Estimation of System Parameters in Discrete Dynamical Systems from Time Series

P. Palaniyandi and M. Lakshmanan*
Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University, Tiruchirapalli - 620 024, India.
(Dated: February 8, 2008)

We propose a simple method to estimate the parameters involved in discrete dynamical systems from time series. The method is based on the concept of controlling chaos by constant feedback. The major advantages of the method are that it needs a minimal number of time series data and is applicable to dynamical systems of any dimension. The method also works extremely well even in the presence of noise in the time series. The method is specifically illustrated by means of logistic and Henon maps.

PACS numbers: 05.45.-a,47.52.+j

In recent years, studies on chaotic dynamical systems have become extremely relevant from a physical point of view due to their potential applications in secure communication [1-5], cryptography [6], and so on. Also, much attention has been given to time series analysis since many physical, chemical, and biological systems exhibit chaotic motion in nature. The main objectives of time series analysis are to identify the structure of the equations which govern the temporal evolution of the dynamical system, the number of independent variables involved, and parameters which control the dynamics of the system [7]. Several methods have been developed for modeling the dynamical systems by different authors [8, 9, 10, 11, 12, 13]. A number of methods have also been proposed for estimating the system parameters based on the concept of synchronization [14, 15, 16, 17, 18], Bayesian approach [19, 20] and least squares approach [21]. In this Letter, a very simple and practical method for estimating the system (control) parameters of discrete dynamical systems from the time series is developed using the concept of controlling chaos [22, 23, 24, 25]. This method is applicable to time series obtained from a discrete system of any dimensions and can be extended to continuous systems also without much difficulty. The method can also be used for the time series which contains considerable amount of noise as well as with scalar time series. Further this method can be used in the field of controlling chaos to find the exact values of controlling constants (κ_i).

Consider an arbitrary N-dimensional discrete chaotic dynamical system (the original map),

$$x_i^{(n+1)} = f_i(x_1^{(n)}, x_2^{(n)}, ..., x_N^{(n)}; \mathbf{p}), \tag{1}$$

where $i = 1, 2, 3, ...N$, $\mathbf{p}$ denotes the system parameters of dimension M to be determined and the discrete index n stands for denoting the iterations. We also assume that the function f is sufficiently smooth. Let us construct a modified discrete dynamical system (the modified map) as

$$y_i^{(n+1)} = f_i(y_1^{(n)}, y_2^{(n)}, ..., y_N^{(n)}; \mathbf{p}) + \kappa_i, \tag{2}$$

where κ_i's are constants. The crucial idea in the construction of the modified map is that the addition of constants κ_i in Eq. (1) will not affect the Jacobian of the original map, but it can change the original map without affecting the parameters ($\mathbf{p}$) into a modified map exhibiting a different stable fixed point solution (other than the unstable fixed point of the original map). Also it is always possible to construct such a modified map by finding a suitable set of contants (κ_i's) which makes the modified map to exhibit a stable *period one* fixed point even for the parameters for which the original map evolves chaotically.

Now let us start the evolution of the original and modified systems from a common set of initial states ($i.e., x_i^{(0)} = y_i^{(0)}$). After one time interval, the dynamics of the modified system can be represented as

$$y_i^{(1)} = f_i(y_1^{(0)}, y_2^{(0)}, ..., y_N^{(0)}; \mathbf{p}) + \kappa_i \tag{3}$$

and the dynamical variables of the original and modified systems can be related as

$$y_i^{(1)} = x_i^{(1)} + c_i^{(1)}, \tag{4}$$

where $c_i^{(1)} = \kappa_i$. After the second interval of discrete time, the dynamics of the modified system can expressed as

$$y_i^{(2)} = f_i(x_1^{(1)} + c_1^{(1)}, x_2^{(1)} + c_2^{(1)}, ..., x_N^{(1)} + c_N^{(1)}; \mathbf{p}) + \kappa_i \tag{5}$$

and, after Taylor expansion, the relation between the variable of the original and modified systems becomes

$$y_i^{(2)} = x_i^{(2)} + c_i^{(2)}, \tag{6a}$$

where

$$\begin{aligned} c_i^{(2)} &= \kappa_i + \sum_{j=1}^{N} c_j^{(1)} \frac{\partial f_i}{\partial x_j}\Big|_{\mathbf{x}^{(1)}} \\ &\quad + \frac{1}{2!} \sum_{j=1}^{N} \sum_{k=1}^{N} c_j^{(1)} c_k^{(1)} \frac{\partial^2 f_i}{\partial x_j \partial x_k}\Big|_{\mathbf{x}^{(1)}} + \cdots \end{aligned} \tag{6b}$$

and $\mathbf{x}$ is the vector of dimension N. Proceeding further, the entire time evolution of the modified system can be obtained from the original system by the relation

$$y_i^{(n)} = x_i^{(n)} + c_i^{(n)}, \qquad n = 0, 1, 2, ... \tag{7a}$$

where

$$c_i^{(n)} = \kappa_i + \sum_{j=1}^{N} c_j^{(n-1)} \frac{\partial f_i}{\partial x_j}\Big|_{\mathbf{x}^{(n-1)}} + \frac{1}{2!} \sum_{j=1}^{N}\sum_{k=1}^{N} c_j^{(n-1)} c_k^{(n-1)} \frac{\partial^2 f_i}{\partial x_j \partial x_k}\Big|_{\mathbf{x}^{(n-1)}} + \cdots \tag{7b}$$

and $c_i^{(0)} = 0$ by our initial assumption $y_i^{(0)} = x_i^{(0)}$.

Next, let $z_i^{(0)}, z_i^{(1)}, ..., z_i^{(m-1)}$ be the m data set points of the given chaotic time series obtained for the original map. Then the trajectory of the modified map (which is constructed by adding a set of constants κ_i with the original map) can be obtained from the above time series by the relation

$$y_i^{(n)} = z_i^{(n)} + c_i^{(n)}, \tag{8a}$$

where

$$c_i^{(n)} = \kappa_i + \sum_{j=1}^{N} c_j^{(n-1)} \frac{\partial f_i}{\partial x_j}\Big|_{\mathbf{z}^{(n-1)}} + \frac{1}{2!} \sum_{j=1}^{N}\sum_{k=1}^{N} c_j^{(n-1)} c_k^{(n-1)} \frac{\partial^2 f_i}{\partial x_j \partial x_k}\Big|_{\mathbf{z}^{(n-1)}} + \cdots \tag{8b}$$

and $\mathbf{z}$ is a vector of dimension N. If the original system is one dimensional, then

$$c^{(n)} = \kappa + c^{(n-1)} \frac{df}{dx}\Big|_{z^{(n-1)}} + \frac{1}{2}(c^{(n-1)})^2 \frac{d^2 f}{d^2 x}\Big|_{z^{(n-1)}} + \cdots \tag{9}$$

Let y_i^* be the *period one* fixed point of the modified map obtained by Eq. (8) for the given time series data. Then the n^{th} and $(n+1)^{th}$ iterations of the map can be expressed as

$$z_i^{(n)} + c_i^{(n)} = y_i^* \quad \text{and} \tag{10}$$

$$z_i^{(n+1)} + c_i^{(n+1)} = y_i^* \tag{11}$$

and by subtracting Eq. (10) from Eq. (11), we get

$$c_i^{(n+1)} - c_i^{(n)} = z_i^{(n)} - z_i^{(n+1)}. \tag{12a}$$

Similarly,

$$c_i^{(n+2)} - c_i^{(n)} = z_i^{(n)} - z_i^{(n+2)}, \tag{12b}$$

where $c_i^{(n+2)}$, $c_i^{(n+1)}$ and $c_i^{(n)}$ are functions of κ_i and $\mathbf{p}$. Thus, we have obtained $2N$ nonlinear simultaneous algebraic equations for $(M+N)$ unknowns (N κ_i's and M $\mathbf{p}$'s), and solving them we can obtain the values of the unknowns $\mathbf{p}$ and κ_i, provided the solution exists. After estimating the unknown parameters, one can also check the accuracy of the estimated parameters as follows. Iterate the Eq. (8a) with estimated values of the parameters in Eq. (8b) till a fixed point solution (y_i^*) is obtained. Then compare the fixed point (y_i^*) obtained by the above iteration using the time series of the original map with the fixed point calculated by Eq. (2) at the estimated parameters. The degree of closeness of these fixed points gives a measure of the accuracy in the estimated parameters.

TABLE I: Convergence of r and κ in the logistic map

Iterations	Using exact time series		Using noisy time series	
	r	κ	r	κ
0	10.00000000	-0.50000000	10.00000000	-0.50000000
1	8.99885997	-0.48521330	8.99885540	-0.48552646
2	7.99761227	-0.48347134	7.99760776	-0.48374876
3	6.99636547	-0.48126778	6.99636104	-0.48151029
4	5.99512049	-0.47835604	5.99511616	-0.47856279
5	4.99387963	-0.47426330	4.99387546	-0.47443116
6	3.99265086	-0.46786433	3.99264699	-0.46798613
7	3.69848466	-0.46349089	3.70025299	-0.46361426
8	3.67047439	-0.46283146	3.67212276	-0.46295672
9	3.67000000	-0.46282092	3.67164967	-0.46294621
10	3.67000000	-0.46282092	3.67164967	-0.46294621

As an example to our method in one dimension, consider the well known logistic map

$$x^{(n+1)} = rx^{(n)}(1 - x^{(n)}), \quad 0 \le x \le 1, \quad 0 \le r \le 4 \tag{13}$$

where r is the unknown system (control) parameter. Then the modified logistic map can be constructed as

$$y^{(n+1)} = ry^{(n)}(1 - y^{(n)}) + \kappa, \tag{14}$$

where κ is a constant to be determined which makes the modified logistic map to exhibit *period one* fixed point solution for the parameter where the original map exhibits chaotic solution.

Let $z^{(0)}, z^{(1)}, z^{(2)},...,z^{(m-1)}$ be the time series data obtained from the logistic map at some arbitrary time interval for a unknown system parameter (r). Assume $z^{(0)}$ be the common initial state for both the original and modified logistic maps (*i.e.* $x^{(0)} = y^{(0)} = z^{(0)}$) and so $c^{(0)} = 0$ by Eq. (8a). Then after substituting three data points $z^{(1)}, z^{(2)}$ and $z^{(3)}$ (one can take any three successive data) of the time series and the values of $c^{(1)}$, $c^{(2)}$ and $c^{(3)}$ calculated by making use of the Eq. (9) into the Eq. (12), we get

$$\kappa r(1 - \kappa - 2z^{(1)}) = z^{(1)} - z^{(2)}, \tag{15a}$$

$$\kappa r(1 - 2z^{(2)})[1 + r(1 - \kappa - 2z^{(1)})] - \kappa^2 r[1 + r(1 - \kappa - 2z^{(1)})]^2 = z^{(1)} - z^{(3)}. \tag{15b}$$

The values of κ and the unknown parameter (r) can be estimated by solving the above two nonlinear simultaneous algebraic equations with an initial guess of κ and r. For illustration purpose, we have used the numerically generated time series of the logistic map for the system parameter $r = 3.67$ and solved the eqns. (15) by globlally convergent Newton's method [26] with an initial guess -0.5 to κ and 10.0 to the parameter r. The convergences of the system parameter r and κ are shown in the Table I and it shows that the estimated value r is 3.67 which is the exact value of parameter at which the time series data of the logistic map is generated.

In order to test the robustness of the method, a noisy time series generated by considering that the system itself produces some error in the data in each iteration was also used in the above illustration. In our analysis a noise of strength 10^{-2} is added with the eqution of the system that generates the time series data. For a particular set of data the estimated value of r is found to be 3.67164967 after solving eqns. (15) for the same initial guess to κ and r as before. Also, the values of parameter (r) estimated from the noisy time series data at various intervals (we have considered 1000 data points) of time is found to be distributed around 3.67. We have also carried out similar analysis for the Moran-Ricker (exponential) map and verified that the system parameter can be identified correctly both in the absence and presence of noise.

TABLE II: Convergence of α, β, κ_1 and κ_2 in the Henon map

Itera-	Using exact time series				Using noisy time series			
tions	α	β	κ_1	κ_2	α	β	κ_1	κ_2
0	2.50000000	1.50000000	-0.10000000	0.00000000	2.50000000	1.50000000	-0.10000000	0.00000000
1	2.25643981	1.18700678	-0.11800073	-0.04888403	2.25664011	1.18682541	-0.11787076	-0.04876757
2	1.99627422	0.88915339	-0.14682621	-0.10147980	1.99662943	0.88880258	-0.14654089	-0.10125436
3	1.72472420	0.60769535	-0.19746953	-0.16839182	1.72522662	0.60715419	-0.19696823	-0.16812574
4	1.44784495	0.35719041	-0.30159310	-0.26710065	1.44850317	0.35640699	-0.30072823	-0.26703903
5	1.32207233	0.28601093	-0.44576804	-0.33566022	1.31986881	0.28336810	-0.44645811	-0.33730112
6	1.41037943	0.30302200	-0.45513595	-0.33541800	1.40939982	0.30087213	-0.45644206	-0.33692901
7	1.40001650	0.30000320	-0.45914419	-0.33391288	1.39875604	0.29768355	-0.46069850	-0.33545169
8	1.40000000	0.30000000	-0.45918942	-0.33389223	1.39873784	0.29767999	-0.46074960	-0.33542922
9	1.40000000	0.30000000	-0.45918942	-0.33389223	1.39873784	0.29767999	-0.46074960	-0.33542922

For the illustration of our method in two dimensional discrete system, we consider the Henon map,

$$x_1^{(n+1)} = 1 + x_2^{(n)} - \alpha (x_1^{(n)})^2, \qquad (16a)$$
$$x_2^{(n+1)} = \beta x_1^{(n)}, \qquad (16b)$$

where α and β are control parameters to be determined, and the modified Henon map can be constructed as

$$y_1^{(n+1)} = 1 + y_2^{(n)} - \alpha (y_1^{(n)})^2 + \kappa_1, \qquad (17a)$$
$$y_2^{(n+1)} = \beta y_1^{(n)} + \kappa_2, \qquad (17b)$$

where κ_1 and κ_2 are constants which force the modified Henon map to exhibit *period one* fixed point solution for a set of parameters where the original Henon map shows chaotic behaviour.

Let $(z_1^{(0)}, z_2^{(0)}), (z_1^{(1)}, z_2^{(1)}),...,(z_1^{(m-1)}, z_2^{(m-1)})$ be the data sets of time series obtained from the Henon map at some arbitrary interval of time for a set of unknown system parameters (α and β). The starting assumption of common initial state $x_1^{(0)} = y_1^{(0)} = z_1^{(0)}$ and $x_2^{(0)} = y_2^{(0)} = z_2^{(0)}$ makes $c_1^{(0)} = c_2^{(0)} = 0$ by Eq. (8a). In the case of Henon map, the substitution of $c_1^{(1)}$, $c_1^{(2)}$, $c_1^{(3)}$, $c_2^{(1)}$, $c_2^{(2)}$ and $c_2^{(3)}$ calculated using Eq. (8b) into Eq. (12) leads to $four$ nonlinear simultaneous algebraic equations as

$$\begin{aligned} &-2\alpha(\kappa_1 - \alpha\kappa_1^2 + \kappa_2 - 2\alpha\kappa_1 z_1^{(1)}) z_1^{(2)} \\ &-\alpha(\kappa_1 - \alpha\kappa_1^2 + \kappa_2 - 2\alpha\kappa_1 z_1^{(1)})^2 \\ &+\beta\kappa_1 + \kappa_2 = z_1^{(1)} - z_1^{(3)}, \qquad (18a) \end{aligned}$$
$$\beta(\kappa_1 - \alpha\kappa_1^2 + \kappa_2 - 2\alpha\kappa_1 z_1^{(1)}) = z_2^{(1)} - z_2^{(3)}, \qquad (18b)$$
$$\kappa_2 - \alpha\kappa_1^2 - 2\alpha\kappa_1 z_1^{(1)} = z_1^{(1)} - z_1^{(2)}, \qquad (18c)$$
$$\beta\kappa_1 = z_2^{(1)} - z_2^{(2)}. \qquad (18d)$$

In this illustration, we have used the numerical time series data of the Henon map generated for the system parameters $\alpha = 1.4$ and $\beta = 0.3$ and solved the above coupled Eq. (18) by the globlally convergent Newton's method [26] with an initial guess $\alpha = 2.5$ $\beta = 1.5$, $\kappa_1 = -0.1$ and $\kappa_2 = 0$. The convergence of α, β, κ_1 and κ_2 are shown in Table II and it also shows that the estimated values of α and β are 1.4 and 0.3 respectively. And these estimated values are in exact agreement with the values of the parameters for which the time series of the Henon map is generated. As in our previous example, we have solved the Eq. (18) using the time series data containing a random noise of strength 10^{-2} for the same initial guess. In this case, the estimated values are found to be $\alpha = 1.39873784$ and $\beta = 0.29767999$ and the values of parameters α and β estimated at various interval of time using the noisy data is found to be distributed around 1.4 and 0.3, respectively. One can also verify that the above system parameters can be obtained from a scalar time series (either z_1's or z_2's) by contructing four algebraic equations suitably from Eq. (12) and making use of the system equations. For example, parameters (α and β) can be estimated from the scalar time series of z_1 using the four algebraic equations which contain z_1 alone in the right hand side, constructed by making use of $c_1^{(1)}$, $c_1^{(2)}$, $c_1^{(3)}$, $c_1^{(4)}$ and $c_1^{(5)}$ in Eq. (12).

At this point, one may raise the question, why not invert directly the map (1) itself using the time series data so as to find the system parameters. While this is certainly possible in the case of exact time series, the extreme sensitiveness of chaotic systems to initial conditions make it an unreliable procedure in the presence of suitable noise. For example, in the case of logistic map the estimated value of r is found to be 3.7 while the original value is 3.67 when an 1% white noise in the range 0 to 1 is introduced in the time series. On the other hand in our method described above, no such difficulty arises.

Next, we wish to point out that it is possible to extend the analysis to identify the system itself in principle, say an N^{th} degree polynomial for the right hand side of Eq. (1). By solving sufficient number of Eqs. (12) one can then identify the form of the map itself. From another point of view, the procedure outlined here also gives a method to obtain the values of the controlling constants (κ_i) for a chaotic system to a desired periodic orbit. Finally, we have also extended the same precedure to continuous dynamical systems for estimating the system parameters by finding a set of differential equations which determine the connection between the original and modified systems. The details will be presented elsewhere.

To conclude, the main advantage of our method is that a very minimal number of time series data is sufficient for the accurate determination of the system parameters. We can check the accuracy of the estimated parameters by comparing the fixed point obtained by Eq. (8) using the time series at estimated parameters with the fixed point of the modified dynamical system at the same parameters. Thus, we have developed a very simple as well as useful method for estimating the unknown system parameters of the discrete dynamical systems of any dimensions and illustrated it by means of logistic and Henon maps.

This work has been supported by the National Board of Higher Mathematics, Department of Atomic Energy, Government of India and the Department of Science and Technology, Government of India through research projects. The authors thank Dr. K. P. N. Murthy, Dr. S. Rajasekar, Dr. K. Murali, Dr. P. Muruganandam and Dr. P. M. Gade for many valuable suggestions.

* Electronic address: lakshman@cnld.bdu.ac.in

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