Non-standard conserved Hamiltonian structures in dissipative/damped systems : Nonlinear generalizations of damped harmonic oscillator

R Gladwin Pradeep, V K Chandrasekar, M Senthilvelan and M Lakshmanan

Centre for Nonlinear Dynamics, School of Physics,

Bharathidasan University, Tiruchirappalli - 620 024, India

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Abstract

In this paper we point out the existence of a remarkable nonlocal transformation between the damped harmonic oscillator and a modified Emden type nonlinear oscillator equation with linear forcing, $\ddot{x} + \alpha x \dot{x} + \beta x^3 + \gamma x = 0$, which preserves the form of the time independent integral, conservative Hamiltonian and the equation of motion. Generalizing this transformation we prove the existence of non-standard conservative Hamiltonian structure for a general class of damped nonlinear oscillators including Liénard type systems. Further, using the above Hamiltonian structure for a specific example namely the generalized modified Emden equation $\ddot{x} + \alpha x^q \dot{x} + \beta x^{2q+1} = 0$, where α, β and q are arbitrary parameters, the general solution is obtained through appropriate canonical transformations. We also present the conservative Hamiltonian structure of the damped Mathews-Lakshmanan oscillator equation. The associated Lagrangian description for all the above systems is also briefly discussed.

I. INTRODUCTION

Dissipative systems are dynamical systems whose phase space volume decreases/varies when the dynamical system evolves in time. This should be contrasted with the conservative systems whose phase space volume remains a constant. A dynamical system, which is represented by a second order ordinary differential equation (ODE) is said to be a dissipative one if the flow function of the equivalent system of first order ODEs turns out to be a negative constant[1,2]. In the case when the flow function becomes zero the underlying system is by definition a conservative one[1,2]. A typical example is the damped harmonic oscillator (DHO) equation, $\ddot{x} + \alpha\dot{x} + \lambda x = 0$, where overdot denotes differentiation with respect to time and α and λ are arbitrary parameters. Working out the flow function Λ for this system one finds that $\Lambda = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = -\alpha$, where f_1 and f_2 are defined by $\dot{x} = y = f_1(x,y); \dot{y} = -\alpha y - \lambda x = f_2(x,y)$, which confirms that the system under consideration is a dissipative one. On the other hand, for a conservative system, we have a set of first order equations, $\dot{x} = \frac{\partial H}{\partial p} = f_1(x,p); \dot{p} = -\frac{\partial H}{\partial x} = f_2(x,p)$, where H is the Hamiltonian function, and the flow function is always equal to zero which can be straightforwardly confirmed by substituting the Hamilton equations of motion. This observation reveals the fact that if one is able to find a time independent integral for a dissipative system then this time independent integral can be correlated to a Hamiltonian function, perhaps in a non-standard form in terms of a new set of canonical variables, for the second order equation under consideration. Such a conserved Hamiltonian description also leads one to further investigations, including quantization[3].

Since the pioneering work of Bateman[4] the quest for a Hamiltonian description for the damped harmonic oscillator equation was pursued by several authors and only very recently conserved Hamiltonian description for this system[5] for all the three parametric regimes have been constructed by three of us . A possible quantization has also been suggested.

In a parallel investigation we have also found that another dissipative type system, namely the modified Emden type equation (MEE), also admits time independent integral and a conservative Hamiltonian description[6]. A question which now naturally arises is whether the aforementioned systems are isolated examples or there exists a wider class of dissipative systems that admit conservative Hamiltonian description. If so, how to isolate and classify them? A more detailed investigation in this direction reveals the fascinating fact that one can map the damped harmonic oscillator equation onto the modified Emden type equation

through a nonlocal transformation. Interestingly the time independent integrals and the Hamiltonian description for the nonlinear system can also be derived from the linear one by simply substituting the same nonlocal transformation at the appropriate places.

In this paper, after presenting the above novel results, we report a rather general transformation which can map the damped harmonic oscillator to a larger class of damped nonlinear systems that admit conservative Hamiltonian description. We present the general nonlinear oscillator equation and explicit forms of the Hamiltonian. The Hamiltonian forms obtained by this procedure are of non-standard (not equal to the standard potential plus kinetic energy) forms. Recently considerable interest has been shown in the classification nonlinear dissipative equations which admit non-standard conservative Hamiltonian structure [7,8,9]. We then consider a specific nonlinear system, $\ddot{x} + \alpha x^q \dot{x} + \beta x^{2q+1} = 0$, where q, α and β are arbitrary parameters, which is a natural generalization of the DHO equation and the MEE and discuss the dynamics/integrability of this equation in some detail. Explicit solution for this equation was constructed for the specific parametric choices[10,11] $\beta = \frac{\alpha^2}{(q+2)^2}$ and $\beta = \frac{\alpha^2}{4(q+1)}$. For the former choice of parameter we have recently shown that this equation can be linearized to the free particle equation through a generalized linearizing transformation[12]. Also the $q = 1$ case (MEE) has been completely integrated by using the underlying Hamiltonian structure[6] for all values of the parameter α and β. However, neither the Hamiltonian structure nor explicit solutions for arbitrary choices of q, α and β for this equation have been reported and we present the results here. Finally we also present the Hamiltonian structure of the damped Mathews-Lakshmanan oscillator, whose undamped version exhibits amplitude-dependent harmonic type oscillatory solutions[13,14].

The plan of the paper is as follows. In section II we discuss the time independent integrals and Hamiltonians of the two dissipative systems, namely the DHO and MEE with linear forcing and deduce the nonlocal transformation which interrelates each other. In section III we introduce a more general nonlocal transformation and substitute it into the DHO equation and construct the nonlinear generalizations of the DHO. We also derive the associated time independent integrals and Hamiltonian for this general nonlinear oscillator equation. In section IV, we consider as a specific example, namely the generalized MEE, and discuss the Hamiltonian structure and obtain its general solution by integrating the canonical equations of motion after suitable canonical transformations. In section V, we present as a second example the time independent integral of motion and conservative Hamiltonian structure of

the damped Mathews-Lakshmanan oscillator. In section VI, we briefly discuss the associated Lagrangian description for all the above systems. Finally, we present our conclusion in section VII.

II. HAMILTONIAN STRUCTURE OF DHO AND MEE

In this section we briefly recall the Hamiltonian dynamics associated with the two dissipative systems, namely the damped harmonic oscillator and the modified Emden type equation, and then show how they are interrelated. Note that the former one is a linear system and latter one is a nonlinear system. One may essentially consider the transformation to be introduced as a linearizing transformation of the latter.

A. Damped harmonic oscillator

To start with let us consider the damped harmonic oscillator (using a different notation for convenience of comparison)

$$y'' + \alpha y' + \lambda y = 0, \qquad \left(' = \frac{d}{d\tau}\right) \tag{1}$$

where α and λ are arbitrary parameters. Recently, we have identified the following time independent integral of motion[5] for the system (1) :

$$I = \begin{cases} \dfrac{(r-1)}{(r-2)}\left(y' + \dfrac{\alpha}{r}y\right)(y' + \dfrac{(r-1)}{r}\alpha y)^{(1-r)}, & \alpha^2 > 4\lambda \\[4mm] \dfrac{y'}{(y' + \frac{1}{2}\alpha y)} - \log[y' + \dfrac{1}{2}\alpha y], & \alpha^2 = 4\lambda \\[4mm] \dfrac{1}{2}\log[y'^2 + \alpha yy' + \lambda y^2] + \dfrac{\alpha}{2\omega}\tan^{-1}\left[\dfrac{\alpha y' + 2\lambda y}{2\omega y'}\right], & \alpha^2 < 4\lambda, \end{cases} \tag{2}$$

where $r = \frac{\alpha}{2\lambda}(\alpha \pm \sqrt{\alpha^2 - 4\lambda})$ and $\omega = \frac{1}{2}\sqrt{4\lambda - \alpha^2}$, for the overdamped, critically damped and the underdamped oscillations, respectively. From these time independent integrals we

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have also derived the following time independent Hamiltonian [5],

$$H = \begin{cases} \dfrac{(r-1)}{(r-2)}(p)^{\frac{(r-2)}{(r-1)}} - \dfrac{(r-1)}{r}\alpha py, & p > 0, \ \alpha^2 > 4\lambda \\[2em] \log(p) - \dfrac{1}{2}\alpha py, & p > 0, \ \alpha^2 = 4\lambda \\[2em] \dfrac{1}{2}\log[y^2 \sec^2(\omega py)] - \dfrac{\alpha}{2}py, & \alpha^2 < 4\lambda \end{cases} \tag{3}$$

where the canonical conjugate momentum p is defined by

$$p = \begin{cases} \left(y' + \dfrac{(r-1)}{r}\alpha y\right)^{(1-r)}, & \alpha^2 \geq 4\lambda \\[2em] \dfrac{1}{\omega y}\tan^{-1}\left[\dfrac{2y' + \alpha y}{2\omega y}\right]. & \alpha^2 < 4\lambda \end{cases} \tag{4}$$

From the Hamiltonian (3)-(4) one can straightforwardly write down the canonical equations of motion for all the three regimes and obtain the known exact solutions straightforwardly. Note that no multivaluedness arises in the system due to the constraints on the momentum.

Case 1,2 $\alpha^2 \geq 4\lambda$

Substituting the first of the expressions in (3) into the canonical equations $y' = \frac{\partial H}{\partial p}$ $p' = -\frac{\partial H}{\partial y}$, one gets the following equivalent system of first order ordinary differential equations for the damped harmonic oscillator in the overdamped and critically damped ($\alpha^2 > 4\lambda$ and $\alpha^2 = 4\lambda$) parametric regimes,

$$y' = (p)^{\frac{1}{(1-r)}} - \frac{(r-1)}{r}\alpha y, \quad p' = \frac{(r-1)}{r}\alpha p. \tag{5}$$

One can easily check that the second order equivalence of this equation coincides exactly with (1) and that the standard form of the solutions for the overdamped and the critically damped cases follow naturally by integrating the above system of first order ODEs. We mention here that the divergence of the flow function of the underlying canonical equations of motion is zero, $\Lambda = \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial p} = 0$, which in turn confirms that the damped harmonic oscillator has a non-standard conservative Hamiltonian description.

Case 3 $\alpha^2 < 4\lambda$

Finally, for the underdamped case, one can write down the Hamilton equations of motion in the form

$$y' = \frac{1}{2}y(2\omega \tan(\omega py) - \alpha), \quad p' = \frac{1}{2}p(\alpha - 2\omega \tan(\omega py)) - \frac{1}{y}. \tag{6}$$

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Again rewriting this equation as a single second order equation in the variable y one ends up in the form (1). The divergence of the flow function in this case is also zero.

The existence of time independent Hamiltonians (3) in a non-standard form and the basic definition of the divergence of flow function lead us to the conclusion that the damped harmonic oscillator has a non-standard conservative Hamiltonian description as well, in spite of its known dissipative nature.

B. Modified Emden type equation

In a parallel investigation we have also found that the MEE, $\ddot{x} + \alpha x \dot{x} + \beta x^3 = 0$, admits a time independent conservative Hamiltonian description[6]. The MEE, which is also known as the Painlevé - Ince equation, is an extensively studied and physically significant equation in the contemporary nonlinear dynamics literature[15−19]. On the other hand the MEE with linear external forcing,

$$\ddot{x} + \alpha x \dot{x} + \beta x^3 + \gamma x = 0, \tag{7}$$

possesses certain unusual nonlinear dynamical properties for the parametric choice $\beta = \frac{\alpha^2}{9}$. We have shown that this system admits nonisolated periodic orbits of conservative Hamiltonian type[9]. These periodic orbits exhibit the unexpected property that the frequency of oscillations is completely independent of amplitude and continues to remain as that of the linear harmonic oscillator[9].

The more general system (7) with α, β and γ being arbitrary parameters also admits the following time independent integrals of motion[21],

$$I = \begin{cases} \dfrac{(r-1)}{(r-2)}\left(\dot{x} + \dfrac{\alpha}{2r}x^2 + \dfrac{r\gamma}{\alpha(r-1)})\right)\left(\dot{x} + \dfrac{\alpha(r-1)}{2r}x^2 + \dfrac{r\gamma}{\alpha}\right)^{(1-r)}, & \alpha^2 > 8\beta \\[2.5em] \dfrac{\dot{x}}{(\dot{x} + \frac{1}{4}\alpha(x^2 + \frac{4\gamma}{\alpha^2}))} - \log[\dot{x} + \dfrac{1}{4}\alpha(x^2 + \dfrac{4\gamma}{\alpha^2})], & \alpha^2 = 8\beta \\[2.5em] \dfrac{1}{2}\log[\dot{x}^2 + \dfrac{\alpha}{2}\dot{x}(x^2 + \dfrac{\gamma}{\beta}) + \dfrac{\beta}{2}(x^2 + \dfrac{\gamma}{\beta})^2] \\ \quad + \frac{\alpha}{2\omega}\tan^{-1}\left[\dfrac{\alpha\dot{x} + 2\beta(x^2 + \frac{\gamma}{\beta})}{2\omega\dot{x}}\right], & \alpha^2 < 8\beta, \end{cases} \tag{8}$$

<div align="center">6</div>

where $r = \frac{\alpha}{4\beta}(\alpha \pm \sqrt{\alpha^2 - 8\beta})$ and $\omega = \frac{1}{2}\sqrt{8\beta - \alpha^2}$. From the time independent integrals we have identified the following Hamiltonian for the system (7),

$$H = \begin{cases} \dfrac{(r-1)}{(r-2)}(p)^{\frac{(r-2)}{(r-1)}} - \dfrac{(r-1)}{2r}\alpha px^2 - \dfrac{pr\gamma}{\alpha}, & \alpha^2 > 8\beta \\[3ex] \log(p) - \dfrac{1}{2}\alpha p(\dfrac{x^2}{2} + \dfrac{4\gamma}{\alpha^2}), & \alpha^2 = 8\beta \\[3ex] \dfrac{1}{2}\log[(x^2 + \dfrac{\gamma}{\beta})^2 \sec^2(\dfrac{\omega p}{2}(x^2 + \dfrac{\gamma}{\beta}))] - \dfrac{\alpha}{4}p(x^2 + \dfrac{\gamma}{\beta}), & \alpha^2 < 8\beta \end{cases} \tag{9}$$

where the canonical conjugate momentum p is defined by

$$p = \begin{cases} \left(\dot{x} + \dfrac{(r-1)}{2r}\alpha x^2 + \dfrac{r\gamma}{\alpha}\right)^{(1-r)}, & \alpha^2 \geq 8\beta \\[3ex] \dfrac{2}{\omega(x^2 + \frac{\gamma}{\beta})}\tan^{-1}\left[\dfrac{4\dot{x} + \alpha(x^2 + \frac{\gamma}{\beta})}{2\omega(x^2 + \frac{\gamma}{\beta})}\right]. & \alpha^2 < 8\beta \end{cases} \tag{10}$$

The Hamilton equations of motion follows straightaway from (9) as

$$Cases\ 1\ \&\ 2 \quad \dot{x} = p^{\frac{1}{1-r}} - \frac{\alpha(r-1)x^2}{2r} - \frac{r\gamma}{\alpha}, \quad \dot{p} = \frac{\alpha(r-1)px}{r} \tag{11}$$

$$Case\ 3 \qquad \dot{x} = \frac{1}{4\beta}(\beta x^2 + \gamma)\left(2\omega\tan\left[\frac{1}{2}p\omega(x^2 + \frac{\gamma}{\beta})\right] - \alpha\right),$$

$$\dot{p} = \frac{x(p\alpha(\beta x^2 + \gamma) - 4\beta - 2p\omega(\beta x^2 + \gamma)\tan[\frac{1}{2}p\omega(x^2 + \frac{\gamma}{\beta})])}{2(\beta x^2 + \gamma)}. \tag{12}$$

Note that in all the above three cases, the divergence of the flow function Λ is zero, corresponding to a a conservative Hamiltonian description. However, unlike the damped harmonic oscillator one cannot integrate the canonical equations (11) and (12) and obtain the solutions straightforwardly. To overcome this difficulty one should introduce suitable canonical transformations and change the Hamilton equations (11) and (12) into simpler forms so that they can be integrated in a straightforward manner as demonstrated in the case of MEE (with $\gamma = 0$) recently by us[6].

C. Transformation connecting DHO and MEE

By comparing the structure of the integral of motion, Hamiltonian function and canonical equations of the damped harmonic oscillator with that of the equation (7), namely equation

(2) with (8), (3)-(4) with (9)-(10) and (5)-(6) with (11)-(12), one can identify that the two systems are transformed into each other through the nonlocal transformation

$$y = \frac{x^2}{2} + \frac{\gamma}{\lambda}, \quad d\tau = xdt, \tag{13}$$

with the identification $\lambda = 2\beta$. One can also check directly that the above transformation also maps the damped harmonic oscillator equation (1) onto the MEE with forcing, equation (7), and vice versa. Consequently one can treat the transformation as a linearizing transformation (albeit nonlocal) of the nonlinear equation (7).

Note that the transformation (13) is not the only possible linearizing transformation at least for specific parametric choices. We mention here that one can also transform the nonlinear system (7) with $\beta = \frac{\alpha^2}{9}$ into a linear harmonic oscillator equation ($\ddot{U} + \gamma U = 0$) by introducing another nonlocal transformation of the form [9,22]

$$U = xe^{\int_0^t \alpha x(\hat{t})d\hat{t}}, \quad \tau = t. \tag{14}$$

The nonlocal transformation (14) is different from (13) in the following respect. In (13) the nonlocality is introduced in the independent variable whereas in (14) the nonlocality is introduced in the dependent variable. We may also add that the nonlocal transformation (13) has some similarity with the well known Kustaanheimo-Stiefel transformation used in atomic physics[23]. Even though both the nonlocal transformations (13) and (14) map the nonlinear equation into a linear one and vice versa, the nonlocal transformation of the type (13) is much useful in identifying Hamiltonian structures associated with the nonlinear system whereas the nonlocal transformation of the type (14) is more useful in constructing general solution for the transformed nonlinear system. For more details about the nonlocal transformation of the type (14) we may refer to Ref. 22. In the following we confine our attention to the nonlocal transformation of the form (13) and its generalization.

III. A GENERAL CLASS OF NONLINEAR DAMPED OSCILLATOR : HAMILTONIAN DESCRIPTION

We find that the transformation (13) is a specific case of a rather general nonlocal transformation of the form

$$y = \int f(x)dx, \quad d\tau = \frac{f(x)}{g(x)}dt. \tag{15}$$

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For example, restricting $f(x) = x$ and $g(x) = 1$ in (15) one gets exactly (13).

The nonlocal transformation (15) modifies the damped harmonic oscillator equation (1) to the general class of nonlinear oscillators of the form,

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + \alpha\frac{f(x)}{g(x)}\dot{x} + \lambda\frac{f(x)}{g(x)^2}\int f(x)dx = 0, \tag{16}$$

where $f(x)$, $g(x)$ are arbitrary function of x.

Applying the above nonlocal transformation (15) to the damped harmonic oscillator equation (1) we obtain the following time-independent integral of motion for the nonlinear system (16), that is,

Case 1. $\alpha^2 > 4\lambda$

$$I = \frac{(r-1)}{(r-2)}\left(g(x)\dot{x} + \frac{\alpha}{r}\int f(x)dx\right)\left(g(x)\dot{x} + \frac{(r-1)}{r}\alpha\int f(x)dx\right)^{(1-r)} \tag{17}$$

Case 2. $\alpha^2 = 4\lambda$

$$I = \frac{g(x)\dot{x}}{g(x)\dot{x} + \frac{1}{2}\alpha\int f(x)dx} - \log[g(x)\dot{x} + \frac{1}{2}\alpha\int f(x)dx] \tag{18}$$

Case 3. $\alpha^2 < 4\lambda$

$$I = \frac{1}{2}\log[g(x)^2\dot{x}^2 + \alpha g(x)\dot{x}\int f(x)dx + \lambda(\int f(x)dx)^2]$$
$$+ \frac{\alpha}{2\omega}\tan^{-1}\left[\frac{\alpha g(x)\dot{x} + 2\lambda\int f(x)dx}{2\omega g(x)\dot{x}}\right], \tag{19}$$

where $\omega = \frac{1}{2}\sqrt{4\lambda - \alpha^2}$ and $r = \frac{\alpha}{2\lambda}(\alpha \pm \sqrt{\alpha^2 - 4\lambda})$.

The Hamiltonian for the equation (16) can also be constructed by simply substituting the transformation (15) into (3) so that the latter reads

$$H = \begin{cases} \frac{(r-1)}{(r-2)}\left(\frac{p}{g(x)}\right)^{\frac{r-2}{r-1}} - \frac{\alpha(r-1)}{r}\frac{p}{g(x)}\int f(x)dx, & \alpha^2 > 4\lambda \\\\ \frac{\alpha}{2g(x)}p\int f(x)dx + \log\left[\frac{g(x)}{p}\right], & \alpha^2 = 4\lambda \\\\ \frac{1}{2}\log\left[(\int f(x)dx)^2\sec^2\left[\frac{\omega p}{g(x)}\int f(x)dx\right]\right] - \frac{\alpha p}{2g(x)}\int f(x)dx, & \alpha^2 < 4\lambda, \end{cases} \tag{20}$$

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where p is the canonical conjugate momentum defined by

$$
p = \begin{cases} \dfrac{g(x)}{\left[g(x)\dot{x} + \frac{(r-1)}{r}\alpha \int f(x)dx\right]^{r-1}}, & \alpha^2 \geq 4\lambda \\[4mm] \dfrac{g(x)}{\omega \int f(x)dx}\tan^{-1}\left[\dfrac{2g(x)\dot{x} + \alpha \int f(x)dx}{2\omega \int f(x)dx}\right], & \alpha^2 < 4\lambda. \end{cases}
\tag{21}
$$

One can easily check that the canonical equations of motion take the form

$$
\alpha^2 \geq 4\lambda \; : \; \dot{x} = \frac{1}{g(x)}\left[\left(\frac{p}{g(x)}\right)^{\frac{1}{1-r}} - \frac{(r-1)}{r}\alpha\int f(x)dx\right],
$$

$$
\dot{p} = \frac{(r-1)}{r}\alpha p\frac{f(x)}{g(x)} + \frac{pg'(x)}{g(x)^2}\left[\left(\frac{p}{g(x)}\right)^{\frac{1}{1-r}} - \frac{\alpha(r-1)}{r}\int f(x)dx\right].
\tag{22}
$$

$$
\alpha^2 < 4\lambda \; : \; \dot{x} = \frac{\int f(x)dx}{2g(x)}\left[2\omega\tan\left[\frac{p\omega}{g(x)}\int f(x)dx\right] - \alpha\right],
$$

$$
\dot{p} = \frac{1}{2g(x)^2 \int f(x)dx}\left[f(x)g(x)\left(2g(x) - \left(\alpha - 2\omega\tan\left[\frac{p\omega \int f(x)dx}{g(x)}\right]\right)\right)\right.
$$

$$
\left. \times p\int f(x)dx\right) + p\left(\int f(x)dx\right)^2\left(\alpha - 2\omega\tan\left[\frac{p\omega \int f(x)dx}{g(x)}\right]\right)g'(x)\right].
\tag{23}
$$

One can straightforwardly check that the second order equivalence of the equations (22) and (23) coincides exactly with (16) in the appropriate parametric regimes. Further, we find that the flow function for the canonical equations of motion is zero in all the three parametric regions, equations (22)-(23), which in turn confirms that (16) or (22)-(23) has a Hamiltonian description. Further, the integrability of (16) is automatically ensured by the existence of the time independent Hamiltonian (20) in the Liouville sense.

We also note here that equation (16) can be transformed into an Abel equation of the second kind[24],

$$
ww' + \frac{g'(x)}{g(x)}w^2 + \alpha\frac{f(x)}{g(x)}w + \lambda\frac{f(x)}{g(x)^2}\int f(x)dx = 0,
\tag{24}
$$

through the transformation $w(x) = \dot{x}$. Then the time independent first integrals of (24) with $g(x) =$ constant can be constructed by the procedure described in Ref. 25. In the case where $\int f(x)dx$ is an invertible function of x, then the procedure given in Ref. 25 to construct the solution will lead to quadratures in terms of certain complicated integrals which however cannot be evaluated in general. For further details one may refer to Ref. 26. Therefore one should adopt a different procedure to construct the general solution of (16).

The effective way to proceed further is to transform the Hamiltonian (20) to a simpler form through suitable canonical transformations as demonstrated recently by us for the case of the MEE[6]. However, to adopt this procedure one should specify the explicit forms of $f(x)$ and $g(x)$. We demonstrate the procedure with a specific example in the following section.

IV. GENERALIZED MEE : HAMILTONIAN STRUCTURE AND GENERAL SOLUTION

To illustrate the ideas given in section 3, we focus our attention on the case $f(x) = x^q$, $g(x) = 1$ so that equation (16) now becomes

$$\ddot{x} + \alpha x^q \dot{x} + \beta x^{2q+1} = 0, \tag{25}$$

where $\beta = \frac{\lambda}{(q+1)}$. The reason for choosing this form of $f(x)$ for illustration is that it provides a natural generalization of the damped harmonic oscillator and the modified Emden type equation.

Substituting $f(x) = x^q$ in equations (17) - (19) we obtain the following time independent integral of motion for the equation (25), that is

Case 1. $\alpha^2 > 4\beta(q+1)$

$$I = \frac{(r-1)}{(r-2)} \left(\dot{x} + \frac{\hat{\alpha}}{r} x^{q+1} \right) \left(\dot{x} + \frac{(r-1)}{r} \hat{\alpha} x^{q+1} \right)^{(1-r)} \tag{26}$$

Case 2. $\alpha^2 = 4\beta(q+1)$

$$I = \frac{\dot{x}}{\dot{x} + \frac{\hat{\alpha}}{2} x^{q+1}} - \log[\dot{x} + \frac{\hat{\alpha}}{2} x^{q+1}] \tag{27}$$

Case 3. $\alpha^2 < 4\beta(q+1)$

$$I = \frac{1}{2} \log[\dot{x}^2 + \hat{\alpha}\dot{x}x^{q+1} + \frac{\beta}{(q+1)} x^{2(q+1)}] + \frac{\alpha}{2\omega} \tan^{-1} \left[\frac{\alpha\dot{x} + 2\beta x^{q+1}}{2\omega\dot{x}} \right], \tag{28}$$

where $r = \frac{\alpha}{2\beta(q+1)}(\alpha \pm \sqrt{\alpha^2 - 4\beta(q+1)})$, $\omega = \frac{1}{2}\sqrt{4\beta(q+1) - \alpha^2}$ and $\hat{\alpha} = \frac{\alpha}{q+1}$. The Hamiltonian for the equation (25) can be deduced from equation (20) which turns out to be

$$H = \begin{cases} \dfrac{(r-1)}{(r-2)}p^{\frac{(r-2)}{(r-1)}} - \dfrac{(r-1)}{r}\hat{\alpha}px^{q+1}, & \alpha^2 > 4\beta(q+1) \\[3mm] \log(p) - \dfrac{\hat{\alpha}}{2}px^{q+1}, & \alpha^2 = 4\beta(q+1) \\[3mm] \dfrac{\hat{\alpha}}{2}px^{q+1} - \dfrac{1}{2}\log\left[x^{2(q+1)}\sec^2[\dfrac{\omega}{(q+1)}x^{q+1}p]\right], & \alpha^2 < 4\beta(q+1), \end{cases} \quad (29)$$

where the canonically conjugate momentum is defined by

$$p = \begin{cases} \dfrac{1}{(r-1)}\left(\dot{x} + \dfrac{(r-1)}{r}\hat{\alpha}x^{q+1}\right)^{(1-r)}, \\[4mm] \dfrac{(q+1)}{\omega x^{q+1}}\tan^{-1}\left[\dfrac{\alpha x^{q+1} + 2(q+1)\dot{x}}{2\omega x^{q+1}}\right]. \end{cases} \quad (30)$$

Here we note that the integrals (26)-(28) can also be derived systematically through various methods available in the recent literature. To name a few, we cite Prelle-Singer procedure[27], method of reducing Liénard equation to the Abel equation form[25], factorization method[28,29] and so on. However, all these methods provide only time independent integrals but neither the integrals nor the Hamilton canonical equations can be integrated straightforwardly.

A.   Method of obtaining general solution

In this section we transform the Hamiltonian (29) into a simpler form, by introducing a suitable canonical transformation and then obtain the solutions.

Case 1: $\alpha^2 > 4\beta(q+1)$

The Hamiltonian for this parametric regime is given by

$$H = \dfrac{(r-1)}{(r-2)}p^{\frac{(r-2)}{(r-1)}} - \dfrac{(r-1)}{r}\hat{\alpha}px^{q+1}, \quad r = \dfrac{(\alpha^2 \pm \alpha\sqrt{\alpha^2 - 4\beta(q+1)})}{2\beta(q+1)} \quad (29)$$

By introducing a canonical transformation

$$x = q\left(\dfrac{U}{P}\right)^{\frac{1}{q}}, \quad p = \dfrac{1}{2}\left(U^{q-1}P^{q+1}\right)^{\frac{1}{q}} \quad (31)$$

the Hamiltonian (29) can be transformed into the form

$$H = \sigma_1 P^{n_1} U^{m_1} + \eta_1 U^2 = E, \tag{32}$$

where we have defined $\sigma_1 = \frac{(r-1)}{(r-2)}\left(\frac{1}{2}\right)^{\frac{(r-2)}{(r-1)}}$, $n_1 = \frac{(q+1)(r-2)}{q(r-1)}$, $m_1 = \frac{(q-1)(r-2)}{q(r-1)}$ and $\eta_1 = \frac{(1-r)}{2r}\hat{\alpha}q^{(q+1)}$.

The canonical equations of motion for the transformed Hamiltonian (32) now reads

$$\dot{U} = \sigma_1 n_1 U^{m_1} P^{n_1-1}, \tag{33a}$$

$$\dot{P} = -(\sigma_1 m_1 P^{n_1} U^{m_1-1} + 2\eta_1 U). \tag{33b}$$

One may observe that for the choice $n_1 = 1$ equation (33) becomes uncoupled and one can integrate the resultant equations straightforwardly. In the following, first we consider the general case and then discuss the particular case $n_1 = 1$ which corresponds to the parametric choice $\beta = \frac{\alpha^2}{(q+2)^2}$.

Substituting (33a) into (32) and rewriting the latter, we get

$$E = \frac{\sigma_1 U^{m_1}}{(\sigma_1 n_1)^{\frac{n_1}{n_1-1}}}\left[\dot{U}^{\frac{n_1}{n_1-1}}U^{\frac{m_1 n_1}{1-n_1}}\right] + \eta_1 U^2. \tag{34}$$

Rewriting equation (34) we obtain

$$\dot{U} = \left[\frac{1}{\hat{\sigma}_1}[E - \eta_1 U^2]\right]^{\frac{n_1-1}{n_1}} U^{\frac{m_1}{n_1}}, \qquad n_1 \neq 1 \tag{35}$$

where $\hat{\sigma}_1 = \frac{\sigma_1}{(\sigma_1 n_1)^{\frac{n_1}{n_1-1}}}$. Integrating the above equation one obtains,

$$t - t_0 = \int \frac{\hat{\sigma}_1^n U^m dU}{[E - \eta_1 U^2]^n}, \tag{36}$$

where $m = \frac{-m_1}{n_1}$ and $n = \frac{n_1-1}{n_1}$. The above integral can be split into two cases depending upon the values of m and n, that is

$$t - t_0 = \begin{cases} \dfrac{U^{m-1}}{\eta_1(2n-m-1)(E-\eta_1 U^2)^{n-1}} \\ \qquad + \dfrac{(m-1)E}{\eta_1(2n-m-1)}\displaystyle\int \dfrac{U^{m-2}}{(E-\eta_1 U^2)^n}dU \qquad m, n > 0 \\ \\ \dfrac{1}{E(1-m)U^{m-1}(E-\eta_1 U^2)^{n-1}} \\ \qquad + \dfrac{\eta_1(m+2n-3)}{E(1-m)}\displaystyle\int \dfrac{1}{U^{m-2}(E-\eta_1 U^2)^n}dU \quad m < 0. \end{cases} \tag{37}$$

The integrals on the right hand sides of (37) can be integrated again and again until all the integrals are exhausted, thereby giving the general solution in an implicit form.

Now we analyze the case $n_1 = 1$ in (33). For this choice, equation (33a) can be integrated to yield U in the form

$$U = [(1 - m_1)(\sigma_1 t + c_1)]^{\frac{1}{1-m_1}}, \tag{38}$$

where c_1 is an integration constant. From equation (32) one can express P in terms of U and E and substituting the latter into the first expression in (31) one gets

$$x = q \left(\frac{\sigma_1 U^{m_1+1}}{E - \eta_1 U^2} \right)^{\frac{1}{q}}, \tag{39}$$

where $\sigma_1 = \frac{q+1}{q} \left(\frac{1}{2} \right)^{\frac{q+1}{q}}$, $\eta_1 = -\alpha \frac{q^{q+1}}{2(q+2)}$ and $m_1 = \frac{q-1}{q+1}$. Substituting (38) into (39) and simplifying the resultant expression we arrive at

$$x(t) = \left(\frac{2^{\frac{q^2-1}{q}} q^{q-1}(q+1)^2(q+2)(\sigma_1 t + c_1)^q}{2E(q+2) + \alpha 2^{q+1} q^{q+1}(\sigma_1 t + c_1)^{q+1}} \right)^{\frac{1}{q}}. \tag{40}$$

with E and c_1 are two arbitrary constants. We mention here that the linearizable case $\alpha^2 = 9\beta$ belongs to this case and the solution exactly agrees with the known one in the literature[10−12].

Case 2 : $\alpha^2 = 4\beta(q+1)$

Using the same canonical transformation, equation (31), one can transform the Hamiltonian, $H = \log(p) - \frac{\hat{\alpha}}{2} p x^{q+1}$, into the form

$$H = \frac{(q-1)}{q} \log(U) + \frac{q+1}{q} \log[P] + \eta_2 U^2 = E \tag{41}$$

where $\eta_2 = -\frac{\hat{\alpha}}{4} q^{(q+1)}$. The corresponding canonical equations become

$$\dot{U} = \frac{(q+1)}{qP}, \tag{42a}$$

$$\dot{P} = \left(\frac{1-q}{q} \right) \frac{1}{U} + 2\eta_2 U. \tag{42b}$$

From (42a) one can express P in terms of $\dot{U}$ and substituting this into (41) one can bring the latter to the form

$$\hat{E} = \frac{(q-1)}{q} \log(U) - \frac{(q+1)}{q} \log(\dot{U}) + \eta_2 U^2 \tag{43}$$

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which in turn gives us

$$\dot{U} = \exp[\frac{q\eta_2 U^2}{(q+1)}]U^{\frac{q-1}{q+1}}E_1, \tag{44}$$

where $E_1 = \exp[\frac{-q\hat{E}}{(q+1)}]$. Integrating the above equation, one obtains the solution in the form

$$
\begin{aligned}
t - t_0 &= \frac{1}{E_1} \int \frac{dU}{\exp[\frac{q\eta_2 U^2}{(q+1)}]U^{\frac{(q-1)}{q+1}}} \\
&= -\frac{1}{2E_1}U^{\frac{2}{q+1}}\left(\frac{q+1}{q\eta_2 U^2}\right)^{\frac{1}{q+1}}\Gamma\left[\frac{1}{q+1}, \frac{q\eta_2 U^2}{q+1}\right],
\end{aligned} \tag{45}
$$

where Γ is the gamma function[30]. For the choice $q = 1$ the integral (45) can be evaluated in terms of error function [6,30].

Case 3 : $\alpha^2 < 4\beta(q+1)$

Now we focus our attention on the underdamped case. Using the canonical transformation (31), we rewrite the underlying Hamiltonian in the form

$$H = \frac{\hat{\alpha}}{4}q^{(q+1)}U^2 - \log\left[\left(\frac{U}{P}\right)^{\frac{(q+1)}{q}}\sec[\frac{\omega}{2(q+1)}U^2]\right]. \tag{46}$$

The associated canonical equations read

$$\dot{U} = \frac{q+1}{qP}, \tag{47a}$$

$$\dot{P} = \frac{2q\omega U^2\tan[\frac{\omega U^2}{2(q+1)}] - ((q+1)q^{q+2}U^2\alpha - 2(q+1)^2)}{2q(q+1)U}. \tag{47b}$$

From (47a) one can write $P = \frac{q+1}{q\dot{U}}$ and substituting this in the Hamiltonian (46) and simplifying we get

$$\tilde{H} = \log[U\dot{U}^{\frac{q+1}{q}}\sec[\frac{\omega}{2(q+1)}U^2]] + \eta_2 U^2 = E. \tag{48}$$

Rewriting equation (48) we have

$$\dot{U} = \frac{1}{U}\frac{\exp[\frac{q}{q+1}(E - \eta_2 U^2)]}{\sec[\frac{\omega}{2(q+1)}U^2]^{\frac{q}{q+1}}}. \tag{49}$$

15

Integrating equation (49), we get

$$
\begin{aligned}
t - t_0 &= \int \frac{U \sec[\frac{\omega}{2(q+1)}U^2]^{\frac{q}{q+1}}}{\exp[\frac{q}{q+1}(E - \eta_2 U^2)]} dU \\
&= F\left[\frac{q}{q+1}, \frac{q(\omega - 2iq_1)}{2(q+1)\omega}, \frac{(3q+2)\omega - 2iqq_1}{2(q+1)\omega}, -\exp\left[\frac{i\omega U^2}{(q+1)}\right]\right] \\
&\quad \times \frac{(q+1)^2 \left[\sec\left[\frac{U^2\omega}{2(q+1)}\right]\left(1 + \exp[\frac{i\omega U^2}{(q+1)}]\right)\right]^{\frac{q}{q+1}}}{q(2(q+1)\eta_2 + i\omega)}(\cosh[q_2] - \sinh[q_2]),
\end{aligned}
\tag{50}
$$

where $q_1 = \eta_2(q+1)$, $q_2 = \frac{q(E - \eta_2 U^2)}{q+1}$ and F is the hypergeometric function[30].

V. DAMPED MATHEWS-LAKSHMANAN OSCILLATOR : HAMILTONIAN STRUCTURE

As a second example we consider the following damped version of the Mathews-Lakshmanan oscillator equation[13],

$$
\ddot{x} - \frac{\lambda_1 x}{1 + \lambda_1 x^2}\dot{x}^2 + \frac{\alpha}{1 + \lambda_1 x^2}\dot{x} + \frac{\lambda x}{1 + \lambda_1 x^2} = 0,
\tag{51}
$$

where we have included a nonlinear damping term to the Mathews-Lakshmanan oscillator. The Mathews-Lakshmanan oscillator (equation (51) with $\alpha = 0$) possess simple trigonometric solution[13,14], $x(t) = A\cos\Omega t$, $\Omega^2 = \frac{\lambda}{1 + \lambda_1 A^2}$ with the Hamiltonian $H = p^2(1 + \lambda_1 x^2) + \frac{\lambda x^2}{1 + \lambda_1 x^2}$. We now find that the damped case (51), with $\alpha \neq 0$ is also integrable and admits a conservative Hamiltonian structure.

Comparing the above equation with (16), we find

$$
g(x) = \frac{1}{\sqrt{1 + \lambda_1 x^2}}, \; f(x) = \frac{1}{(1 + \lambda_1 x^2)^{\frac{3}{2}}}.
\tag{52}
$$

The integral of motion is then

$$
I = \begin{cases}
\dfrac{(r-1)}{(r-2)}\dfrac{(r\dot{x} + \alpha x)}{r\sqrt{1 + \lambda_1 x^2}}\left(\dfrac{r\dot{x} + \alpha x(r-1)}{r\sqrt{1 + \lambda_1 x^2}}\right)^{1-r}, & \alpha^2 > 4\lambda \\[3mm]
\dfrac{2\dot{x}}{2\dot{x} - x} - \log\left[\dfrac{2\dot{x} + \alpha x}{2\sqrt{1 + \lambda_1 x^2}}\right], & \alpha^2 = 4\lambda \\[3mm]
\dfrac{\alpha}{2\omega}\tan^{-1}\left[\dfrac{\alpha\dot{x} + 2\lambda x}{2\omega\dot{x}}\right] + \dfrac{1}{2}\log\left[\dfrac{\dot{x}^2 + \alpha x\dot{x} + \lambda x^2}{(1 + \lambda_1 x^2)}\right], & \alpha^2 < 4\lambda.
\end{cases}
\tag{53}
$$

Substituting the above forms of $g(x)$ and $f(x)$ in the expression for the Hamiltonian (20), we get

$$H = \begin{cases} \dfrac{(r-1)}{(r-2)}\left(p\sqrt{1+\lambda_1 x^2}\right)^{\frac{r-2}{r-1}} - \dfrac{\alpha(r-1)}{r}px, & \alpha^2 > 4\lambda \\[3mm] \dfrac{\alpha}{2}px + \log\left[p\sqrt{1+\lambda_1 x^2}\right], & \alpha^2 = 4\lambda \\[3mm] \dfrac{1}{2}\log\left[\dfrac{x^2}{(1+\lambda_1 x^2)}\sec^2[\omega px]\right] - \dfrac{\alpha p}{2}x, & \alpha^2 < 4\lambda, \end{cases} \tag{54}$$

where the canonically conjugate momentum

$$p = \begin{cases} \dfrac{1}{\sqrt{1+\lambda_1 x^2}}\left(\dfrac{r\dot{x} + \alpha x(r-1)}{r\sqrt{1+\lambda_1 x^2}}\right)^{1-r}, & \alpha^2 \geq 4\lambda \\[3mm] \dfrac{1}{\omega x}\tan^{-1}\left[\dfrac{\alpha + 2\dot{x}}{2\omega x}\right], & \alpha^2 < 4\lambda. \end{cases} \tag{55}$$

Using the canonical transformation $P = \dfrac{1}{2}\log(x^2)$, $U = px$, one can rewrite the Hamiltonian for the underdamped case $(\alpha^2 < 4\lambda)$ as

$$H = \frac{1}{2}\left(\log[\sec^2(\omega U)] - \log[e^{-2P} + \lambda_1] - \alpha U\right) \equiv E. \tag{56}$$

Following the same procedure illustrated in the previous example, one can arrive at the following equation

$$\dot{U} = 1 - \cos^2(\omega U)\exp[2E + \alpha U]. \tag{57}$$

Obviously this can be rewritten as the quadrature

$$t - t_0 = \int \frac{dU}{1 - \cos^2(\omega U)\exp[2E + \alpha U]}. \tag{58}$$

The above integration can be easily performed for the choice $\alpha = 0$ and one can recover the known solution of the Mathews-Lakshmanan oscillator. For $\alpha \neq 0$, the integration is nontrivial and requires further investigation. Similarly, for the overdamped and the critically damped cases, one can use the canonical transformation $U = \frac{1}{\sqrt{\lambda_1}}\sinh^{-1}(\sqrt{\lambda_1}x)$, $P = p\sqrt{1+\lambda_1 x^2}$ to reduce their corresponding Hamiltonian to simpler forms. However, again we are left with a quadrature for the choice $\alpha \neq 0$. Work is in progress to find other suitable canonical transformations to find the general solution explicitly for the case $\alpha \neq 0$.

VI.   LAGRANGIAN DESCRIPTION

We now present the equivalent Lagrangian description for the systems studied so far. One can follow two different procedures. Since both the Hamiltonian, H, and the corresponding canonical momentum, p, are available for each one of the systems studied in the previous sections, one can write down the associated Lagrangian as

$$L = p\dot{x} - H. \tag{59}$$

Alternatively, from the known form of the Lagrangian for the damped harmonic oscillator given by us in Ref. 5, and applying the nonlocal transformation (15), one can deduce the corresponding Lagrangian for the nonlinear damped oscillator equations. One can easily check that both the methods give rise to the same Lagrangian.

Considering now the damped harmonic oscillator (1) and its Hamiltonian form (3) along with the conjugate momentum (4), the Lagrangian can be expressed as

$$L = \begin{cases} \left(y' + \dfrac{(r-1)}{r}\alpha y\right)^{(2-r)}, & \alpha^2 > 4\lambda \\[2em] \log\left[y' + \dfrac{1}{2}\alpha y\right], & \alpha^2 = 4\lambda \\[2em] \dfrac{1}{2\omega}\left(\alpha \tan^{-1}\left[\dfrac{\alpha y' + 2\lambda y}{2\omega y'}\right] - \dfrac{2y'}{y}\tan^{-1}\left[\dfrac{\alpha y + 2y'}{2\omega y}\right]\right) \\ \quad + \dfrac{1}{2}\log\left[y'^2 + \alpha y' y + \lambda y^2\right], & \alpha^2 < 4\lambda. \end{cases} \tag{60}$$

Now applying the nonlocal transformation (15) to the above Lagrangian (60), one can readily obtain the Lagrangian associated with the general nonlinear damped oscillator equa-

tion (16) as

$$
L = \begin{cases}
\left(g(x)\dot{x} + \dfrac{(r-1)}{r}\alpha \displaystyle\int f(x)dx \right)^{(2-r)}, & \alpha^2 > 4\lambda \\[4ex]
\log\left[g(x)\dot{x} + \dfrac{1}{2}\alpha \displaystyle\int f(x)dx \right], & \alpha^2 = 4\lambda \\[4ex]
\begin{aligned}
&\dfrac{\alpha}{2\omega}\tan^{-1}\left[\dfrac{\alpha g(x)\dot{x} + 2\lambda \int f(x)dx}{2\omega g(x)\dot{x}} \right] \\
&-\dfrac{g(x)\dot{x}}{\omega \int f(x)dx}\tan^{-1}\left[\dfrac{\alpha \int f(x)dx + 2g(x)\dot{x}}{2\omega \int f(x)dx} \right] \\
&+\dfrac{1}{2}\log\left[g(x)^2\dot{x}^2 + \alpha g(x)\dot{x}\int f(x)dx + \lambda(\int f(x)dx)^2 \right],
\end{aligned} & \alpha^2 < 4\lambda.
\end{cases}
\tag{61}
$$

Naturally, the same form follows from the Hamiltonian (20) and the conjugate momentum (21) using (59). The Lagrangian for the specific examples considered in Secs. IV and V can be deduced by specifying the forms of $f(x)$ and $g(x)$ in equation (61). It may be noted that all the above Lagrangians are of non-standard type, that is of the forms which cannot be written in the standard form as 'kinetic energy' minus 'potential energy'. In particular, one can readily check that the non-standard Lagrangian

$$
L = \frac{1}{\dot{x} + \frac{2}{3}\int b(x)dx},
\tag{62}
$$

deduced by Musielak[7] for the nonlinear ODE

$$
\ddot{x} + b(x)\dot{x} + \frac{2}{9}b(x)\int b(x)dx = 0,
\tag{63}
$$

follows from the above general form (61) by choosing $f(x) = b(x)$, $g(x) = 1$, $\alpha = 1$ and $\lambda = \frac{2}{9}$ in equation (16). Note that the above equation (63) itself includes the MEE discussed in Refs. [8,9] as a special case with $b(x) = kx$.

VII.   CONCLUSION

In this paper, we have investigated a class of nonlinear dissipative systems which admit a non-standard time independent Hamiltonian description through a novel nonlocal transformation. The procedure is simple and straightforward. By introducing a nonlocal transformation in the 'source equation', namely the DHO equation, we are able to generate

a class of 'target equations', namely the nonlinear generalizations of DHO. We have used the same nonlocal transformation to deduce the time independent Hamiltonian for the nonlinear equation. The nonlocal transformation introduced in this paper is different from the one which we have adopted in Ref. 22 and has certain salient features in identifying the time independent Hamiltonian structure. To illustrate the procedure we have considered two specific systems, namely the generalized MEE and the damped Mathews-Lakshmanan oscillator equation. We obtained the solution of generalized MEE by integrating the canonical equations of motion after introducing a suitable transformation. The associated Lagrangian description for all the above systems is also briefly discussed. A similar analysis can also be performed in the case of two degrees of freedom systems. The details will be presented separately.

VIII.   ACKNOWLEDGEMENTS

[1] P. G. Drazin *Nonlinear Systems* (Cambridge University Press, Cambridge, 1992)

[2] G. Nicolis *Introduction to Nonlinear Science* (Cambridge University Press, Cambridge, 1995)

[3] C. I. Um, K. H. Yeon, T. F. George *Phys. Rep.* **362** 63 (2002)

[4] H. Bateman *Phys. Rev.* **38**, 815 (1931)

[5] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan *J. Math. Phys.* **48**, 032701 (2007)

[6] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan *J. Phys. A: Math. Theor.* **40**, 47171 (2007)

[7] Z. E. Musielak *J. Phys. A : Math. Theor.* **41**, 055205 (2008)

[8] J. F. Carinena and M. F. Ranada *J. Math. Phys.* **46**, 062703 (2005)

[9] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan *Phys. Rev. E* **72**, 066203 (2005)

[10] M. R. Feix, C. Geronimi, L. Cairo, P. G. L. Leach, R. L. Lemmer and S. E. Bouquet *J. Phys. A Math. Gen.* **30**, 7437 (1997)

[11] S. E. Bouquet, M. R. Feix and P. G. L. Leach *J. Math. Phys.* **32**, 1480 (1991)

[12] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan *J. Phys. A: Math. Gen.* **39**, L69 (2006)

[13] P. M. Mathews and M. Lakshmanan *Quart. Appl. Maths.* **32**, 315 (1974)

[14] M. Lakshmanan and S. Rajasekar *Nonlinear Dynamics : Integrability, Chaos and Patterns* (Springer-Verlag, New York, 2003)

[15] E. L. Ince *Ordinary Differential Equations* (Dover, New York, 1956)

[16] H. T. Davis *Introduction to Nonlinear Differential and Integral Equations* (Dover, New York, 1962)

[17] I. C. Moreira *Hadronic. J* **7**, 475 (1984)

[18] P. G. L. Leach *J. Math. Phys.* **26**, 2510 (1985)

[19] V. J. Erwin, W. F. Ames and E. Adams *Wave Phenomena: Modern Theory and Applications* ed C. Rogers and J. B. Moodie (Amsterdam, 1984)

[20] S. Chandrasekhar *An introduction to the Study of Stellar Structure* (Dover, New York, 1957)

[21] V. K. Chandrasekar S. N. Pandey, M. Senthilvelan and M. Lakshmanan *J. Math. Phys.* **47**, 023508 (2006)

[22] V. K. Chandrasekar, M. Senthilvelan, A. Kundu and M. Lakshmanan *J. Phys. A. Math. Gen.* **39**, 9743 (2006)

[23] P. Kustaanheimo and E. Stiefel J Reine *Angew. Math.* **218**, 204 (1965)

[24] G. M. Murphy *Ordinary Differential Equations and Their Solutions* (Affilitated East-West Press, New Delhi, India, 1960)

[25] R. Iacono *J. Phys. A : Math. Theor.* **41**, 068001 (2008)

[26] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan *J. Phys. A: Math. Theor.* **41**, 068002 (2008)

[27] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan *Proc. R. Soc. London, Soc. A* **461**, 2451 (2005)

[28] M. A. Reyes and H. C. Rosu *J. Phys. A : Math. Theor.* **41**, 285206 (2008)

[29] A. R. Janzen *Private Communication* (2007)

[30] I. S. Gradshteyn and I. M. Ryzhik *Table of Integrals, Series and Products* (Academic Press, London, 1980)