0]). \quad (80)$$

Also one can write down a Lax pair for (73) itself (Takhtajan, [1977])

$$L = iS \text{ and } B = -iS \frac{\partial^2}{\partial x^2} - iS_x \frac{\partial}{\partial x}, \quad S = \vec{S} \cdot \vec{\sigma}, \quad (81)$$

($\vec{\sigma}$: Pauli matrices)

so that an IST analysis can be performed directly thereby again proving the complete integrability of the spin system.

Further the above analysis also shows that the spin system (73) is geometrically equivalent to the nonlinear Schrödinger equation (Lakshmanan, [1977]) and that their eigenvalue problems are gauge equivalent (Zakharov and Takhtajan, 1979). Since then the analysis of nonlinear excitations in magnetic system has been drawing considerable interest. One finds that even in the presence of additional external magnetic fields or uniaxial or biaxial anisotropies the system continues to be completely integrable (for details see for example, recent reviews of Mikeska & Steiner, [1991]; Kosevich et al, [1990]). One can use the presence of such stable excitations to develop suitable statistical mechanics and obtain structure factors to compare with experimental results. Effects of further interactions leading to integrability and nonintegrability and damping effects, etc., are some of the important topics of current interest which are being pursued vigorously.

4.7.2 Solitons in optical fibers

It is well known now that during the past 20 years or so optical fibers have revolutionized the world's telephone system. By transforming speech into pulses of light and sending these pulses along ultra-clear glass fibres, communication engineers can pack thousands of telephone conversations into a single fiber. But in optical fiber communication, both dispersion and fiber loss are two important variables which decide the information capacity and distance of transmission. An effective technology to overcome the problem of dispersion and fiber loss is being developed starting from the idea of soliton based communications first proposed by Hasegawa in 1973 [Hasegawa, 1989]. Solitons in optical fiber was first observed by Mollenauer [1990] and the soliton laser was developed in 1984. Since optical soliton arises as the balance between the nonlinear effect and the group velocity dispersion effect as in the case of any other soliton phenomenon, no distortion of the pulse takes place to a first order as a consequence of the dispersion. As a result the optical soliton technology will be expected to make revolution in international communications in the next few years. However, when the light intensity of the soliton decreases due to the fiber loss, the pulse width of the soliton transmission system also requires suitable reshaping of the pulse by suitable amplifiers.

Mathematically, optical soliton is the stationary solution of the initial boundary value problem of the nonlinear Schrödinger equation for the light intensity $E(z, t)$

$$iE_z + i\kappa\kappa'E_t - \frac{\kappa''}{2}E_{tt} + \frac{\omega n_2}{c}|E|^2E = 0, \quad (82)$$

where $\kappa' (= \frac{1}{v_g})$ is the inverse of the group velocity, $\kappa'' = \frac{\partial^2 \kappa}{\partial^2 \omega^2}$ is the group velocity dispersion coefficient and n_2 is the nonlinear refraction coefficient, c is the velocity of light and ω is the

carrier frequency. Under the transformation $\tau = t - \frac{z}{v_g}$ eq.(82) can be written as

$$iE_z - \frac{\kappa''}{2}E_{\tau\tau} + \frac{\omega n_2}{c}|E|^2E = 0. \quad (83)$$

The one soliton solution then can be given as

$$E(z, t) = \sqrt{\frac{c^2}{\omega n_2}} \cdot \eta \cdot \text{sech} \eta (\tau + \kappa z - \theta_0) \cdot \exp \left\{ -i\kappa\tau + \frac{i}{2}(\eta^2 - \kappa^2)z \right\}, \quad (84)$$

where η , θ_0 and κ are constant parameters. The actual derivation of eq.(83) follows by starting from Maxwell's equations for the propagation of intense electromagnetic radiation along the optical fibre, which is a silica dielectric medium. Using appropriate slowly varying envelope approximations for propagation along the fibre direction and taking into account the necessary nonlinearity for the refractive index, eq.(83) can be obtained. (For details, see for example Agarwal [1995]).

4.7.3 Solitons in Liquid Crystals

Liquid crystal is a state of matter intermediate between liquid and crystal. The material is optically anisotropic and can flow atleast in one spatial dimension. The molecules of the organic compound showing liquid crystal phases can be either rod-like, disc-like or bowl like in shape. At low temperature, both the orientations and positions of the molecules are in order (long-range) and so we have the crystal phase. At high temperature, both types of degrees of freedom are in disorder and the material is in the isotropic liquid phase. However, within a certain temperature range, there may exist an intermediate phase (mesophase) in which the orientations are in order, but the positions are in disorder. Such an intermediate phase (mesophase) is called "nematic" and the other known mesophases are smectic, cholesteric and ferroelectric.

In liquid crystals, since the molecules have both orientational and translational degrees of freedom, the hydrodynamic equations of motion are coupled nonlinear equations on \vec{n} and \vec{v} where ' \vec{n} ' is the director, a unit vector representing the average orientation of the molecules and \vec{v} is the velocity of the centre of mass of the molecules and both \vec{n} and \vec{v} are functions of space and time. The orientation of the molecules can be detected optically as it is localized and orientational waves can be observed easily by the naked eye. This provides a convenient means of measuring the wave. Thus, the identification of solitons in liquid crystals play an important role in the switching mechanism of some ferroelectric crystal displays. (For details see for example, Lam and Prost [1991]; Lam [1995]).

(i) Similar to domain (Bloch) wave in a ferromagnet, a soliton in a liquid crystal is a smooth, localized state linking up two uniform states at the two far ends.

(ii) For propagating a soliton in a liquid crystal, the damping of the director is heavy resulting in the overdamped case in the equation of motion for the director. Thus, liquid crystals are generally nonintegrable systems and the solitons involved are usually just solitary waves.

(iii) In liquid crystals, reorientation of the molecules can generally induce fluid flows.

Considering an one dimensional shear of nematic, the orientation of the molecule θ obeys the overdamped sine-Gordon equation

$$k\theta_{xx} - r_1\theta_t + \frac{s}{2}(r_1 - r_2 \cos 2\theta) = 0. \quad (85)$$

(Here $s = \frac{dv}{dx}$ is the shear and r_1, r_2 are viscosities, k is the elastic constant). The above equation has two uniform steady states, $\pm\theta_0$, $\theta_0 = \frac{1}{2} \cos^{-1} \frac{r_1}{r_2}$. It is then possible to have a localized configuration of the molecules. When this configuration travels without distortion, we have a soliton.

As liquid crystals are also nonlinear optical materials (dielectrics), optical solitons can also be found in liquid crystals. When liquid crystals are subjected to an electric field E perpendicular to \vec{n} , they tend to align parallel to the field only if the electric field is greater than a threshold value ($E > E_{th}$) and the corresponding equation of motion is given by

$$A_{yy} + a_2\omega^2|A|^2A + 2i\omega a_1A_t - 2ikA_z = 0, \quad (86)$$

where E is assumed to be linearly polarized along x direction, propagating along z so that $E = A(y, z, t)e^{i(\omega t - kz)}$. When $A_t = 0$, the above equation reduces to NLS equation (self-focussing case). When $A_z = 0$ which corresponds to the case of self-modulation, equation (86) is scaled with a new time variable to NLS equation. In both the cases optical solitons exist.

4.7.4 Solitons in particle physics

The various models to describe different types of elementary particles and their interactions are generically nonlinear field models at a classical level (recall Einstein's view quoted in Sec. 2.5 in this regard), which then needs to be second quantized. In the (1+1) dimensional cases they often reduce to the well known integrable soliton systems such as the sine-Gordon equation, σ -model equation, Lund-Regge equation and so on. One may cite many examples, where nonlinearity arises (Note that we have already discussed the Skyrme model briefly in Sec.3), which are only illustrative and not exhaustive.

1. Higgs mechanism: In the so called standard model of electroweak gauge theory in order to give masses for the (weak) gauge bosons, a complex field with Lagrangian $L = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \mu^2\phi^2 + \lambda\phi^4$ is introduced. Looking at the vacuum solution $\langle\phi\rangle = \sqrt{-\frac{\mu}{\lambda}}$ and expanding $\phi \rightarrow \langle\phi\rangle + \phi$, the fluctuation ϕ becomes the Higgs field. A search for the Higgs particle is still elusive.

2. Non-abelian theory: It is a manifestly nonlinear theory. The $SU(2)$ gauge fields A_μ^a ($a = 1, 2, 3$) or $SU(3)$ gluons A_μ^a ($a = 1, \dots, 8$) have cubic and quartic interactions. The field strength is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc}A_\mu^b A_\nu^c \quad (87)$$

where f 's are the structure constants of the gauge group. Even in the absence of fermions, a free Yang-Mills theory is difficult to solve. In this case the Lagrangian is

$$L = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} \quad (88)$$

and the equation of motion is

$$D_\mu^{ab} F_{\mu\nu}^b = 0, \quad D_\mu^{ab} = \partial_\mu \delta^{ab} + f^{abc} A_\mu^c. \quad (89)$$

An interesting special case is the self-dual Yang-Mills equation where

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} - [A_\mu, A_\nu]. \quad (90)$$

There has been considerable work on the integrability of this equation and it is conjectured that all or at least most of the soliton equations are special reductions of it [see for example, Ward, 1985; 1986]. These four dimensional equations arise in the study of field theory and relativity. The SDYM equation is regarded as being “integrable” as a consequence of the so called “twistor” representation relating their solutions to certain holomorphic vector bundles.

3.Gravitation theory: Of course the gravitational field equations even in the absence of matter ($T_{\mu\nu} = 0$)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \quad (91)$$

is highly nonlinear. Classical solutions such as Schwarzschild, Kerr, etc. are well known. Recently Einstein’s field equations with axial symmetry in the form of Ernst equation was found to be integrable. There are other solitons bearing cases also known in the literature.

4.String theory: Much work has been carried out on string theory for quantum chromodynamics. These are highly complex nonlinear field equations and sophisticated mathematical results have come out of these studies. However we do not consider any details here. Interested reader may refer, for example, [Witten, 1985].

4.8 Solitons in higher dimensions

Do solitons or their localized counterparts exist in higher spatial dimensions too? If so, what are their characteristic features and when do they occur and what are their ramifications? These are of paramount physical importance as most natural systems are higher spatial dimensional in nature. Following the original works of Zakharov & Manakov [1985] and the references therein, Ablowitz, Fokas & coworkers [1983] and references therein, there has been intense activity in understanding nonlinear dispersive wave equations in higher dimensions during the past ten years or so. The Kadomtsev- Petviashvili(K-P), Davey-Stewartson(D-S) and Ishimori equations are some of the well studied (2+1) dimensional systems which are interesting generalizations of the (1+1) dimensional KdV, nonlinear Schrödinger and Heisenberg ferromagnetic spin (vide Table II) equations. Depending on the sign of the coefficients of certain terms, these equations are also further classified as KPI, KP II, DSI and DSII, etc., Naturally, these (2+1) dimensional equations are richer in structures, where boundary conditions play a crucial role [Ablowitz & Clarkson, [1991]]. Some of the (2+1) dimensional nonlinear coherent structures which have been invented in recent times are

(i) line solitons (for example KP and DS) which do not decay in all directions, but there exists certain lines on which the solutions are bounded but nondecaying.

(ii) lump solitons (for example KPI and DSII) which are localized but decay algebraically and do not in general suffer a phase-shift under collision.

(iii) **dromions** (for example DSI) which are driven by boundary effects, being localized and exponentially decaying excitations which in general undergo amplitude and velocity changes under collision but whose total number and energy are conserved [Fokas & Santini, 1990]. The explicit forms of some of these excitations are also given in Table II and some of them are displayed in Figs. 10-12.

We have noted above that certain nonlinear evolution equations in higher spatial dimensions are also linearizable. Thus they can also be analyzed through the three-step procedure indicated earlier. However, scattering analysis in higher dimensions is much more complicated than that in one dimension, and a new approach is required to deal with them. The d -bar approach [Ablowitz & Fokas, 1983; Beals & Coifman, 1989; Zakharov & Manakov, 1985] treats the scattering problem in any dimensions as a d -bar problem of analytic functions in complex variable theory. With this new interpretation, it is possible to approach the inverse scattering analysis for evolutions both in (1+1)- and (2+1)-dimensions in a unified way.

4.8.1 The d -bar problem

Given an analytic function

$$f(z) = u(x, y) + iv(x, y), z = x + iy, u, v, x, y \in \mathcal{R}, \quad (92)$$

the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad (93)$$

can be recast in the so-called d -bar form

$$\frac{\partial f}{\partial \bar{z}} = \bar{\partial} f = 0, \quad (94)$$

where $\partial/\partial \bar{z} = \bar{\partial}z = \bar{\partial} = (1/2)[(\partial/\partial x) + i(\partial/\partial y)]$. Thus $\bar{\partial}f = g(\neq 0)$ is a measure of the nonanalyticity of the function f . (Example: $f(z) = \bar{z}$).

Then the d -bar problem is whether, given the d -bar data $\bar{\partial}f$, one can invert it to get the function $f(z)$. The answer is, in general, yes. The procedure is to make use of the generalized Cauchy integral formula

$$f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(z - \zeta)} \frac{\partial f}{\partial \bar{\zeta}} d\zeta_R d\zeta_I + \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(z - \zeta)} d\zeta, \quad (95)$$

where the last term is typically the identity if f is normalized to unity for large z .

4.8.2 General set up of IST method

Considering the linear eigenvalue problems associated with the NLEEs in (2+1) dimensions, they may be written as

$$P(\lambda)\psi(\mathbf{x}, \lambda) = Q(\mathbf{x})\psi(\mathbf{x}, \lambda), \quad \mathbf{x} = (x, y), \quad (96)$$

where $P(\lambda)$ is a matrix-valued linear differential operator in $x, y \in \mathcal{R}^2$ and *analytic* in $\lambda \in \mathcal{C}$. Here $Q(\mathbf{x})$ is the “potential” and we assume $Q(\mathbf{x}) \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$. The (1+1) dimensional case is obviously a special case of (96). For example

$$P(\lambda) = \frac{d^2}{dx^2} + \lambda^2, \quad (97)$$

for K-dV equation, and

$$P(\lambda) = \frac{\partial^2}{\partial x^2} + i \frac{\partial}{\partial y}, \quad (98)$$

for KPI.

Since the potential $Q \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$, we can easily find the asymptotic form $\psi(\mathbf{x}, \lambda) \rightarrow e_\lambda(\mathbf{x})$ so that

$$P(\lambda)e_\lambda(\mathbf{x}) = 0, \quad |\mathbf{x}| \rightarrow \infty. \quad (99)$$

Then letting

$$\psi(\mathbf{x}, \lambda) = m(\mathbf{x}, \lambda)e_\lambda(\mathbf{x}), \quad (100)$$

so that

$$m(\mathbf{x}, \lambda) \rightarrow I \text{ as } |\mathbf{x}| \rightarrow \infty, \quad (101)$$

we have the modified eigenvalue problem

$$\mathcal{P}_\lambda m(\mathbf{x}, \lambda) = Q(\mathbf{x})m(\mathbf{x}, \lambda). \quad (102)$$

For example, for K-dV this reads

$$m_{xx} + 2i\lambda m_x = -um, \quad (103)$$

and for KP-I we have

$$m_{xx} + im_y + 2i\lambda m_x = -um. \quad (104)$$

4.8.3 Direct scattering

Considering the Green’s function associated with the eigenvalue problem (96),

$$\mathcal{P}_\lambda G_\lambda(\mathbf{x} - \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), \quad (105)$$

subject to the boundary condition (101), we have formally

$$m(\mathbf{x}, \lambda) = I + \int_\Omega G_\lambda(\mathbf{x} - \mathbf{x}')Q(\mathbf{x}')m(\mathbf{x}', \lambda)d\mathbf{x}' = I + G_\lambda * (Qm), (* : \text{convolution}) \quad (106)$$

where I is the unit matrix and Ω is the domain of integration. Rewriting (106), we have

$$(I - G_\lambda * Q)m = I. \quad (107)$$

Assuming now that formally the inverse of $(I - G_\lambda * Q)$ exists (which requires control over G and the potential Q), we can write the formal expression for the eigenfunction,

$$m(\mathbf{x}, \lambda) = (I - G_\lambda * Q)^{-1}.I. \quad (108)$$

Considering the analytic behaviour of the eigenfunction in the complex λ plane, its nonanalyticity is given by the d -bar data

$$\bar{\partial}m = \bar{\partial}[I - G_\lambda * Q]^{-1}.I = (I - G_\lambda * Q)^{-1}(\bar{\partial}G_\lambda) * (Qm). \quad (109)$$

On the other hand, we have from Eq.(102)

$$\mathcal{P}_\lambda.\bar{\partial}m = Q.\bar{\partial}m, \quad (110)$$

using the analyticity property of \mathcal{P}_λ . Thus $m(\mathbf{x}, \lambda)$ and $\bar{\partial}m$ are simultaneous eigenfunctions of \mathcal{P}_λ . From the completeness property of m , we have

$$\bar{\partial}m = Tm, \quad (111)$$

where T is the scattering operator obtained through a superposition of eigenfunctions. Comparing (109) and (111), T can be expressed as

$$T = [I - G_\lambda * Q]^{-1}(\bar{\partial}G_\lambda) * Q. \quad (112)$$

In typical one-dimensional problems such as the K-dV, sine-Gordon, etc. equations associated with the Z-S and AKNS eigenvalue problems, Eq.(112) turns out to be a local equation in λ and $m(\mathbf{x}, \lambda)$ is sectionally holomorphic (in the upper and lower half λ planes with finite number of isolated singular points), so that we have a local Riemann- Hilbert problem with singularities. Consequently T turns out to be the scattering matrix in these cases [Ablowitz & Fokas, 1983].

In the case of two-dimensional systems such as the KPI, DSI, Ishimori I equations, (112) turns out to be nonlocal even though m is sectionally holomorphic and as a result we have a nonlocal Riemann-Hilbert problem [Ablowitz & Clarkson, 1991].

On the other hand, in problems like KP II and DSII, $m(\mathbf{x}, \lambda)$ is nowhere analytic and so one has to deal with a full d -bar problem in these cases[Ablowitz & Clarkson, 1991].

4.8.4 Time evolution

Given the ‘‘potential’’ matrix $Q(\mathbf{x}, 0)$ in (96), which is available from the initial data, Eq.(112) defines the scattering data. In order to find the evolution of the scattering data, one essentially uses the time evolution of the eigenfunction of the form $\psi_t = B\psi$, where B is a matrix linear differential operator in which the unknown $Q(\mathbf{x}, t)$ occurs as coefficients.

For asymptotically vanishing potentials, $Q \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$, $\mathbf{B} \rightarrow \mathbf{B}_0$, which is independent of Q , as a result T evolves in a simple way, satisfying linear ordinary/partial differential equations. Consequently, the scattering data operator can always be determined in principle at an arbitrary time without any difficulty in all the cases where the potential vanishes at ∞ , once they are known at $t = 0$.

4.8.5 Inverse scattering

Given the scattering data operator T and the nonanalyticity of the function $\bar{\partial}m$, one can carry out an inverse scattering analysis[Beals & Coifman, 1989] to retrieve the function m by

invoking the generalized Cauchy integral formula (95), from which Q itself can be obtained uniquely.

Formally this can be done as follows. Let C be the inverse of $\bar{\partial}m$. Then taking into account the normalization of m as $\lambda \rightarrow \infty$, we can write

$$\begin{aligned}
m(\mathbf{x}, t, \lambda) &= I + C.\bar{\partial}m \\
&= I + C.Tm \\
&= I + \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(\lambda - \zeta)} \times \left(\frac{\partial m}{\partial \bar{\zeta}} \right) d\zeta_R d\zeta_I \\
&= I + \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(\lambda - \zeta)} \times (Tm) d\zeta_R d\zeta_I.
\end{aligned} \tag{113}$$

When T is known, Eq.(113) gives the eigenfunction $m(\mathbf{x}, t, \lambda)$. Finally, to obtain the potential $Q(\mathbf{x}, t)$, one looks for the asymptotic expansion of $m(\mathbf{x}, t, \lambda)$ for large λ in the form

$$m(\mathbf{x}, t, \lambda) = I + \frac{m_1(\mathbf{x}, t)}{\lambda} + O\left(\frac{1}{\lambda^2}\right). \tag{114}$$

When this is used in the eigenvalue problem (96), one can express the potential Q in terms of the coefficient $m_1(\mathbf{x}, t)$. On the other hand, using (114) in (113), m_1 and hence the potential can be expressed in terms of the scattering data. Thus the inverse problem can be solved uniquely, thereby solving the Cauchy initial value problem in both (1+1)- and (2+1)-dimensions. Some of the simplest solutions obtained are given in Table II for KP and DS equations.

Finally what about extension of IST to (3+1) dimensions? There seems to be serious difficulties in the inverse procedure due to certain constraints on the scattering data. The problem remains open at present.

4.9 Bifurcation and chaos in physical systems

Integrable nonlinear systems like the soliton systems described above, or other finite degrees of freedom systems (for details see for example, Lakshmanan & Sahadevan [1993]) or diffusive systems such as the Burger's equation, are all though very important from a physical point of view still relatively rare and almost measure zero in number compared to the totality of nonlinear systems. Often they are said to be exceptions rather than rule, in spite of the invention of a large number of them along with their nice properties. Under perturbations most of these integrable systems become non-integrable. For finite degrees of freedom Hamiltonian systems, often KAM theorem becomes relevant under such circumstances (see for example Lichtenberg & Lieberman, [1983]). However most systems behave in a much more intricate way when the nonlinearity is increased or the KAM theorem is violated. We have already seen earlier in sec. 3.5 that Hamiltonian systems such as the Henon-Heiles model can show sensitive dependence on initial conditions depending upon the strength of nonlinearity, exhibiting chaotic motions. Similarly, systems like Lorenz system (sec. 3.6) show dissipative chaos (Lichtenberg & Lieberman [1983]). During the past two decades, an explosion of research has firmly led to the acceptance of chaos as an ubiquitous and robust

nonlinear phenomenon frequently encountered in nature (Drazin [1992], McCauley[1993], Mullin[1993]) and the concept has permeated almost all branches of science and technology. The field is growing into a stage where the initial surprises associated with the phenomenon are waning and new understandings are appearing, while actual controlling and harnessing of chaos are being contemplated.

The net result of the investigations on chaotic nonlinear dynamical systems since 1970 is that the notion of complete predictability has given way to deterministic chaos or randomness for suitable nonlinear dynamical systems. Given an N-particle system with masses $m_i (i = 1, 2, \dots, N)$ and (constraint free) forces \vec{F}_i acting on them, the state of the system is in general expected to be uniquely specified by solving the set of second order ordinary differential equations

$$m_i \frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i(t, \vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, \frac{d\vec{r}_1}{dt}, \dots, \frac{d\vec{r}_N}{dt}), \quad i = 1, 2, \dots, N, \quad (115)$$

for prescribed $6N$ initial conditions $\vec{r}_i(0)$ and $\left. \frac{d\vec{r}_i}{dt} \right|_{t=0}$. This Laplace dictum that for “a superintelligence nothing could be uncertain and the future as the past, would be present to its eyes” [Gleick, 1987] is already flawed: (i) when N is large, one requires a statistical description and (ii) when quantum effects are present uncertainties can arise even in the simultaneous prescription of initial conditions necessitating a quantum description. Now the further advance in our understanding of the dynamics is that even when the above two limitations are absent, depending on the nature of the forces in eq.(70), that is whether \vec{F}_i is linear or nonlinear, new uncertainties can arise leading to deterministic randomness or chaos. For appropriately chosen nonlinear force \vec{F}_i the system can show sensitive dependence on initial conditions, which can never happen when \vec{F}_i is linear, leading to exponential divergence of nearby trajectories, a possibility already foreseen by Poincaré (Sec.2.3) during the beginning of the century. The physical consequence is the butterfly effect of Lorenz(sec 3.6).

Since in this meeting, the various individual aspects of chaos are well discussed, we include here only a very brief account of them for completeness. For details the readers are referred to the various other articles in this book.

4.9.1 Chaos in dissipative and conservative nonlinear systems

We have already seen in sec.3 that the Henon-Heiles and Lorenz systems are prototype of Hamiltonian and dissipative chaos. Since then much understandings have been achieved on both types of chaos.

(i) Dissipative systems: The time evolution of these systems contracts volume in phase-space(the abstract space of state variables) and consequently trajectories approach asymptotically either a chaotic or a non-chaotic attractor. The latter may be a fixed point, a periodic limit cycle or a quasiperiodic attractor. These and the chaotic attractors are bounded regions of phase-space towards which the trajectory of the system, represented as a curve, converges in the course of long time evolution. Bifurcation or qualitative changes of periodic attractors can occur leading to more complicated and chaotic structures as a control parameter is varied. Different routes to the onset of chaos have been identified(see for example, Lichtenberg & Lieberman [1983], Drazin [1992]).

The chaotic attractor is typically neither a point nor a curve but a geometrical structure having a self-similar and fractal (often multifractal) nature. Such chaotic attractors are called strange attractors. Many physically and biologically important nonlinear dissipative systems, both in low and higher dimensions, exhibit strange attractors and chaotic motions. Typical examples are the various damped and driven nonlinear oscillators, the Lorenz system, the Brusselator model, the Bonhoeffer- van der Pol oscillator, the piecewise linear electronic circuits and so on (see for example, Lakshmanan & Murali [1996]).

(ii) Conservative or Hamiltonian systems: Nonlinear systems of conservative or Hamiltonian type also exhibit often chaotic motions. But here the phase space volume is conserved and so no strange attractor is exhibited. Instead, chaotic orbits tend to visit all parts of a subspace of the phase-space uniformly. The dynamics of a nonintegrable conservative system is typically neither entirely regular nor entirely irregular, but the phase-space consists of a complicated mixture of regular and irregular components. In the regular region the motion is quasiperiodic and the orbits lie on tori while in the irregular regions the motion appears to be chaotic but they are not attractive in nature. Typical examples include coupled nonlinear oscillators, Henon-Heiles system, anisotropic Kepler problem and so on (see for example Drazin [1992], McCauley [1993]).

4.9.2 Quantum chaos

The deterministic randomness or chaos exhibited by generic nonlinear dynamical systems has been found to present significant practical and philosophical implications, and probably limitations as well, in the description of microscopic world. There is no doubt that quantum theory is a more accurate description of nature. However, Bohr's correspondence principle requires that in the appropriate limit the remnant of signatures of (classical) chaos (of macroscopic world), namely the exponential divergence of nearby trajectories and the intrinsic uncertainty due to nonlinearity, should follow, barring unforeseen singularities in the $\hbar \rightarrow 0$ limit (\hbar :Planck's constant), which might prevent the smooth transition from quantum mechanics to classical mechanics. Search for such quantum manifestations of classical chaos in the practical sense, which goes by the terminology 'quantum chaos' or 'quantum chaology', has recently attracted considerable interest (for details see for instance, Gutzwiller[1990], Nakamura[1993], Reichl[1992]).

For example, one might look for possible fingerprints of chaos in the eigenvalue spectrum, wavefunction patterns and so on. In particular, by looking at the short range correlations between energy level (spacings) of a large class of quantal systems such as billiards of various types, coupled anharmonic oscillators, atomic and molecular systems, it has been realized that there exists generically a universality in the spacing distribution of the quantum version of classically integrable as well as chaotic systems. For regular systems nearest-neighbour spacings follow a Poisson distribution, while chaotic systems follow either one of the three universality classes, depending on the symmetry and spin. These universality classes correspond to Gaussian Orthogonal Ensemble (GOE) or Wigner statistics, Gaussian Unitary Ensemble (GUE) statistics or Gaussian Symplectic Ensemble (GSE) statistics, similar to ones which occur in random matrix theory of nuclear physics. For near integrable and intermediate cases the level distributions are found to satisfy Brody or Berry-Robnik or Izrailev statistics(Reichl[1992]).

In recent times it has been found that highly excited Rydberg atoms and molecules (which are effectively one-electron systems) under various external fields are veritable goldmines for exploring the quantum aspects of chaos. These systems are particularly appealing as they are not merely mathematical models but important physical systems which can be realized in the laboratory. Particular examples are the hydrogen atom in external magnetic fields, crossed electric and magnetic fields, van der Waals force, periodic microwave radiation and so on. These studies seem to have much relevance to the understanding of the so called mesoscopic systems and such investigations are of high current interest in nonlinear physics.

4.9.3 Controlling, synchronization and secure communication

The above studies make it clear that chaos is ubiquitous in nature and that it is intrinsically unpredictable and sensitively dependent on initial conditions so that nearby trajectories diverge exponentially. Consequently the phase trajectories (in the phase space) can take complicated geometrical structures, for example a fractal structure for typical dissipative chaotic system. Naturally one would expect such a complex motion cannot be controlled or altered by minimal efforts unless drastic changes are made to the structure of the system. Surprisingly, recent investigations in this direction have clearly demonstrated (see for example, Shinbrot et al [1993], Lakshmanan & Murali [1996]) that not only can chaotic systems be tamed or controlled by minimal preassigned perturbations to avoid any harmful effects to the physical system under consideration but controlling can be effected in a purposeful way to make the system evolve towards a goal dynamics. Numerous control algorithms have been devised in recent times, many of which have been experimentally verified, for controlling chaos and they broadly fall under two categories, (i) feedback and (ii) nonfeedback methods, which effectively use the fact that the chaotic attractor contains infinite number of unstable periodic orbits which can then be chosen suitably and controlled for regular motion.

Another but related consequence of sensitive dependence on initial conditions is that two identical but independently evolving chaotic systems can never synchronize to be in phase and in amplitude, as any infinitesimal deviation in the starting conditions (or the system specification) can lead to exponentially diverging trajectories making synchronization impossible. Contrast this with the case of linear systems (and also regular motions of nonlinear systems), where the evolution of two identical systems can be very naturally synchronized. In this connection, the recent suggestion of Pecora and Carroll [1993] that it is possible to synchronize even chaotic systems by introducing appropriate coupling between them has changed our outlook on chaotic systems, synchronization and controlling of chaos, paving ways for new and exciting technological applications: spread spectrum communications of analog and digital signals (For details see for example, Lakshmanan & Murali, [1996]).

5 Outstanding problems and future outlook

In the earlier sections, we have traced briefly the salient features of the development of various topics in nonlinear physics and their ramifications, ultimately leading to the twin concepts of solitonic and chaotic structures. While the soliton excitations are predominant in one space and one time dimensional systems, the chaos phenomenon is well studied (at least

numerically) for low degrees of freedom systems. It is obvious that one has touched only the tip of the iceberg as far as nonlinear systems are concerned and our present understanding is confined to a narrow range of them. The nature of excitations in physically relevant higher spatial dimensional systems, the transition from integrable regular systems to nonintegrable and chaotic systems and the formation of spatio-temporal patterns on perturbations of soliton systems are some of the important problems to be tackled in the next few decades. Also the definition of integrability, particularly in the complex plane and its relation to real time dynamical behaviour, is one of the most important fundamental notions to be understood and extended to nonintegrable and chaotic situations. Many new technologies which are in the process of unfolding as a result of the various applications of the notions of solitons and chaos remain to be harnessed to their full potentialities, in such areas as magnetoelectronics, information technology, secure communications and so on. In this section, we will focus briefly on some of these topics with a view to point out the problems and potentialities of nonlinear physics in the next few decades.

5.1 Integrability and chaos

We pointed out in the previous section that soliton equations may be considered as completely integrable infinite dimensional Hamiltonian systems. From another point of view we also saw that solutions are meromorphic and free from movable critical singular manifolds (Painlevé property). From yet another point of view, the existence and uniqueness of their solutions can be established. It is not only the Hamiltonian type soliton equations which are known to be integrable. The nonlinear diffusive Burger's equation

$$u_t + uu_x = \gamma u_{xx} \quad (116)$$

is linearizable in the sense that the celebrated Cole-Hopf transformation (see for example, Sachdev [1987])

$$u = -2\gamma \frac{v_x}{v} \quad (117)$$

converts eq.(116) into the linear heat equation

$$v_t + \gamma v_{xx} = 0 \quad (118)$$

and so may be considered to be integrable. Similarly the nonlinear diffusive equation

$$u_t + u^2 u_x + Du^2 u_{xx} = 0 \quad (119)$$

is known to be linearizable and possesses infinite number of Lie-Bäcklund symmetries (Fokas & Yorsos [1982]). Eqs.(116) and (119) also satisfy the Painlevé property. Similarly for finite degrees of freedom, there exists both integrable Hamiltonian and dissipative systems (Lakshmanan & Sahadevan [1993]; Ramani, Grammaticos & Bountis [1989]), obeying the Painlevé property.

Examples:

1) Two coupled anharmonic oscillators:

$$\ddot{x} + 2(A + 2\alpha x^2 + \delta y^2)x = 0, \quad (120a)$$

$$\ddot{y} + 2(B + 2\beta y^2 + \delta x^2)y = 0. \quad (120b)$$

Integrable cases:

i) $A = B, \alpha = \beta, \delta = 6\alpha$

ii) $\alpha = \beta, \delta = 2\alpha$

iii) $A = 4B, \alpha = 16\beta, \delta = 12\beta$

iv) $A = 4B, \alpha = 8\beta, \delta = 6\beta$

2) Lorenz system: Eq.(29)

Integrable case:

i) $\sigma = \frac{1}{2}, b = 1, r = 0$

So what is integrability? When does it arise? When is a given system nonintegrable? What distinguishes nearintegrable ones and chaotic systems and so on? These are some of the paramount questions which arises as far as nonlinear systems are concerned. Systematic answers will pave the way for a meaningful understanding of nonlinear systems in general and the role of nonlinearity in particular.

As far as integrable systems are concerned the earlier discussions seem to point out at least the following broad definitions:

1) Integrability in the complex plane: Integrable - integrated with sufficient number of arbitrary constants or functions; nonintegrable - proven not to be integrable. This loose definition can be related to the existence of single-valued, analytic solutions a concept originally advocated by Fuchs, Kovalevskaya, Painlevé and others, thereby leading to the notion of “integrability in the complex plane” and to the Painlevé property mentioned in the previous section. For real valued coordinates, this can lead to integration methods such as Bäcklund transformations, Hirota’s bilinearization and ultimately Lax pair and inverse scattering analysis to completely solve the Cauchy initial value problem.

2) Integrability - Existence of integrals of motion: One looks for sufficient number of single - valued, analytic integrals of motion. For example, N integrals for Hamiltonian systems with N degrees of freedom, involutive and functionally independent. Then the equation of motion can be integrated by quadratures, leading to Liouville integrability. Such a possibility leads to strong association with symmetries, generalized Lie symmetries and Lie-Bäcklund symmetries.

3) Integrability: Existence and uniqueness of solutions: Mathematicians often call complete integrability as related to the existence and uniqueness of solutions.

How are all these and other possible such concepts interrelated? Which is the ultimate definition of integrability? In each of these definitions there are pitfalls and one might construct some counterexamples, even if they are pathological in nature. Then can one construct algorithmic ways of isolating integrable nonlinear systems (whatever it ultimately means) and then analyse their dynamics systematically?

At least the definition of integrability in the complex plane seems to offer such a possibility, however unsatisfactory the present status of Painlevé analysis method is. Its applicability seems to be wide: it successfully isolates integrable cases in nonlinear difference equations, nonlinear odes (both of dissipative and Hamiltonian type), nonlinear pdes in (1+1), (2+1), nonlocal, integro-differential equations and so on. It also captures other integrability properties such as linearization, bilinearization, Bäcklund transformations and so on. However

there are many unanswered or partially answered questions, whose understanding can dramatically alter our understanding of nonlinear systems.

1. Why does the method is successful in isolating integrable cases? Integrability is something which we relate to real (space-) time dynamics. Why should the properties in the complex plane/manifold determine the real time behaviour?

2. Why is that certain type of singularities are bad while others are admissible? Movable pdes of finite order are admissible but movable branch points and essential singularities are associated with nonintegrability and chaos? Then the P-property is defined to within a transformation. Which ones are allowed?

3. Why are fixed singularities of all type including essential singularities are allowed while movable critical singularities are not allowed? If denseness of branching is to be a criterion why branching around fixed singular points are allowed?

4. In nonintegrable cases, one observes that it is essential in typical cases to develop the solutions as double infinite series around a movable singular point or manifold. Can one extract real time behaviour from these asymptotic forms? Do the complex patterns of singularities and apparent fractal structure have any connection with real time behaviour? [Bountis, 1992]. Typical examples of Duffing oscillator and Duffing-van der Pol oscillator are given in Figs.13-16(see also Lakshmanan & Murali[1996]).

5. Can any criterion for the denseness of branching be given? It may be possible to develop the ‘PolyPainlevé test’ proposed by Martin Kruskal [Kruskal & Clarkson, 1992] so as to understand the effect of denseness of branching.

6. What is the connection between the nature of solutions around the singular points/manifolds with the integrals of motion and existence of solutions?

It appears that in the next few decades determined efforts to understand integrability and nonintegrability aspects along these lines can throw much light on the nature of nonlinear systems, which then will lead to algorithmic handles to deal with such systems in a general sense. This will open up many paths to analyse nonlinear physical systems under very many new circumstances. Thus singularity structure may turn out to be the key to unlock nonlinear dynamical systems in the next few decades.

5.2 Nonlinear excitations in higher spatial dimensions

In the earlier section 4.8 we noted that richer physical structures can arise in (2+1) dimensions. While in (1+1) dimension we have seen the possibility that both localized coherent structures as well as chaotic structures exist, one would like to know whether these excitations survive in higher dimensions too and whether there are new elementary excitations and new phenomena lurking in higher spatial dimensions. Unfortunately the natural physical extensions of (1+1) dimensional soliton equations such as sine-Gordon, nonlinear Schrödinger or Heisenberg ferromagnetic spin equations of the form

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \sin \phi = 0, \quad (121)$$

$$iq_t + \nabla^2 q + |q|^2 q = 0, \quad (122)$$

$$\vec{S}_t = \vec{S} \times \nabla^2 \vec{S}, \quad \vec{S}^2 = 1, \quad (123)$$

etc., where $\bar{\nabla}^2$ is the 2-dimensional or 3-dimensional Laplacian, do not seem to possess straightforward extensions of solitonlike exponential localized structures. In fact, it is expected that the solutions of these equations even might develop singularities, sometimes called collapse. These equations sometimes may possess under special geometries interesting classes of particular solutions, like time-independent spherically symmetric, axially symmetric, instanton, vortex, monopole, hedgehog and so on solutions (Rajaraman [1980]; Makhankov et al [1993]). But the nature of their general solutions is simply not known.

On the other hand, at least in (2+1) dimensions, introduction of additional nonlocal terms or effectively additional scalar fields giving rise to boundary effects can offset the tendency for leakage of energy in the second spatial direction so as to make the system admit exponentially localized solutions, namely dromions, in addition to algebraically decaying lumps (mentioned in Sec.4). The K-P, D-S and Ishimori equations and their typical solutions are given in Sec.4, Table II. In addition, we have some of the other interesting (2+1) dimensional equations, each one with nonlocal terms, admitting exponentially localized solutions (see Radha & Lakshmanan [1995]).

1) (2+1) dimensional KdV

$$u_t + u_{\xi\xi\xi} = 3(u\partial_{\eta}^{-1}u_{\xi})_{\xi}. \quad (124)$$

2) Niznik-Novikov-Veselov equation

$$u_t + u_{\xi\xi\xi} + u_{\eta\eta\eta} + au_{\xi} + bu_{\eta} = 3(u\partial_{\eta}^{-1}u_{\xi})_{\xi} + 3(u\partial_{\xi}^{-1}u_{\eta})_{\eta}. \quad (125)$$

3) Generalized NLS

$$iq_t = q_{xy} + Vq, \quad (126a)$$

$$V_x = 2\partial_y|q|^2. \quad (126b)$$

4) (2+1) dimensional sG

$$\theta_{\xi\eta t} + \frac{1}{2}\theta_{\eta}\rho_{\xi} + \frac{1}{2}\theta_{\xi}\rho_{\eta} = 0, \quad (127a)$$

$$\rho_{\xi\eta} = \frac{1}{2}(\theta_{\xi}\rho_{\eta})_t. \quad (127b)$$

5) (2+1) dimensional simplest scalar equation

$$iq_t + q_{xx} - 2\lambda q \int_{-\infty}^y |q|_x^2 dy' = 0, \quad \lambda = \pm 1. \quad (128)$$

6) 2D long dispersive wave equation

$$\lambda q_t + q_{xx} - 2qv = 0,$$

$$\lambda r_t + r_{xx} + 2rv = 0,$$

$$(qr)_x = v_{\eta}, \quad \partial_{\eta} = \partial_x - \partial_y. \quad (129)$$

So if boundary contributions are important for localized solutions to exist, then what about the nature of the excitations in the physically important systems such as (121-123) given above. Numerical investigations are time consuming and require considerable effort.

For example, for nonlinear σ -model Zakrzewski and coworkers [1995] have obtained interesting numerical results, including scattering, elastic and inelastic collisions. Typical results are given in Fig.(16). The following problems need urgent attention, which might point towards new vistas in nonlinear physics.

1. What are all the possible stable structures (special solutions) in (2+1) and (3+1) dimensions? How stable are they? What are their collision properties? If unstable, are they metastable? If not do they give rise to new spatio-temporal pattern? What is the effect of external forces including damping?

2. Can one develop techniques to solve the Cauchy initial value problem of physically important (2+1) dimensional extensions of (1+1) dimensional soliton equations? Or can one obtain enough informations about the nature of general excitations through numerical analysis? How can the numerics be simplified to tackle such problems?

3. Is it possible to perceive something similar to FPU experiments in (2+1) and (3+1) dimensions? What new phenomena are in store here? Can actual analog simulations be made with suitable miniaturization of electronic circuits so that these (2+1) and (3+1) dimensional systems can be analysed systematically?

4. Is it possible to extend the inverse scattering formulation to (3+1) dimensional systems also as in the case of (2+1) dimensional evolution equations? The main difficulty seems to arise in the inverse analysis due to certain nonuniqueness arising from constraints on the scattering data (see for example, Ablowitz & Clarkson [1991]).

It is very certain that the future of nonlinear physics will be much concentrated around such higher dimensional nonlinear systems, where new understandings and applications will arise in large numbers. A long term sustained numerical and theoretical analysis of (2+1) and (3+1) dimensional nonlinear evolution equations both for finite and continuous degrees of freedom will be one of the major tasks for several decades to come which can throw open many new nonlinear phenomena. Also one might consider discretization and analog simulation of these systems, to which nonlinear electronics community can contribute much.

5.3 Nonintegrable systems, spatio-temporal patterns and chaos

In the earlier sections we considered integrable nonlinear systems. However these are far fewer in number. Most natural systems are nonintegrable: however many of them may be considered as perturbations of integrable nonlinear systems. Examples: condensed matter systems including magnetic, electronic and lattice systems, optoelectronic systems, hydrodynamical systems and so on. The perturbing forces could be space-time inhomogeneities and modulations, external forces of different origins, damping and dissipative as well as diffusive forces and so on. Thus it is imperative to study the effect of these various additional forces with reference to the basic nonlinear excitations of integrable systems.

Such an analysis needs to consider the different length scales of the perturbation (both space and time) with respect to the nonlinear excitations of the unperturbed case [Scharf, 1995; Kivshar & Spatcheck, 1995]. Depending on such scales, the original entities might survive albeit necessary deformations or may undergo chaotic or complex motions or deformations may give rise to interesting spatio-temporal patterns. Some preliminary studies on such soliton perturbations are available in the literature (see for example, Scharf[1995], Kivshar & Spatcheck [1995]). In fig.17, a typical soliton perturbation in the case of the

nonlinear Schrödinger equation

$$iu_t + u_{xx} + 2u|u|^2 = \epsilon u \cos(kx) \quad (130)$$

is seen to give rise to a spatio-temporal pattern.

Detailed classification of the types of perturbations and the resulting coherent and chaotic structures and spatio-temporal patterns can be used as dictionaries to explain different physical situations in condensed matter, fluid dynamics, plasma physics, magnetism, atmospheric physics and so on. Further such studies in (2+1) dimensional systems, wherein any stable entity when perturbed by additional weak forces can lead to exciting new structures corresponding to realistic world description. A concerted effort through analytical and numerical investigations to tackle the nonintegrable systems using integrable structures portends to provide rich dividends in the next century.

5.4 Micromagnetics and magnetoelectronics

Micromagnetics is the subject which is concerned with the study of detailed magnetization configurations and the magnetization reversal process in magnetic materials (Brown Jr. [1963]). Particularly it encompasses the study of ferromagnets and ferromagnetic thin films (used in magnetic thin film sensors and devices). The theory considers the ferromagnetic free energy in the ferromagnetic material to consist of

- i) the ferromagnetic exchange energy,
- ii) the magnetic anisotropy energy,
- iii) the magnetoelastic energy,
- iv) the magnetostatic energy and
- v) the magnetic potential energy due to external magnetic fields.

The magnetization orientation $\vec{M}(\vec{r}, t)$ follows the Landau-Lifshitz-Gilbert equation (Landau & Lifshitz [1935]; Lakshmanan & Nakamura [1985])

$$\frac{\partial \vec{M}}{\partial t} = -\gamma \vec{M} \times \vec{F}_{eff} - \frac{\lambda}{M} \vec{M} \times (\vec{M} \times \vec{F}_{eff}), \quad (131)$$

where γ is the gyromagnetic ratio and λ is the damping constant and \vec{F}_{eff} is the effective magnetic field. The first term in the equation describes the the gyromagnetic motion (precession of \vec{M} about \vec{F}_{eff}), and the second describes the rotation of the effective field. Note that $|\vec{M}| = \text{constant}$. The Heisenberg ferromagnet equations discussed earlier in Sec.4.7.1 are then essentially special cases of the above Landau-Lifshitz equation when the damping vanishes and \vec{F}_{eff} takes special forms. The complex magnetization patterns and the detailed spin structures within the domain boundaries are obtained by solving eq.(131). The structures so obtained can then be used for different applications (See also the special issue on “Magnetoelectronics”-Physics Today, April 1995).

1. Magnetoresistive recording

Over a century now, for magnetic recording most systems have used an inductive head for writing and reading which employ coils to both induce a magnetic field (write mode) and sense a recorded area (read mode). Recently, a more powerful reading head called

the magnetoresistance (MR) head, has been introduced into disk products which employs a sensor whose resistance changes in the presence of a magnetic field. Its performance gain has enhanced the density of storage by upto 50% commercial conversion to MR heads is only just starting.

2. Magneto-optical recording

Again instead of using the conventional inductive head for recording, optical pulses or lasers have become usage. Optical recording is expected to increase in capacity and transfer rate by a factor about 20 over the next decade. Rewritable systems will be based largely on magneto optical technologies that exploit the smaller mark sizes made possible by new short wavelength lasers.

In the above two models, it is mainly the interaction between the magnetization of the medium and the electrical field of the head in the case of magnetoresistive head and the electromagnetic (optic or laser) field in the case of magneto optical head that play important roles. For efficient storage, excitation of the magnetization of the medium (which may be due to thermal or due to other external disturbances) if any should be localized. Theoretically speaking, these different magnetic interactions can be accomodated in appropriate spin Hamiltonian models.

3. Magnetic films for better recording

The fundamental magnetization process in thin films can be characterized by the formation, motion and annihilation of magnetization vortices. When a sufficiently strong external magnetic field is applied, magnetization reversal takes place and these vortices move across the film. If the intergranular exchange coupling in magnetic films is large the size of the vortices will be larger and travel more freely over larger distances. Thus the intergranular exchange coupling has a significant impact on the properties of recorded bits in thin films because in the case of large vortices the recording noise is large and in the case of low vortices the noise is low.

Vortices form an interesting class of solutions to multidimensional nonlinear evolution equations. Hence here also solving higher dimensional Landau- Lifshitz equations with intergranular exchange coupling and interaction with large external fields for vortex like solutions is an important task for future.

4. Study of single domain magnets

The behaviour of individual magnetic domains has become important for technology. The problem of media noise which is one of the fundamental present day problems of magnetic storage can be avoided if we use individual magnetic domains to store each bit. The study of single domain magnets (mesoscopic magnets) has accelerated the development of new theoretical approaches to magnetic dynamics. Thus for the next few years attention has to be paid how quantum mechanical effects influence the properties of all these small systems.

5.5 Optical soliton based communication: perspectives and potentialities

Soliton based optical-fiber communication is imminent, as we have noted in Sec.4.7.2. The experimental works of Mollenauer and co-workers [1990] has clearly demonstrated the successful soliton transmission over more than 10,000kms in a dispersion-shifted fiber. With

such an exciting possibility, it is important to study further technical effects related to soliton propagation in optical fibers. We mention here a few of them.

1. The assumption of instantaneous nonlinear response amounts to neglecting the contribution of molecular vibrations in the higher-order nonlinear effect. In general, both electrons and nuclei respond to the optical field in a nonlinear manner. For silica fiber, in the femto-second region, the higher-order nonlinear effects (higher-order nonlinear dispersion effect, self-induced Raman scattering effect) become important. Further in the near zero group velocity dispersion region, higher-order dispersion term becomes essential. Typically the NLS equation (83) gets modified to the form

$$iE_z - \frac{\kappa''}{2}E_{\tau\tau} + \frac{n_2\omega}{c}|E|^2E - \frac{i\kappa'''}{6}E_{\tau\tau\tau} + i\gamma(|E|^2E)_\tau + i\gamma_s(|E|^2)_\tau E = 0, \quad (132)$$

where $\kappa''' = \frac{\partial^3 \kappa}{\partial \omega^3}$ describes third order dispersion and γ describes nonlinear dispersion and γ_s stands for self-induced Raman scattering effect. Much works need to be done in the analysis of the above type of equations. Similarly for erbium doped fibers, which are quite useful from the point of view of self-amplification, one has to analyse the full soliton dynamics of Maxwell-Bloch equations.

2. In most experimental situations of propagation of light pulses in nonlinear medium, pulsed laser is used as a source for excitation, especially when high-power operation is involved. In these circumstances, a beam of light will experience both diffraction and dispersion, in addition to the self-focussing (defocussing) and self-phase modulation that results from the nonlinearity. The corresponding evolution equation is the higher dimensional NLS equation of the form given in eq.(132). What kind of pulses do such equations admit which can be considered as non-diffractive and non-dispersive pulses of experimental relevance? Can one have a stable light bullet (soliton) which can be used experimentally?

3. Since SiO_2 is a symmetric molecule, second order nonlinear effect vanishes. Nevertheless the electric-quadrupole and magnetic-dipole moments can generate weak second-order nonlinear effects. Defect or color centers inside the fiber core can also contribute to second-harmonic generations under certain conditions. What is the role of such second-order nonlinear effect on the optical soliton propagation?

4. In many circumstances, a more complete description of the propagation would rather involve an interaction between two (or more) coupled modes. For example, birefringence will give rise to two nondegenerate polarization modes. The coupling could also be between the modes of two optical guides as in dual-core fiber. Coupled nonlinear Schrödinger family of equations are used to study these phenomena. Investigation of these types of models is a very important current and future direction in this field as pulse transmission devices that rely on coupling between fibers(e.g, switches, directional couplers) are often components of optical fiber communication systems. In addition, investigation of nonlinear processes in coupled optical waveguides will aid in the design of various optical computers and sensor elements.

In short, there are very many pressing problems to be investigated on the nonlinear Schrödinger family of equations, depending upon the physical situation in which the light propagation in nonlinear fibers takes place. Sustained investigations can throw much insight into the basic phenomenon, besides helping the actual soliton based information technology

to develop into a favoured method of information transmission in the coming decades.

6 Conclusions and Outlook

Nonlinear physics (and nonlinear science) has come a long way from a position of insignificance to a central stage in physics and even in science as a whole. While the pace of such a development was rather slow, an eventful golden era which ensued during the period 1950-70 saw the stream-rolling of the field into an interdisciplinary topic of great relevance of scientific endeavour. We have tried to present here a rather personal perspective of some of these developments and the outstanding tasks urgently need to be carried out in the forthcoming decades and the possible dividends they can bring in. Of course these forecasts are based on the speaker's own understanding and knowledge of the specific areas he is familiar with. Though the motivation has been to cover both integrable and chaotic nonlinear systems, specific emphasis was given to integrable systems and associated coherent structures, since the topic of chaos is covered in many of the other lectures of the meeting.

Of course it is clear that the future developments and directions we indicate here depend mostly on the present status of the concerned topics. But one is also strongly aware of the fact that path breaking new ideas and directions can arise from nowhere suddenly and dramatically and without forewarnings at anytime in the future. The past and immediate present developments described in this article definitely substantiate such unforeseen possibilities. As stressed earlier there was no field called nonlinear science or nonlinear physics fifty years ago and we cannot naturally foresee exactly how the field would have transformed in the next fifty years.

However, we have tried to stress that much new physics can come out by 1) clearly understanding the concepts of integrability and nonintegrability from a unified point of view, 2) by analysing nonlinear structures in (2+1) and (3+1) dimensional spaces, which are more realistic, and 3) through indepth analysis of the effect of perturbation on integrable nonlinear systems and analysis of other nonintegrable systems and classifying the types of novel spatio-temporal structures which might arise. We have also tried to point out some of the tasks and potentialities in certain emerging technology oriented topics such as magneto-electronics, optical soliton based communications and so on, which are the off-shoots of progress at the fundamental level.

There are numerous important topics which we have not touched upon or discussed their future developments here, including such topics as nonlinearities in plasma physics, acoustics, biological physics, many areas of condensed matter physics, astrophysics, gravitational theory, detailed quantum aspects and so on. Probably experts in these topics will cover such areas in future. Similarly the quest towards the ultimate theory of matter in particle physics, whichever form it may take, will ultimately be a nonlinear one. There is no doubt that nonlinearity will rule the world for many more years to come and there is scope for everybody to try his hand in the field for a better understanding of Nature.

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Figure Captions

Fig.1 The circuit diagram of the van der Pol oscillator

Fig.2 Subharmonics of the van der Pol oscillator

Fig.3 Energy sharing in the FPU anharmonic lattice between the various modes

Fig.4 Zabusky-Kruskal numerical analysis of the KdV equation: birth of solitons

Fig.5 Two-soliton scattering in the KdV equation

Fig.6 Schematic diagram of the IST method

Fig.7 Hamiltonian chaos in Henon-Heiles system: Poincaré surface of section for different energies

Fig.8 A chaotic attractor of the Lorenz system

Fig.9 Methods for solutions of soliton equations

Fig.10 A line soliton solution for the KP-I equation

Fig.11 Lump-soliton solution of KP-I equation

Fig.12 Dromion solution of the DS-I equation

Fig.13 Square lattice of pole singularities in the complex t -plane of a free undamped Duffing oscillator ($\ddot{x} + \omega_0^2 x + \beta x^3 = 0$, $\omega_0^2 = 1$, $\beta = 5$)

Fig.14 Singularity distribution in the complex t -domain of the damped Duffing oscillator equation ($\ddot{x} + 0.1\dot{x} + \omega_0^2 x + \beta x^3 = 0$, $\omega_0^2 = 1$, $\beta = 5$)

Fig.15 Singularity clustering for the driven (chaotic) Duffing oscillator ($\ddot{x} + x + 5x^3 = f \cos \omega t$)

Fig.16 Scattering of two skyrmions [Piete & Zakrzewski, 1995]

Fig.17 Complicated spatio-temporal behaviour of the soliton in the perturbed NLS equation [Scharf, 1995]