Delay time modulation induced oscillating synchronization and intermittent anticipatory/lag and complete synchronizations in time-delay nonlinear dynamical systems

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Abstract

Existence of a new type of oscillating synchronization that oscillates between three different types of synchronizations (anticipatory, complete and lag synchronizations) is identified in unidirectionally coupled nonlinear time-delay systems having two different time-delays, that is feedback delay with a periodic delay time modulation and a constant coupling delay. Intermittent anticipatory, intermittent lag and complete synchronizations are shown to exist in the same system with identical delay time modulations in both the delays. The transition from anticipatory to complete synchronization and from complete to lag synchronization as a function of coupling delay with suitable stability condition is discussed. The intermittent anticipatory and lag synchronizations are characterized by the minimum of similarity functions and the intermittent behavior is characterized by a universal asymptotic $-\frac{3}{2}$ power law distribution. It is also shown that the delay time carved out of the trajectories of the time-delay system with periodic delay time modulation cannot be estimated using conventional methods, thereby reducing the possibility of decoding the message by phase space reconstruction.

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Synchronization of chaos is one of the most fundamental phenomena exhibited by coupled chaotic oscillators. Recent studies on chaotic synchronization has also focussed on nonlinear time-delay systems in view of their hyperchaotic nature. Further, the concept of delay time modulation has been introduced in order to understand dynamical systems with time dependent topology such as internet, world wide web, population dynamics, neurology, etc. It has also been shown that nonlinear delay systems with time dependent delay can exhibit more complex dynamics. Consequently, studies on synchronization of such systems with time dependent delay becomes very important in order to understand their cooperative dynamics. From this point of view, in this paper we have considered simple scalar piecewise linear time-delay systems with unidirectional time-delay coupling in order to explore the various types of synchronized behaviours and their transitions. The introduction of simple fully rectified sinusoidal modulation in these systems with constant coupling delay can lead to the existence of a new type of oscillating synchronization that oscillates between three different types of synchronizations, namely complete, lag and anticipatory synchronizations. With delay time modulation in both the coupling and feedback delay, the coupled system displays intermittent lag/anticipatory and complete synchronizations for suitable ranges of values of the delay times and modulational frequencies. It has also been shown that the decrease in the value of modulational frequency leads to exact and then to lag/anticipatory synchronizations from their intermittent nature. The existence of different synchronizations are corroborated by suitable stability condition based on Krasovskii-Lyapunov theory and their corresponding similarity functions. The intermittent regimes are characterized by a universal asymptotic $-\frac{3}{2}$ power law distribution.

I.   INTRODUCTION

Chaos synchronization has been receiving a great deal of interest for more than two decades in view of its potential applications in various fields of science [1, 2, 3, 4, 5, 6, 7]. Since the identification of chaotic synchronization, different kinds of synchronizations have been proposed in interacting chaotic systems, which have all been identified both

theoretically and experimentally. Complete synchronization refers to a perfect locking of chaotic trajectories so as to remain in step with each other in the course of time, $X(t) = Y(t)$ [1, 2, 3, 4]. Generalized synchronization is defined as the presence of some functional relationship between the states of the drive and the response, $Y(t) = F(X(t))$ [8, 9, 10]. Phase synchronization is characterized by entrainment of the phases of the two signals, whereas their amplitudes remain uncorrelated [11, 12]. Lag synchronization implies that there is an exact time shift between the evolution of drive and response systems, where the response lags the state of the drive, $Y(t) = X(t-\tau)$ [13, 14, 15]. Anticipating synchronization also appears as a coincidence of shifted-in-time states of the two systems, where the response anticipates the state of drive, $Y(t) = X(t+\tau)$ [16, 17, 18, 19, 20]. Recently intermittent lag synchronization has also been identified [21, 22, 23]. Also it has been shown very recently in a three-element network module of semiconductor laser system that dynamical relaying can lead to zero-lag synchronization even in the presence of coupling delays [24].

The notion of time dependent delay (TDD) with stochastic or chaotic modulation in time-delay systems was introduced by Kye *et al* [25] to understand the behaviour of dynamical systems with time dependent topology. These authors have reported that in a time-delay system with TDD, the reconstructed phase trajectory does not collapse to a simple manifold, a property different from that of delayed systems with fixed delay time (which is considered to be a serious drawback of the later type of systems). It has been shown very recently that a distributed delay enriches the characteristic features of the delayed system over that of the fixed delay systems [26]. Based on these considerations, current studies on chaotic synchronization in time-delay systems are focused towards time-delay systems with time dependent delay [27, 28, 29, 30]. In this connection it is also of considerable interest to study the effect of simple modulations such as periodic modulation [29, 30] on the nature of the chaotic attractor.

Recently, we have studied chaotic synchronization in a system of two unidirectionally coupled odd piecewise linear time-delay systems [31] with two different constant delay times: one in the coupling term and the other in the individual systems, namely, feedback delay. We have shown that there is a transition from anticipatory to lag synchronization through complete synchronization as a function of a system parameter with suitable stability criterion. The present work was motivated by the fact that whether there arises any new phenomenon due to the introduction of periodic delay time modulation in the coupled time-delay system

we have studied earlier and its effects on the various synchronization scenario. Interestingly, we have found that even with simple periodic modulation, the time-delay system cannot be collapsed into a simple manifold and that the delay time cannot be extracted using standard methods. More interestingly, we have also found that the fully rectified sinusoidal modulation of delay time introduces a new type of oscillating synchronization that oscillates between anticipatory, complete and lag synchronizations for the case of constant coupling delay. This is further corroborated by suitable stability condition based on Krasovskii-Lyapunov theory. Intermittent anticipatory and lag synchronizations are also found to exist in the present system for the case of identical modulation in both the coupling and feedback delay, for a range of modulational frequencies. In addition, we also find that there exist regions of exact anticipatory and lag synchronizations for lower values of modulational frequencies. The results have been corroborated by the nature of similarity functions and the intermittent behavior by the probability distribution of the laminar phase, satisfying universal $-\frac{3}{2}$ power law behavior of on-off intermittency [32, 33].

The plan of the paper is as follows. In Sec. II, we introduce the scalar piece-wise linear time-delay system with delay time modulation and explore the dynamical change in the time series of the time-delay system due to delay time modulation. In Sec. III we have introduced a unidirectional time-delay coupling with delay time modulation between the two scalar time-delay systems and we have identified the condition for stability of the synchronized state following Krasovskii-Lyapunov theory. A new type of oscillating synchronization that oscillates between anticipatory, complete and lag synchronizations and vice versa is shown to exist in Sec. IV for the case of constant coupling delay and with modulated feedback delay. In Sec. V, we have pointed out the existence of intermittent anticipatory synchronization when the strength of the coupling delay is less than that of the feedback delay with identical modulations, while in Sec. VI, complete synchronization is realized when the two delays are equal. Intermittent lag synchronization is shown to set in when the coupling delay exceeds the feedback delay in Sec. VII. In Sec. VIII we very briefly indicate the possibility of more complicated oscillating synchronizations in the case of nonidentical modulations. Finally in Sec. IX, we summarize our results.

II. PIECEWISE LINEAR TIME-DELAY SYSTEM WITH DELAY TIME MODU-
LATION AND DYNAMICAL CHANGES

At first, we will introduce the single scalar time-delay system with piecewise linearity
in the presence of delay time modulation, which has been studied in detail for its chaotic
dynamics in references [34, 35, 36] with constant time-delay. Then some measures to estimate
the delay time will be discussed both in the presence and in the absence of delay time
modulation to show that the imprints of the delay time carved out of the time series of the
chaotic attractor are completely wiped out by the modulation of delay time.

A. The scalar delay system

We consider the following first order delay differential equation introduced by Lu and
He [36] and discussed in detail by Thangavel *et al.* [34],

$$\dot{x}(t) \; = \; -ax(t) + bf(x(t-\tau)), \tag{1}$$

where a and b are parameters, τ is the constant time-delay and f is an odd piecewise linear
function defined as

$$f(x) = \begin{cases} 0, & x \leq -4/3 \\ -1.5x - 2, & -4/3 < x \leq -0.8 \\ x, & -0.8 < x \leq 0.8 \\ -1.5x + 2, & 0.8 < x \leq 4/3 \\ 0, & x > 4/3 \end{cases} \tag{2}$$

Recently, we have reported [35] that system (1) exhibits hyperchaotic behavior for the pa-
rameter values $a = 1.0, b = 1.2$ and $\tau = 25.0$ and the hyperchaotic nature was confirmed by
the existence of multiple positive Lyapunov exponents (see Figures 1 and 2 in ref. [35]).

Now, we wish to replace the time-delay parameter τ as a function of time for our present
study, instead of the constant time-delay, in the form [29, 30]

$$\tau(t) \; = \; \tau_0 + \tau_a \left| \sin(\omega t) \right|, \tag{3}$$

where τ_0 is the zero frequency component, τ_a is the amplitude and ω/π is the frequency of the
modulation. Note that in the delay term, we have introduced the fully rectified sinusoidal

modulational form (absolute of the sine term) so as to keep the delay time positive even for values of $\tau_a > \tau_0$ so as to avoid acausality problem in Eq. (1) for negative values of τ when $\tau_a > \tau_0$. However, for values of τ_0 sufficiently greater than τ_a the rectification in the modulation (3) is not required.

B. Estimation of the effect of delay time modulation

Recently, the concept of time dependent delay with stochastic or chaotic modulation was introduced by Kye *et al.* [25] in the time-delay systems and they have shown in the case of Mackey-Glass system that the delay time carved out of time series of the time-delay system is undetectable by the conventional measures and hence any reconstruction of phase space of the delayed system is hardly possible. This fact has motivated some authors [27, 28, 29, 30] to look for delay systems with delay time modulation as an ideal candidate for secure communication.

Interestingly we find here that even with a fully rectified sinusoidal delay time modulation of the form (3), system (1) exhibits the properties studied by Kye *et al* with stochastic or chaotic modulation. In order to demonstrate the effect of fully rectified sinusoidal delay time modulation of the form (3) on the time series of the piecewise linear time-delay system which we have considered here, we will calculate *(1) filling factor* [37], *(2) length of polygon line* [38] and *(3) average mutual information* [25, 39, 40] both in the presence and in the absence of delay time modulation and show how periodic modulation removes any imprints of the time-delay.

1. Filling factor

Now we will compute the filling factor [37] for the chaotic trajectory $x(t)$ of the time-delay system (1) by projecting it onto the pseudospace $(x, x_{\hat{\tau}}, \dot{x})$ with P^{3N} equally sized hypercubes, where the delayed time series $x_{\hat{\tau}} = x(t - \hat{\tau})$ is constructed from $x(t)$ for various values of $\hat{\tau}$. The filling factor is the number of hypercubes which are visited by the projected trajectory, normalized to the total number of hypercubes, P^{3N}. Figure 1a shows the filling factor for constant delay $\tau_0 = 10$ when $\tau_a = 0$ in Eq. (3), where one can identify the existence of an underlying time-delay induced instability [37] which induces local minima

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in the filling factor at $\hat{\tau} \approx n\tau_0$, $n = 1, 2, 3....$ From the later, one can identify the value of the time-delay parameter τ of the system (1) under consideration. Figure 1b shows filling factor with delay time modulation of the form (3) with $\tau_0 = 10, \tau_a = 90$ and $\omega = 0.0001$, where no local minima occurs. Figure 1c is plotted for $\tau_0 = 100$ and $\tau_a = 0$ to show that the disappearance of local minima in Fig. 1b is not due to large delays but only because of delay time modulation. From the figures one can realize that the imprints of the delay time embedded in the projected trajectory is completely ironed out due to the presence of delay time modulation.

2. Length of polygon line

Next, to calculate the length of polygon line [38], the trajectory in $(x, x_{\hat{\tau}}, \dot{x})$ space is restricted to a two dimensional surface. The restriction in dimension is effected by intersecting the projected trajectory with a surface $k(x, x_{\hat{\tau}}, \dot{x}) = 0$. Consequently the number of times the trajectory traverses the surface and the corresponding intersection points can be calculated. One then orders the points with respect to the values of $x_{\hat{\tau}}$, and a simple measure for the alignment of the points is the length L of polygon line connecting all the ordered points. Figure 2a shows length of polygon line L with constant delay $\tau_0 = 10$, where the local minima correspond to the delay time of the system we have considered. Figure 2b shows length of polygon line L with delay time modulation where there is no remnance of information about delay time from the trajectory, whereas Fig. 2c is plotted for $\tau_0 = 100, \tau_a = 0$, to show that the imprints of delay time carved out in the trajectory vanishes in Fig. 2b only due to the delay time modulation and not because of large delay.

3. Average mutual information

As a final example, we will calculate average mutual information defined by (see for example, [25, 39, 40] and references therein)

$$I(\hat{\tau}) = \sum_{x(n), x(n+\hat{\tau})} P(x(n), x(n + \hat{\tau})) \times \log_2 \left[\frac{P(x(n), x(n + \hat{\tau}))}{P(x(n)) P(x(n + \hat{\tau}))} \right], \tag{4}$$

where $P(x(n), x(n + \hat{\tau}))$ is the joint probability density for measurements in the chaotic time series $X = (x(1), x(2), ..., x(m))$ and in the constructed delay time series $X_{\hat{\tau}} = (x(1 +$

$\hat\tau), x(2 + \hat\tau), ..., x(m + \hat\tau))$ by varying $\hat\tau$, resulting in values $x(n)$ and $x(n + \hat\tau)$. $P(x(n))$ and $P(x(n + \hat\tau))$ are the individual probability densities for the measurements of X and $X_{\hat\tau}$. Figure 3 shows the average mutual information for the cases of constant delay time with $\tau_0 = 10$ (Fig. 3a) and with delay time modulation (Fig. 3b). Figure 3c is plotted for $\tau_0 = 100$ to show that the absence of local peaks in Fig. 3b is due to delay time modulation and not because of large delay. For fixed delay time the average mutual information shows local peaks at the time-delay $\hat\tau = \tau_0$ (or multiples of it $\hat\tau = n\tau_0$) of the system, whereas for the case of delay time modulation the average mutual information has no such peaks to identify the delay time of the delayed system.

One can also obtain similar results with other measures such as autocorrelation function, onestep prediction error and average fitting error [37, 38, 41, 42]. However, we are not presenting these results here for convenience. In order to perform the phase space reconstruction, the first step is to find out the delay time for the projected trajectories. By introducing the delay time modulation the imprints of delay time in the projected trajectory is completely removed as seen above for the present system, inhibiting any possibility of phase space reconstruction. This is essentially consequent of the fact that when the delay time is modulated by the fully rectified sine term, the delay time effectively gets increased in which case the number of positive Lyapunov exponent also increases (as noted in Fig. 2 in Ref. [35]. Consequently study of chaos synchronization in a system of such coupled delay time modulated oscillators will be of considerable interest.

III. COUPLED SYSTEM AND THE STABILITY CONDITION IN THE PRESENCE OF DELAY TIME MODULATION

Now let us consider the following unidirectionally coupled drive $x_1(t)$ and response $x_2(t)$ systems with two different modulated time-delays $\tau_1(t)$ and $\tau_2(t)$ as feedback and coupling time-delays, respectively (hereafter we write $\tau_1(t)$ and $\tau_2(t)$ simply as τ_1 and τ_2 respectively),

$$\dot{x}_1(t) = -ax_1(t) + b_1 f(x_1(t - \tau_1)), \tag{5a}$$

$$\dot{x}_2(t) = -ax_2(t) + b_2 f(x_2(t - \tau_1)) + b_3 f(x_1(t - \tau_2)), \tag{5b}$$

where b_1, b_2 and b_3 are constants, $a > 0$, and $f(x)$ is of the same form as in Eq. (2) with

$$\tau_1 = \tau_{10} + \tau_{1a} \left| \sin(\omega_1 t) \right|, \tag{6a}$$

$$\tau_2 = \tau_{20} + \tau_{2a} \left| \sin(\omega_2 t) \right|, \tag{6b}$$

where τ_{10} and τ_{20} are the zero frequency components of feedback delay and coupling delay, τ_{1a} and τ_{2a} are the amplitudes of the time dependent components of τ_1 and τ_2, respectively, and ω_1/π and ω_2/π are the corresponding frequencies of their modulations.

Now we can deduce the stability condition for synchronization of the two time-delay systems, Eqs. (5a) and (5b), in the presence of the delay coupling $b_3 f(x_1(t - \tau_2))$ with time delay modulation in both the feedback delay and coupling delay. The time evolution of the difference system with the state variable $\Delta = x_{1\tau_2 - \tau_1} - x_2$, where $x_{1\tau_2 - \tau_1} = x_1(t - (\tau_2 - \tau_1))$, can be written for small values of Δ by using the evolution Eqs. (5) as

$$\dot{\Delta} = -a\Delta + (b_2 + b_3 - b_1) f(x_1(t - \tau_2)) + b_2 f'(x_1(t - \tau_2))\Delta_{\tau_1}, \quad \Delta_\tau = \Delta(t - \tau). \tag{7}$$

Then $\Delta = 0$ corresponds to anticipatory synchronization when $\tau_2 < \tau_1$, identical or complete synchronization for $\tau_2 = \tau_1$ and lag synchronization when $\tau_2 > \tau_1$. In order to study the stability of the synchronization manifold as in the case of constant time delay case [31], we choose the parametric condition,

$$b_1 = b_2 + b_3, \tag{8}$$

so that the evolution equation for the difference system Δ becomes

$$\dot{\Delta} = -a\Delta + b_2 f'(x_1(t - \tau_2))\Delta_{\tau_1}. \tag{9}$$

The synchronization manifold is locally attracting if the origin of this equation is stable. Following Krasovskii-Lyapunov functional approach [43, 44], we define a positive definite Lyapunov functional of the form

$$V(t) = \frac{1}{2}\Delta^2 + \mu \int_{-\tau_1(t)}^{0} \Delta^2(t + \theta)d\theta, \tag{10}$$

where μ is an arbitrary positive parameter, $\mu > 0$. Note that $V(t)$ approaches zero as $\Delta \to 0$.

To estimate a sufficient condition for the stability of the solution $\Delta = 0$, we require the derivative of the functional $V(t)$ along the trajectory of Eq. (9),

$$\frac{dV}{dt} = -a\Delta^2 + b_2 f'(x_1(t - \tau_2))\Delta\Delta_{\tau_1} + \mu \left[\Delta_{\tau_1}^2 \tau_1' + \Delta^2 - \Delta_{\tau_1}^2 \right], \tag{11}$$

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to be negative. Note that in the case of constant modulation $\tau_1' = \frac{d\tau_1}{dt}$ vanishes. The above equation can be rewritten as

$$\frac{dV}{dt} = -\mu\Delta^2\Gamma(X,\mu), \tag{12}$$

where $\Gamma = \left[\left((a-\mu)/\mu\right) - \left(b_2 f'(x_1(t-\tau_2))/\mu\right)X + X^2/(1-\tau_1')\right]$, $X = \Delta_{\tau_1}/\Delta$. In order to show that $\frac{dV}{dt} < 0$ for all Δ and Δ_τ and so for all X, it is sufficient to show that $\Gamma_{min} > 0$. One can easily check that the absolute minimum of Γ occurs at $X = b_2 f'(x_1(t-\tau_2))/2\mu(1-\tau_1')$ with $\Gamma_{min} = \left[4\mu(a-\mu)(1-\tau_1') - b_2^2 f'^2(x_1(t-\tau_2))\right]/4\mu^2(1-\tau_1')$. Consequently, we have the condition for stability as

$$a > \frac{b_2^2 f'^2(x_1(t-\tau_2))}{4\mu(1-\tau_1')} + \mu = \Phi(\mu). \tag{13}$$

Again $\Phi(\mu)$ as a function of μ for a given $f'(x)$ has an absolute minimum at $\mu = \left(\left|\frac{b_2 f'(x_1(t-\tau_2))}{2\sqrt{(1-\tau_1')}}\right|\right)$ with $\Phi_{min} = \left|\frac{b_2 f'(x_1(t-\tau_2))}{\sqrt{(1-\tau_1')}}\right|$. Since $\Phi \geq \Phi_{min} = \left|\frac{b_2 f'(x_1(t-\tau_2))}{\sqrt{(1-\tau_1')}}\right|$, from the inequality (13), it turns out that the sufficient condition for asymptotic stability is

$$a > \left|\frac{b_2 f'(x_1(t-\tau_2))}{\sqrt{(1-\tau_1')}}\right| \tag{14}$$

along with the condition (8) on the parameters b_1, b_2 and b_3.

Now from the form of the piecewise linear function $f(x)$ given by Eq. (2), we have,

$$|f'(x_1(t-\tau_2))| = \begin{cases} 1.5, & 0.8 \leq |x_1| \leq \frac{4}{3} \\ 1.0, & |x_1| < 0.8 \end{cases} \tag{15}$$

Consequently the stability condition (14) becomes $a > 1.5 \left|\frac{b_2}{\sqrt{(1-\tau_1')}}\right| > \left|\frac{b_2}{\sqrt{(1-\tau_1')}}\right|$ along with the parametric restriction $b_1 = b_2 + b_3$.

Thus one can take $a > \left|\frac{b_2}{\sqrt{(1-\tau_1')}}\right|$ as a less stringent condition for (14) to be valid, while

$$a > 1.5 \left|\frac{b_2}{\sqrt{(1-\tau_1')}}\right| \tag{16}$$

can be considered as the most general condition specified by (14) for asymptotic stability of the synchronized state $\Delta = 0$. The condition (16) indeed corresponds to the stability condition for exact anticipatory/lag as well as exact complete synchronizations for a given value of the coupling delay τ_2 in a global sense. It may be noted that the stability condition

(16) is valid irrespective of the nature of the coupling delay, that is whether it is constant or modulated. However, when the feed back delay τ_1 is constant the condition (16) reduces to $a > 1.5|b_2|$ as discussed in ref. [31]. In the following, we will consider both the possibilities of constant $(\tau_2 = \tau_{20})$ and periodically modulated $(\tau_2 = \tau_{20}+\tau_{2a}|\sin(\omega_2 t)|)$ coupling delays with a periodically modulated feedback delay $(\tau_1 = \tau_{10} + \tau_{1a}|\sin(\omega_1 t)|)$. We demonstrate through detailed numerical analysis that there exists oscillating synchronization that oscillates between anticipatory, complete and lag synchronizations for the case of constant coupling delay $\tau_2 = \tau_{20}$. Intermittent anticipatory/lag and complete synchronizations are shown to exist for the case of coupling delay with delay time modulation $\tau_2 = \tau_{20} + \tau_{2a}|\sin(\omega_2 t)|$, when $\tau_{2a} = \tau_{1a}$ and $\omega_1 = \omega_2$. For $\tau_{2a} \neq \tau_{1a}$ and $\omega_1 \neq \omega_2$, more complicated oscillating type synchronizations occur.

IV. OSCILLATING SYNCHRONIZATION

At first we consider the constant coupling delay, $\tau_2 = \tau_{20}$, and show that there exists oscillating synchronization that oscillates between anticipatory, complete and lag synchronizations as a function of time for suitable range of parameters.

Now we will choose the delay time modulation in the form (6a) for the feedback delay $\tau_1 (= \tau_{10}+\tau_{1a}|\sin(\omega_1 t)|)$ with $\tau_{10} = 10, \tau_{1a} = 90$ and $\omega_1 = 10^{-4}$. We have fixed the value of $\tau_{2a} = 0$ in (6b), so that the coupling delay becomes constant $\tau_2 = \tau_{20} = 45$ with the parameters $a = 1, b_1 = 1.2$ in Eq. (5) and the values of b_2 and b_3 are chosen according to the parametric restriction (8) depending upon the stability condition to be satisfied. For the chosen values of τ_{10} and τ_{1a}, one can find that τ_1 oscillates between $(\tau_1(t) = \tau_{10} + \tau_{1a}|\sin(\omega_1 t)| = 10 + 90|\sin(\omega_1 t)|)$ the values 10 and 100. With the chosen value of constant coupling delay $\tau_2 = 45$ and time dependent feedback delay τ_1, as time evolves one finds that the feedback delay $\tau_1(t)$ is lesser than the value of constant coupling delay τ_2 initially for some time (in which case $\tau(t) = \tau_2 - \tau_1(t) > 0$, so that there exists lag synchronization $x_1(t - \tau(t)) = x_2(t)$ with varying lag time $\tau(t) = \tau_2 - \tau_1(t)$). As time evolves, $\tau_1(t)$ increases eventually and it approaches $\tau_1 = 45$ at a certain later time $(T = \pi/\omega_1)$, where $\tau(t) = \tau_2 - \tau_1(t) = 0$, so that $x_1(t) = x_2(t)$ and a complete synchronization occurs at a specific value of time. As $\tau_1(t)$ increases further above the value of $\tau_2 = 45$, the delay time $\tau(t)$ becomes negative, $\tau(t) = \tau_2 - \tau_1(t) < 0$ with $x_1(t - \tau(t)) = x_2(t)$) and there exists anticipatory synchronization

with varying anticipating time $\tau(t) = \tau_2 - \tau_1(t)$. This anticipatory synchronization continues till the value of time dependent feed back delay $\tau_1(t)$ decreases to appraoch the value of the constant coupling delay $\tau_2 = 10$ after reaching its maximum value of 100. Therefore as time evolves there is oscillation between lag, complete and anticipatory synchronizations with time dependent anticipating and lag times.

Figure 4a shows the evolution of the drive $x_1(t)$ and the response $x_2(t)$ at the transition between lag to anticipatory synchronization via complete synchronization for the value of $b_2 = 0.1$, where the general stability condition (16) is satisfied, whereas Fig. 4b shows the evolution of the drive $x_1(t)$ and the response $x_2(t)$ at the next transition between anticipatory to lag via complete synchronization. In Figs. 5a and 5b, the difference signals $x_1(t - \tau) - x_2(t), \tau > 0$ and $x_1(t - \tau) - x_2(t), \tau < 0$ are plotted respectively for the value of parameters satisfying the general stability condition corresponding to the Fig. 4, confirming the transition between lag to anticipatory synchronization. Thus as a consequence of delay time modulation there exists a new type of oscillating synchronization that oscillates between anticipatory, complete and lag synchronizations with varying anticipating and lag times.

V.   INTERMITTENT ANTICIPATORY SYNCHRONIZATION

Now we consider the coupled time-delay system (5) with delay time modulation of the form (6) in both the feedback and coupling delays for further studies. We have fixed the values of the parameters as $a = 1, b_1 = 1.2, \tau_{1a} = \tau_{2a} = 90, \omega_1 = \omega_2 = 10^{-5}$ (identical modulations) and the values of b_2 and b_3 are chosen according to the parametric restriction $b_1 = b_2 + b_3$ depending upon the stability condition to be satisfied. For τ_1, the zero frequency component of amplitude is fixed as $\tau_{10} = 10$ and for τ_2, it is fixed as $\tau_{20} = 5$, so that a constant difference is maintained between the feedback and the coupling time delays throughout the time evolution. With the coupling delay $\tau_2(= 5 + 90\,|\sin(10^{-5}t)|)$ being less than the feedback delay $\tau_1(= 10 + 90\,|\sin(10^{-5}t)|)$, that is $\tau_2(t) < \tau_1(t)$, the value of the anticipating time $\tau = \tau_2 - \tau_1$ turns out to be negative such that the relation between drive $x_1(t)$ and the response $x_2(t)$ now becomes $x_1(t - \tau) = x_2(t), \tau < 0$, demonstrating anticipatory synchronization, provided the stability condition (16) is satisfied with the parametric restriction specified by Eq. (8).

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Now let us choose the parameter b_2 as the control parameter, whose value determines the stability condition given by Eq. (14).

1. For $b_2 = 0.7$, $1.5 \left| \frac{b_2}{\sqrt{1-\tau_1'}} \right| > a > \left| \frac{b_2}{\sqrt{1-\tau_1'}} \right|$, the less stringent condition is satisfied with $\sqrt{1-\tau_1'} \approx 1$ for the chosen values of ω and τ_a. One can observe intermittent anticipatory synchronization as shown in Fig. 6, exhibiting typical features of *on-off* intermittency [32, 33] with the *off* state near the laminar phase and the *on* state showing a random burst. For this value of b_2 the amplitude of the laminar phase corresponding to the synchronized state is approximately zero (of the order 10^{-5}).

2. For $b_2 = 0.1$, $a > 1.5 \left| \frac{b_2}{\sqrt{1-\tau_1'}} \right| > \left| \frac{b_2}{\sqrt{1-\tau_1'}} \right|$, the general stability condition (16) is satisfied and correspondingly the numerical analysis reveals that here the intermittent anticipatory synchronization is such that the amplitude of the laminar phases corresponding to the synchronized state is exactly zero (in the sense that the difference $\Delta = x_1(t-\tau) - x_2(t), \tau < 0$ is of the order 10^{-16} in the laminar phases) as shown in Fig. 7.

To analyze the statistical features associated with the intermittent nature in Fig. 7 for the value of $b_2 = 0.1$, we have calculated the distribution of laminar phases $\Lambda(t)$ with the amplitude less than a threshold value $\Delta < 10^{-10}$ and we have observed a universal asymptotic $-\frac{3}{2}$ power law distribution as shown in Fig. 8, which is quite typical for on-off intermittency [32, 33]. One can also find a similar power law distribution for the value of $b_2 = 0.7$ discussed above but now with a *larger* threshold value ($\Delta < 10^{-4}$) of the laminar region.

Now using the notion of similarity function introduced by Rosenblum *et al.* [11] to characterize lag synchronization, one can also characterize anticipatory synchronization. Similarity function for anticipatory synchronization is defined as the time-averaged difference between the variables x_1 and x_2 (with mean values being subtracted) taken with the time shift $|\tau|$,

$$S_a^2(\tau) = \frac{\langle [x_1(t+|\tau|) - x_2(t)]^2 \rangle}{[\langle x_1^2(t) \rangle \langle x_2^2(t) \rangle]^{1/2}}, \tag{17}$$

where, $\langle x \rangle$ means time average over the variable x. If the minimum value of $S_a(\tau)$ reaches zero, that is $S_a(\tau) = 0$, then there exists a time shift $|\tau|$ between the two signals $x_1(t)$ and

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$x_2(t)$ such that $x_1(t + |\tau|) = x_2(t)$, demonstrating the existence of anticipatory synchronization between the drive x_1 and the response x_2 signals. Figure 9 shows the similarity function $S_a(\tau)$ as a function of the difference between the feedback and the coupling delays, $\tau = \tau_2 - \tau_1$ for three different values of b_2, the parameter whose value determines the stability condition (14). In Fig. 9, the Curve 3 is plotted for the value of $b_2 = 1.1$, $(1.5 \left| \frac{b_2}{\sqrt{1-\tau_1'}} \right| > \left| \frac{b_2}{\sqrt{1-\tau_1'}} \right| > a)$, where both the less stringent condition and the most general condition are violated. From the curve 3 one can find that the minimum value of $S_a(\tau)$ is greater than zero for all values of τ, resulting in the lack of exact time shift (anticipating time) between the drive and the response signals. On the other hand the curve 2 corresponds to the value of $b_2 = 0.7$ such that the less stringent condition is satisfied while the general stability criterion (16) is violated as seen above. Curve 2 shows that the minimum of similarity function $S_a(\tau)$ is approximately zero (of the order 10^{-4}) for $\tau < 0$, as may be seen in the inset of Fig. 9, indicating the existence of intermittent anticipatory synchronization with the amplitude of the laminar phases of the difference signal $\Delta = x_1(t-\tau) - x_2(t), \tau < 0$, being approximately zero $(< 10^{-5})$. On the other hand, the curve 1 is plotted for $b_2 = 0.1$, satisfying the general stability criterion Eq. (16), which shows that the minimum of similarity function is much closer to zero (of order 10^{-8}), $\tau < 0$, indicating that there exists an intermittent anticipatory synchronization with the amplitude of the laminar phase of the difference signal becoming exactly zero with the anticipating time equal to the difference between the two time delays $\tau = \tau_2 - \tau_1$.

Next, by reducing the value of the modulational frequencies $\omega = \omega_1 = \omega_2$ further, we find that the lengths of the laminar phases increase gradually with a corresponding decrease in the number of turbulent phases. Finally at an appropriate value of the modulational frequency all the turbulent phases disappear and there exists only exact anticipatory synchronization without any intermittent bursts provided the exact stability condition is satisfied. Correspondingly the similarity function $S_a(\tau)$ becomes zero exactly (which is of the order 10^{-16}) for $\tau < 0$ in this case, as shown in [31].

VI. COMPLETE SYNCHRONIZATION

Complete synchronization follows the anticipatory synchronization when the value of the coupling time-delay τ_2 equals the feedback time-delay τ_1, that is $\tau_2 = \tau_1$, where the antici-

pating time becomes $\tau = \tau_2 - \tau_1 = 0$. Here also, the same stability criterion Eq. (16) holds good with the same parametric restriction specified by (8). In this case of complete synchronization ($\tau_2 = \tau_1$), the delay time modulation does not induce any intermittent nature in the dynamical behavior of the coupled systems for any value of the modulational frequency ($\omega_1 = \omega_2$) as inferred from Eq. (6). Figure 10a shows as an illustration the plot of $x_1(t)$ vs $x_2(t)$ for the values of $b_2 = 0.7$ and $\omega_1 = \omega_2 = 10^{-5}$, such that the less stringent condition is satisfied and the general stability criterion (16) is violated. The plot shows small deviations from the localized diagonal line implying an approximate synchronization, whereas Fig. 10b shows an entirely localized sharp diagonal line for the value of $b_2 = 0.1$, where the general stability condition (16) is satisfied, indicating the complete synchronization.

VII. INTERMITTENT LAG SYNCHRONIZATION

When the value of the coupling delay τ_2 is increased above the value of the feedback delay $\tau_1(\tau_2 > \tau_1)$, then the value of the retarded time $\tau = \tau_2 - \tau_1$ turns out to be positive such that the relation between the drive $x_1(t)$ and the response $x_2(t)$ now becomes $x_1(t - \tau) = x_2(t), \tau > 0$, depicting the existence of lag synchronization, provided the general stability condition (16) is satisfied along with the parametric condition (8).

We have fixed the same values for all the parameters as in the case of intermittent anticipatory synchronization except for the zero frequency component τ_{20} of coupling delay τ_2 which is fixed at $\tau_{20} = 15$. Figure 11 shows the intermittent lag synchronization for the value of $b_2 = 0.7$, in which case only the less stringent stability condition is satisfied, where the laminar phase has an amplitude which is nearly zero (of the order 10^{-5}). Figure 12 shows intermittent lag synchronization for the value of $b_2 = 0.1$, where the amplitude of the laminar phase vanishes exactly. In the later case the most general stability criterion (16) is satisfied. The statistical behavior associated with the intermittent nature in this case of intermittent lag synchronization is also characterized by the probability distribution of laminar phases having amplitude less than a threshold value $\Delta < 10^{-10}$ corresponding to a universal asymptotic $-\frac{3}{2}$ power law distribution as shown in the Fig. 13.

The figure shows the probability distribution $\Lambda(t)$ of intermittent lag synchronization for the value of $b_2 = 0.1$. One can also verify that the intermittent lag synchronization for the value of $b_2 = 0.7$ has also similar power law distribution for *larger* threshold value

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$(\Delta < 10^{-4})$of amplitude of the laminar phases.

The existence of intermittent lag synchronization is also characterized by a similarity function $S_l(\tau)$ defined as

$$S_l^2(\tau) = \frac{\langle [x_1(t - |\tau|) - x_2(t)]^2 \rangle}{[\langle x_1^2(t) \rangle \langle x_2^2(t) \rangle]^{1/2}}. \tag{18}$$

Figure 14 shows the similarity function $S_l(\tau)$ for intermittent lag synchronization as a function of the retarded time $\tau = \tau_2 - \tau_1$. Curve 3 is plotted for the value of $b_2 = 1.1$ (which is greater than both $a\sqrt{1 - \tau_1'}$ and $a\sqrt{1 - \tau_1'}/1.5$), where the minimum of similarity function $S_l(\tau)$ occurs at a finite value of $S_l(\tau) > 0$ and hence there is a lack of lag synchronization between the drive and the response signals indicating asynchronization. Curve 2 corresponds to the value of $b_2 = 0.7$, (which is less than $a\sqrt{1 - \tau_1'}$ but greater than $a\sqrt{1 - \tau_1'}/1.5$), where the minimum of similarity function $S_l(\tau)$ is approximately zero (of the order of 10^{-4}, as may be seen in the inset of Fig. 14) indicating the existence of intermittent lag synchronization with the amplitude of the laminar phase being approximately zero. However, for the value of $b_2 = 0.1$, for which the general condition (16) is obeyed, the minimum of similarity function for Curve 1 becomes much closer to zero (of the order 10^{-8}) which corresponds to intermittent lag synchronization with exact time shift between the two signals during the laminar phase.

Next, as in the case of intermittent anticipatory synchronization, by reducing the value of modulational frequency one can find that the lengths of the laminar phases increase with vanishing turbulent phases and finally at an appropriate value of the modulational frequency there exists exact lag synchronization without any intermittent bursts provided the exact stability condition is satisfied. Correspondingly the similarity function $S_l(\tau)$ becomes zero exactly (which is of the order 10^{-16}) for $\tau > 0$ in this case.

VIII.   COMPLEX OSCILLATING SYNCHRONIZATION

Finally, when $\tau_{1a} \neq \tau_{2a}$ or/and $\omega_1 \neq \omega_2$ the frequencies as well as amplitudes of the modulated feedback delay $\tau_1(t)(= \tau_{10} + \tau_{1a}|\sin(\omega_1 t)|)$ and the modulated coupling delay $\tau_2(t)(= \tau_{20} + \tau_{2a}|\sin(\omega_2 t)|)$ differ from each other resulting in a more complicated variation of the anticipating/lag time $\tau(t) = \tau_2(t) - \tau_1(t)$. This in turn results in the existence of more complex oscillating synchronization than the one presented in Sec. IV. It is clear that

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one can also introduce other kinds of modulations instead of periodic modulation to obtain varying forms of oscillating synchronizations.

IX. SUMMARY AND CONCLUSIONS

In this paper, we have shown that there exists a new type of oscillating synchronization that oscillates between anticipatory, complete and lag synchronization and vice versa for the case of constant coupling delay with varying anticipating and lag times. We have also shown that there exists regions of intermittent anticipatory synchronization, intermittent lag synchronization and complete synchronization in the parameter space of ω and τ_2 with appropriate stability condition for the synchronized state in a system of two piecewise linear time-delay systems with delay time modulation in both the feedback and coupling delay. For a fixed value of ω, we have shown that there is a transition from intermittent anticipatory to intermittent lag synchronization through complete synchronization with the coupling delay τ_2 as the only control parameter, while all the other parameters are kept fixed. We have also found that on further reducing the value of ω, one can observe transition towards exact anticipatory/lag synchronizations from their intermittent behaviour. The signature of the intermittent behavior in both the intermittent anticipatory and intermittent lag synchronizations are characterized by probability distribution of laminar phases satisfying a universal asymptotic $-\frac{3}{2}$ power law distribution. The existence of intermittent anticipatory and intermittent lag synchronizations are characterized by their corresponding similarity functions.

Further, we have observed that in the region where the stringent stability condition (16) is satisfied, the minimum of the similarity functions $S_a(\tau)$ and $S_l(\tau)$ approaches very closely zero for all values of $\tau_2 < \tau_1$ and $\tau_2 > \tau_1$, respectively. The range of zero values corresponding to the minimum of similarity functions $S_a(\tau)$ and $S_l(\tau)$ shows the existence of anticipatory and lag synchronizations for the values of coupling delay τ_2 below and above the feedback delay τ_1, respectively. We have also shown that the estimation of the delay time carved out of the time series of the delayed system even with delay time modulations of fully rectified sinusoidal type is very difficult by conventional methods in the present system for suitable choice of the parameters (in contrast to the chaotic or stochastic delay time modulation as studied by Kye *et al* [25] in Mackey-Glass delay system) and so the messages encoded in such

systems can be expected to be less amenable for extraction by phase space reconstruction. We have also confirmed numerically that the phenomena reported in this paper occur in other time-delay systems such as Mackey-Glass and Ikeda systems also to corroborate the generic nature of the results. Also the model system discussed in the present manuscript is amenable for experimental realization in terms of suitable nonlinear electronic circuits in view of its piecewise linear nature and we are pursuing the experimental verification of it. It is hoped that the study of such modulated systems will lead to a better understanding of the dynamics of systems with time dependent topologies such as neural networks, world wide web, etc.

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Figure captions

1. Filling factor as a function of delay time $\hat{\tau}$ (of delayed time series $x_{\hat{\tau}}$). (a) with constant delay $\tau_0 = 10$ when $\tau_a = 0$, (b) with delay time modulation of the form (3) with $\tau_0 = 10, \tau_a = 90$ and $\omega = 10^{-4}$ and (c) with large constant delay $\tau_0 = 100(\tau_a = 0)$.

2. Length of polygon line as a function of delay time $\hat{\tau}$ (of delayed time series $x_{\hat{\tau}}$). (a) with constant delay $\tau_0 = 10(\tau_a = 0)$, (b) with delay time modulation of the form (3) with the same parameters as in Fig. 1 and (c) with large constant delay $\tau_0 = 100(\tau_a = 0)$.

3. Average mutual information as a function of delay time $\hat{\tau}$ (of delayed time series $x_{\hat{\tau}}$). (a) with constant delay $\tau_0 = 10, \tau_a = 0$, (b) with delay time modulation of the form (3) with the same parameters as in Fig. 1 and (c) with large constant delay $\tau_0 = 100, \tau_a = 0$.

4. Oscillating synchronization for the constant coupling delay $\tau_2 = 45$ with time dependent feedback delay of the form (6a) with $\tau_{10} = 10, \tau_{1a} = 90$ and $\omega = 10^{-4}$. (a) Oscillating from lag to anticipatory synchronization via complete synchronization in the region $t \in (3970, 4020)$ and (b) Oscillating from anticipatory to lag synchronization at the next transition in the region $t \in (27400, 27450)$.

5. (a) Difference between $x_1(t - \tau), \tau > 0$ and $x_2(t)$, showing lag synchronization for certain time and (b) difference between $x_1(t-\tau), \tau < 0$ and $x_2(t)$, showing anticipatory synchronization for the following period of time for $b_2 = 0.1$ satisfying the general stability condition (16). Note that complete synchronization occurs in the transition regime.

6. The time series $x_1(t - \tau) - x_2(t), \tau < 0$, for $b_2 = 0.7$ and $b_3 = 0.5$ (so that the less stringent condition $a > |b_2/\sqrt{1 - \tau'_1}|$ is satisfied while (16) is violated) corresponding to intermittent anticipatory synchronization with the amplitude of the laminar phase approximately zero.

7. The time series $x_1(t - \tau) - x_2(t), \tau < 0$, for $b_2 = 0.1$ and $b_3 = 1.1$. Here the general stability criterion (16) is satisfied corresponding to intermittent anticipatory synchronization with the amplitude of the laminar phase exactly zero.

8. The statistical distribution of laminar phase satisfying $-\frac{3}{2}$ power law scaling for $b_2 = 0.1$ and $b_3 = 1.1$, where the general stability criterion (16) is satisfied.

9. Similarity function for intermittent anticipatory synchronization $S_a(\tau)$ for different values of b_2, the other system parameters are $a = 1.0, b_1 = 1.2$ and $\omega = 10^{-5}$. (Curve 1: $b_2 = 0.1, b_3 = 1.1$, Curve 2: $b_2 = 0.7, b_3 = 0.5$ and Curve 3: $b_2 = 1.1, b_3 = 0.1$).

10. Complete synchronization between $x_1(t)$ vs $x_2(t)$ when $\tau_{20} = \tau_{10}$. (a)Approximate complete synchronization for $b_2 = 0.7$ and (b) Exact complete synchronization for $b_2 = 0.1$.

11. The time series $x_1(t - \tau) - x_2(t), \tau > 0$, for $b_2 = 0.7$ and $b_3 = 0.5$ (so that the less stringent condition $a > |b_2/\sqrt{1 - \tau_1'}|$ is satisfied while (16) is violated) corresponding to intermittent lag synchronization with the amplitude of the laminar phase approximately zero.

12. The time series $x_1(t - \tau) - x_2(t), \tau > 0$, for $b_2 = 0.1$ and $b_3 = 1.1$. Here the general stability criterion (16) is satisfied corresponding to intermittent lag synchronization with the amplitude of the laminar phase exactly zero.

13. The statistical distribution of laminar phase satisfying $-\frac{3}{2}$ power law scaling for $b_2 = 0.1$ and $b_3 = 1.1$, where the general stability criterion (16) is satisfied.

14. Similarity function for intermittent lag synchronization $S_l(\tau)$ for different values of b_2, the other system parameters are $a = 1.0, b_1 = 1.2$ and $\omega = 10^{-5}$. (Curve 1: $b_2 = 0.1, b_3 = 1.1$, Curve 2: $b_2 = 0.7, b_3 = 0.5$ and Curve 3: $b_2 = 1.1, b_3 = 1.0$).
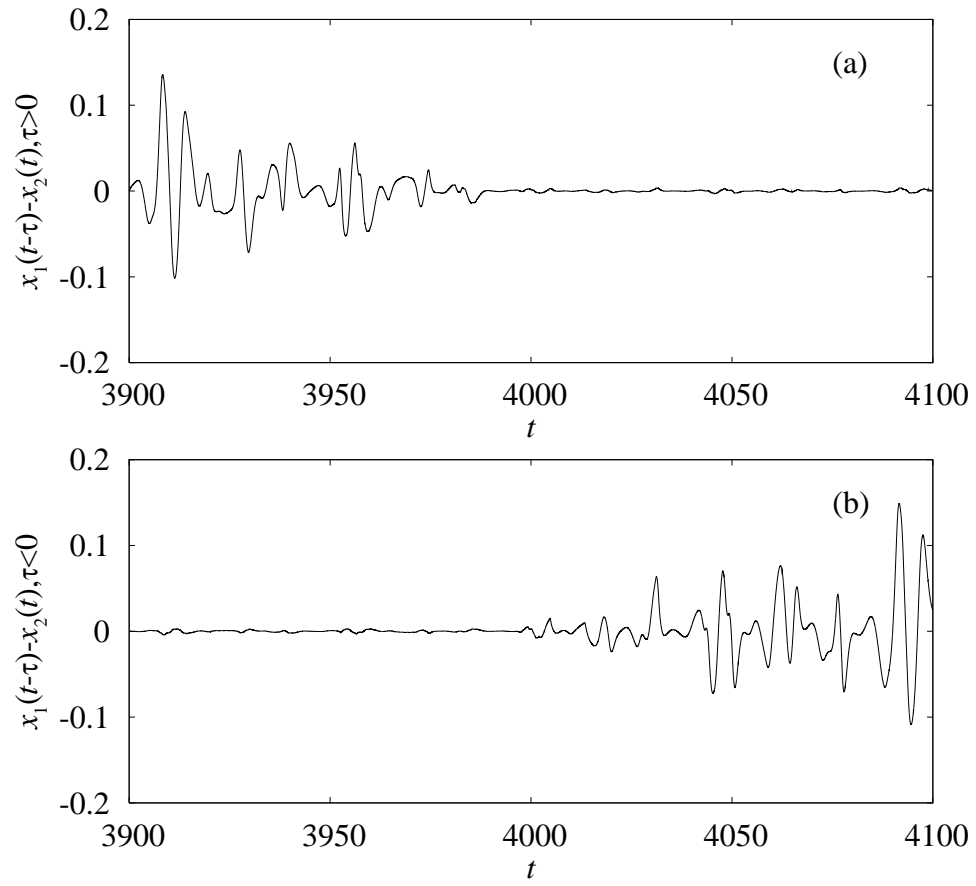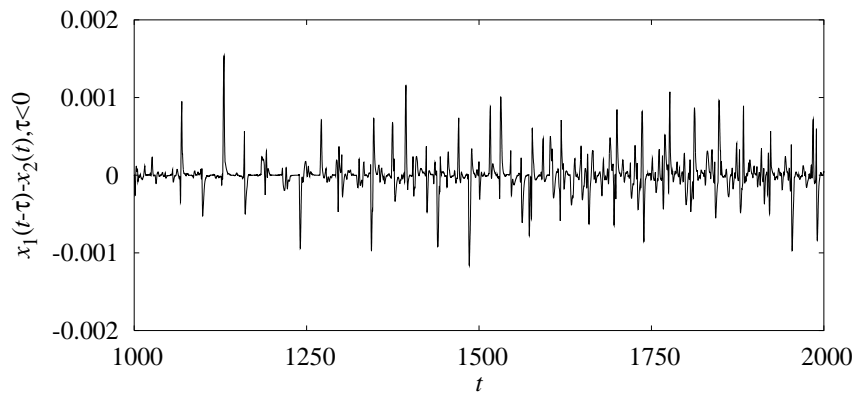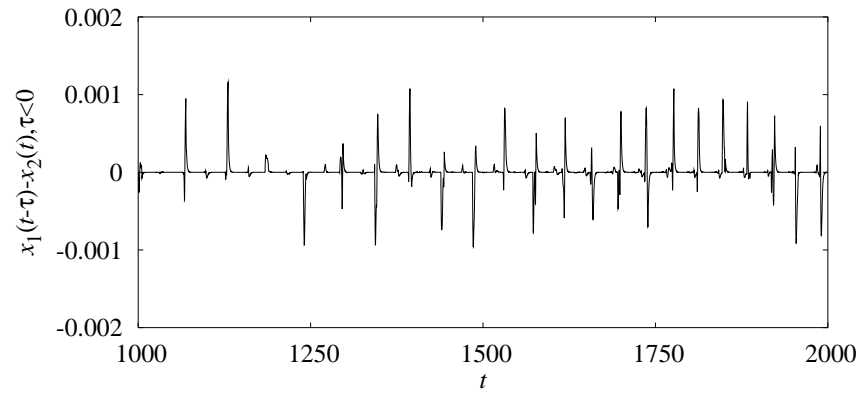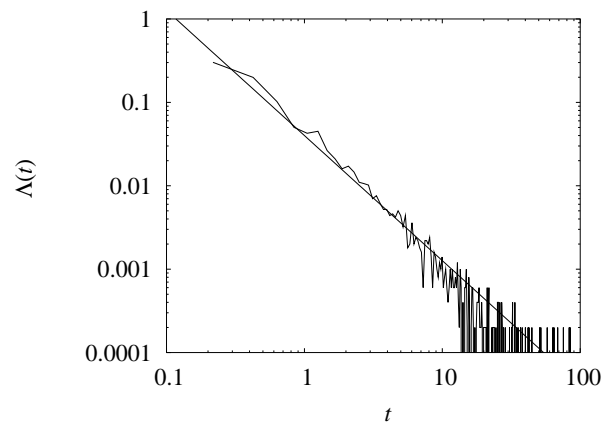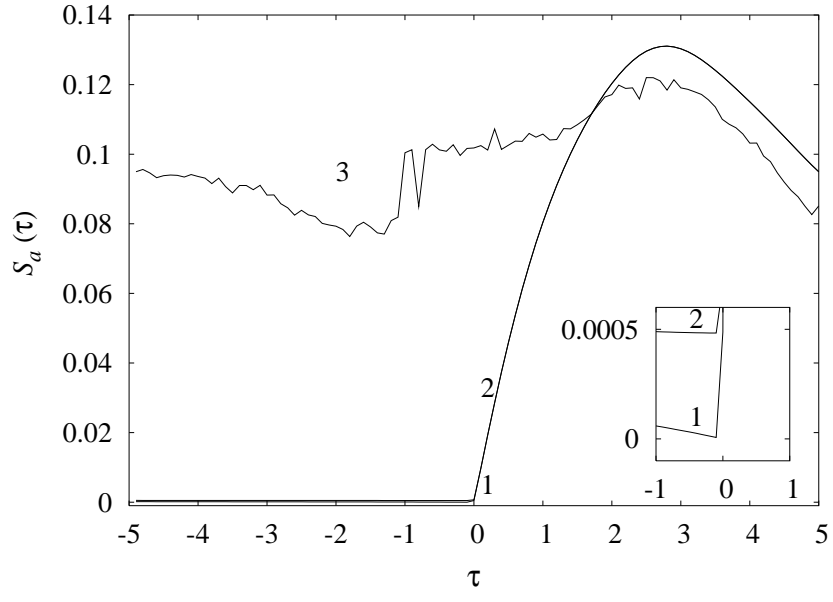
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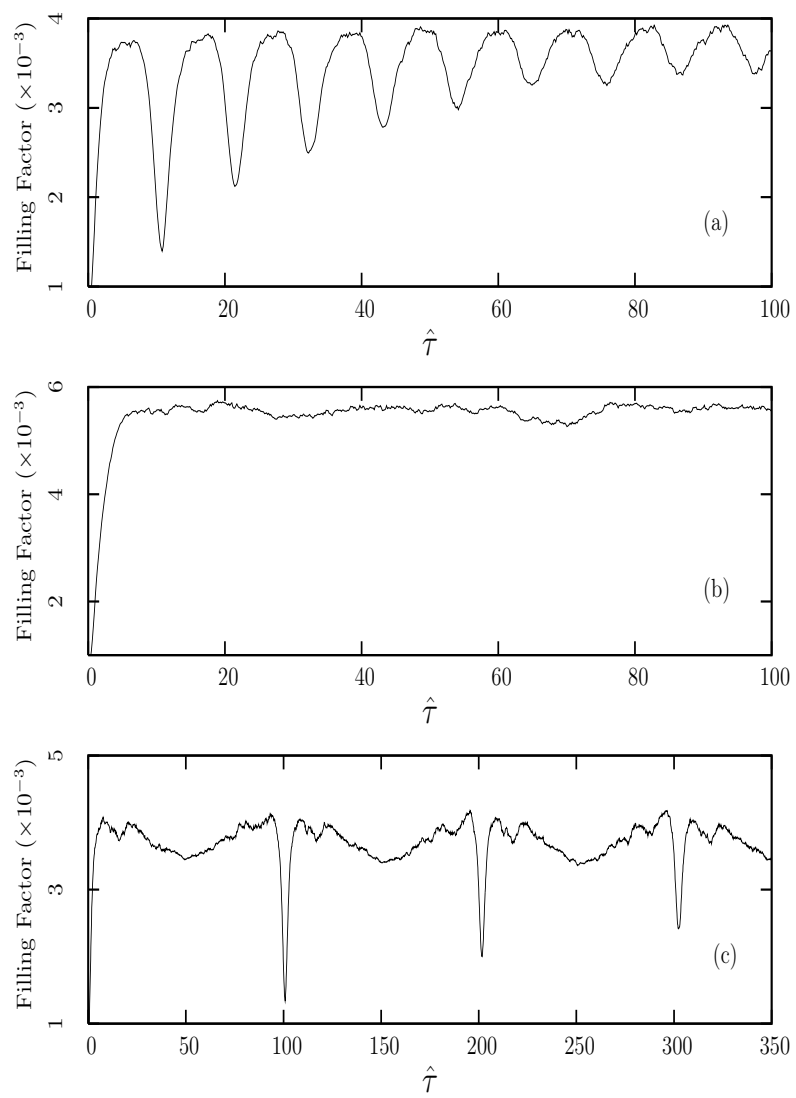
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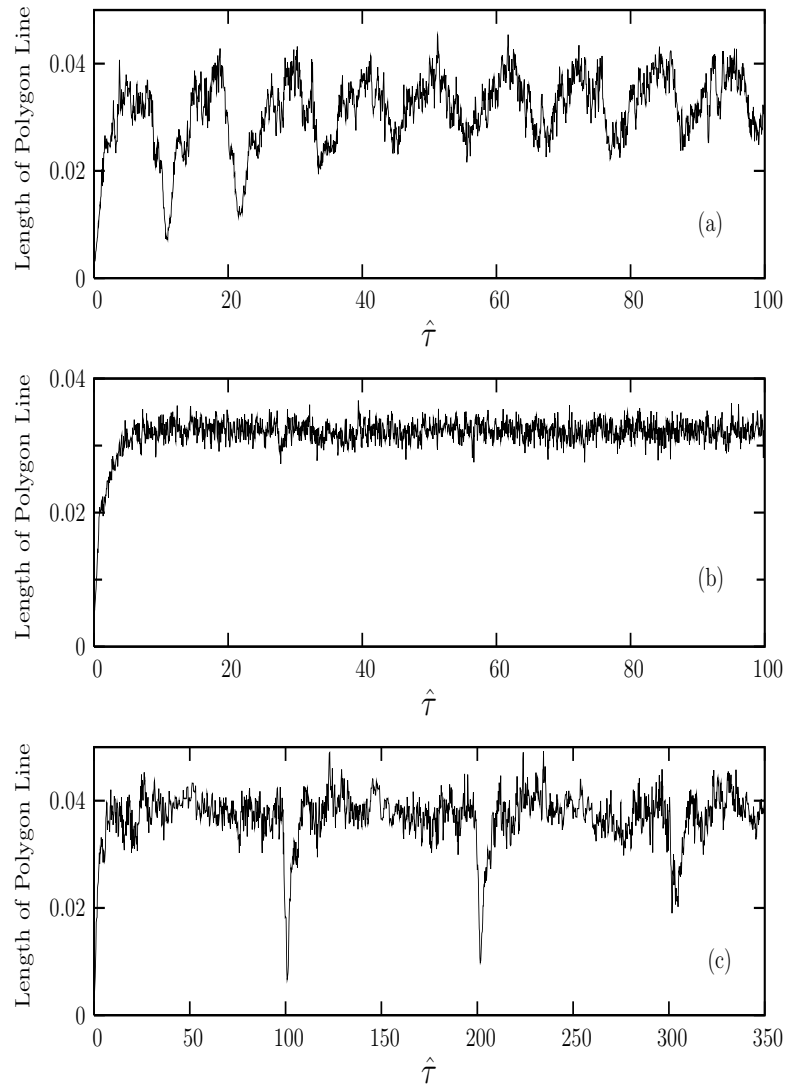
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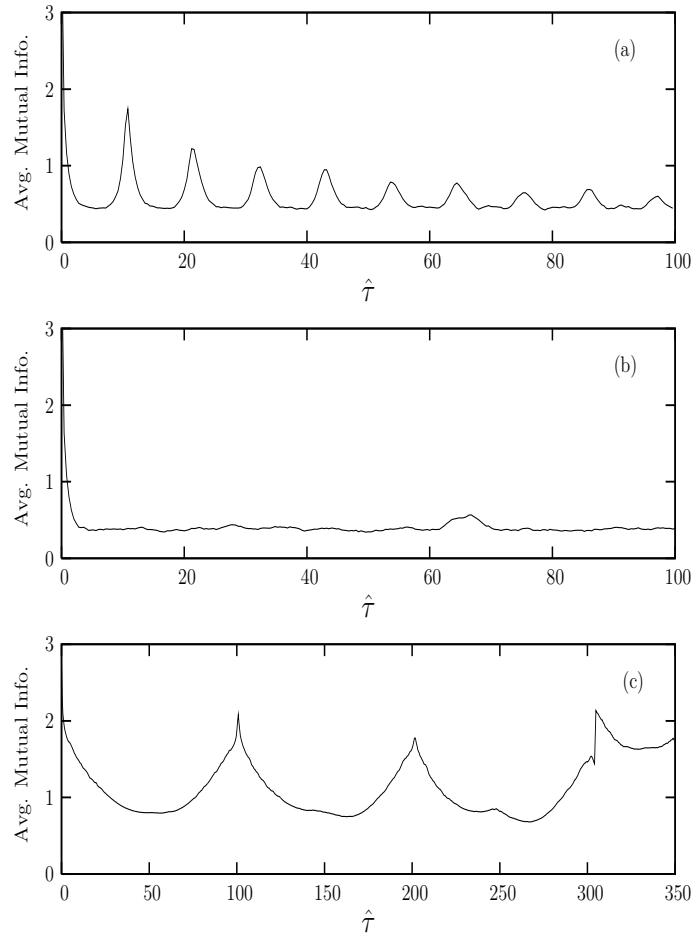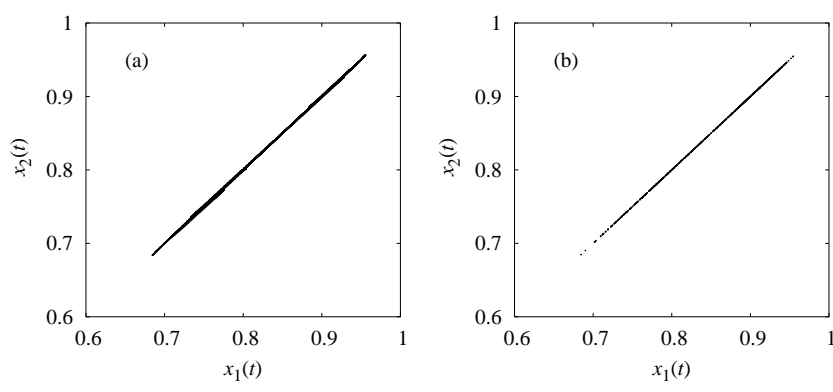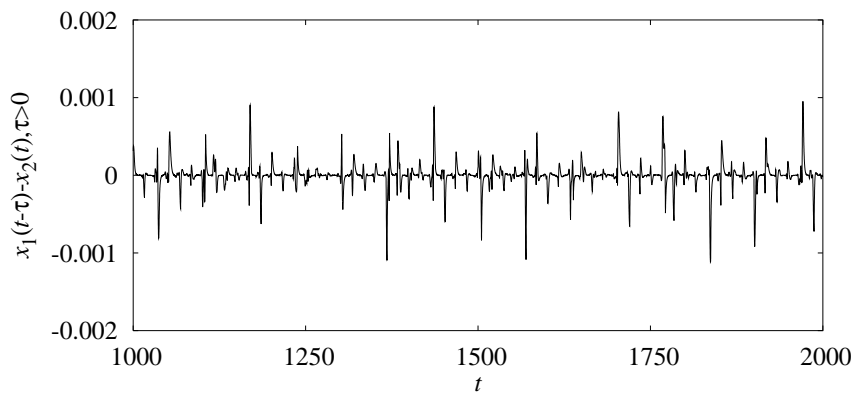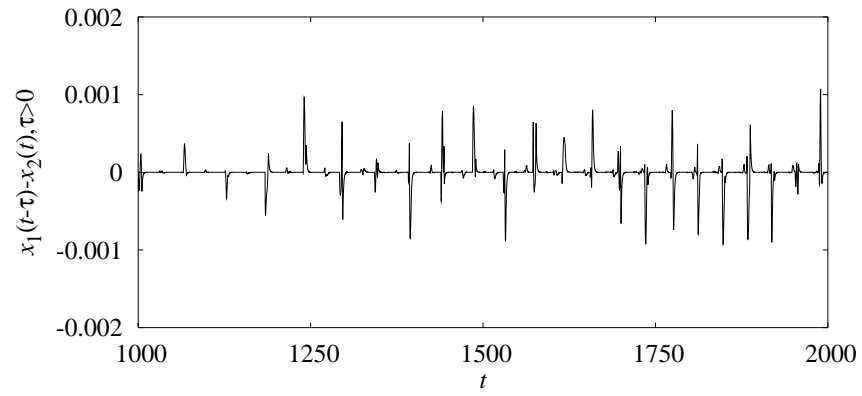
FIG. 3:

FIG. 4:

FIG. 5:
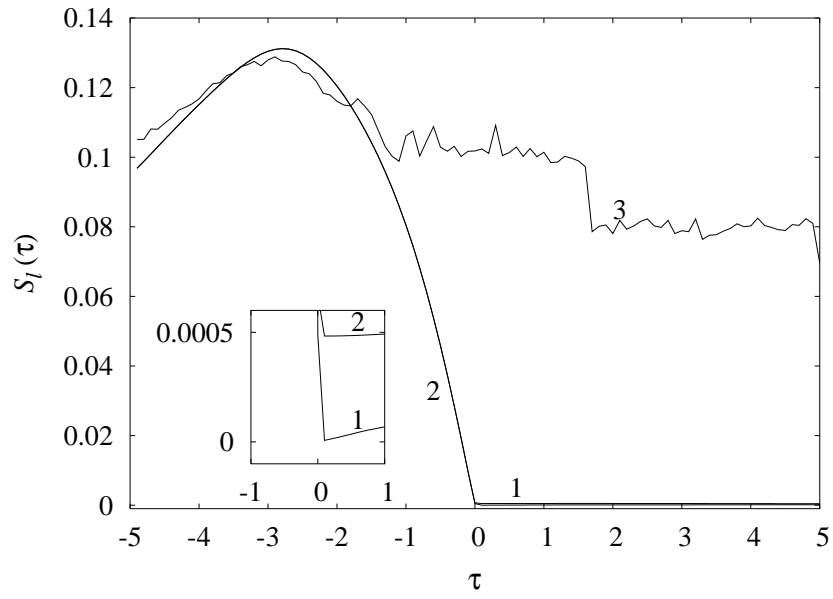

FIG. 6:

FIG. 7:

FIG. 8:

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FIG. 9:

FIG. 10:

FIG. 11:

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FIG. 12:

FIG. 13:

FIG. 14: