On the general solution for the modified Emden type equation $\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0$

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Abstract. In this paper, we demonstrate that the modified Emden type equation (MEE), $\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0$, is integrable either explicitly or by quadrature for any value of α and β. We also prove that the MEE possesses appropriate time-independent Hamiltonian function for the full range of parameters α and β. In addition, we show that the MEE is intimately connected with two well known nonlinear models, namely the force-free Duffing type oscillator equation and the two dimensional Lotka-Volterra (LV) equation and thus the complete integrability of the latter two models can also be understood in terms of the MEE.

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1. Introduction

One of the well discussed models in nonlinear dynamics is the modified Emden type equation (MEE), also called the modified Painlevé-Ince equation,

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0, \tag{1}$$

where over dot denotes differentiation with respect to time and α and β are arbitrary parameters. This equation has received attention from both mathematicians and physicists for more than a century [1–5]. For example, in the nineteenth century, Painlevé had studied this equation and identified general solution for two parametric choices, namely, (i) $\beta = \frac{\alpha^2}{9}$ and (ii) $\beta = -\alpha^2$ [1, 2]. The above differential equation (1) arises in a variety of mathematical problems such as univalued functions defined by second order differential equations [6] and the Riccati equation [7]. On the other hand physicists have shown that this equation arises in different contexts: for example, it occurs in the study of equilibrium configurations of a spherical gas cloud acting under the mutual attraction of its molecules and subject to the laws of thermodynamics [8, 9] and in the modelling of the fusion of pellets [10]. It also governs spherically symmetric expansion or collapse of a relativistically gravitating mass [11]. This equation can also be thought of as a one-dimensional analogue [7, 12] of the boson 'gauge-theory' equations introduced by Yang and Mills. Apart from the above, for the past two decades or so the invariance and integrability properties of this equation alone have been studied in detail by a number of authors, see for example Ref. [13–23]. In a nutshell, the MEE (1) has been found to possess explicit general solution only for the following parametric choices, that is, (*i*) $\alpha = 0$, (*ii*) $\beta = 0$, (*iii*) $\beta = \frac{\alpha^2}{9}$ and (*iv*) $\beta = -\alpha^2$. While the cases (i) and (ii) can be integrated trivially, in the third case the equation is linearizable to a free particle equation and in the fourth case the general solution can be expressed in terms of Weierstrass elliptic function [1–5, 13–24]. Equation (1) has also been noted to possess Painlevé property only for certain values of $r = \frac{\alpha}{4\beta}(\alpha \pm \sqrt{\alpha^2 - 8\beta})$ [17, 19]. Finally, we mention that equation (1) admits a two parameter Lie point symmetry group for arbitrary values of α and β, while for the chioce $\beta = \frac{\alpha^2}{9}$ the equation (1) possesses eight-parameter Lie point symmetries [13]. However, the general solution for the equation (1) with α and β are arbitrary is yet to be explored.

Very recently [23], the present authors have studied a generalized version of this equation from a different perspective and shown that the equation (1), for $\alpha^2 \geq 8\beta$, possesses time independent integrals and admits a Hamiltonian formalism which in turn ensures its complete integrability [23]. However, due to the complicated form of the first integral the general solution was not obtained in the previous work. Keeping in mind the historical importance and popularity of this model we kept exploring the general solution of equation (1) in phase-space. Based on our investigation, in this paper, we construct the time independent integrals for the equation (1) for arbitrary values of α and β (including the case $\alpha^2 < 8\beta$). Since the first integrals are not in simple polynomial forms it is difficult to obtain the general solution by just directly integrating them. In order to overcome this difficulty, first we identify time independent Hamiltonians from these time independent integrals and making use of suitable canonical transformations we convert the Hamiltonians into standard forms. We then integrate the new Hamiltonians and obtain the general solutions or reduce to quadratures. In this way we report the general solution for the equation (1) for arbitrary values of α and β for the first time. Our motivation to explore this solution is, as we see below, also due to the fact that the MEE (1) is not a stand-alone model. It is intimately connected to force-free Duffing oscillator type equation (vide equation (31) below) and two dimensional Lotka-Volterra (LV) equation (vide equation (33) below). The popularity and importance of these two models need no emphasis [23, 25–29]. Thus exploring the general solution for the equation (1) also serves to establish the complete integrability of these two models for appropriate parameters besides understanding the dynamics of the other models mentioned in the introduction.

The plan of the paper is as follows. In Section 2, we give the time independent integrals and correspponting Hamiltonians for the MEE (1). Using suitable canonical transformations, we obtain the general solutions for the parameter ranges $\alpha^2 = 8\beta$, $\alpha^2 > 8\beta$ and $\alpha^2 < 8\beta$ separately in Section 3. In Section 4, we show that the MEE (1) is intimately connected with two other well known nonlinear models, namely, the force-free Duffing oscillator type equation and two dimensional Lotka-Volterra equation. Finally, in Section 5 we summarize our results.

2. Time independent integrals and Hamiltonian description

As pointed out in the introduction, the MEE (1) cannot be straightforwardly integrated. Making use of the modified Prelle-Singer method developed by us recently [22, 23] following the earlier work of Duarte et al [30], in this section we first identify the first integrals separately for each of the three ranges (i) $\alpha^2 = 8\beta$, (ii) $\alpha^2 > 8\beta$ and (iii) $\alpha^2 < 8\beta$. Then we identify suitable canonical Hamiltonian description for each of these cases.

2.1. Time independent integrals

In Ref. [23], the time independent integrals for the equation (1) with $\alpha^2 \geq 8\beta$ has been reported using the modified Prelle-Singer procedure. However, improving the ansatz given in Ref. [23] one can obtain the time independent integrals for all values of α and β and the method of deriving the integrals for the equation (1) with α and β arbitrary is given in Appendix A. Using this method we identify the following time independent first integrals for the equation (1) with α and β arbitrary, that is,

Case 1: $\alpha^2 = 8\beta \;\; (r = 2)$

$$I = \log(\alpha^2 x^2 + 4\alpha\dot{x}) - \frac{4\alpha\dot{x}}{\alpha^2 x^2 + 4\alpha\dot{x}} \tag{2}$$

Case 2: $\alpha^2 > 8\beta \;\; (r \neq 0, 1, 2)$

$$I = \frac{(r-1)}{(r-2)}\Big(\dot{x} + \frac{(r-1)}{2r}\alpha x^2\Big)^{1-r}\Big(\dot{x} + \frac{\alpha}{2r}x^2\Big) \tag{3}$$

Case 3: $\alpha^2 < 8\beta$

$$I = \frac{1}{2}\log(2\dot{x}^2 + \alpha x^2\dot{x} + \beta x^4) + \frac{\alpha}{\omega}tan^{-1}\Big[\frac{\alpha\dot{x} + 2\beta x^2}{\omega\dot{x}}\Big], \tag{4}$$

where $\omega = \sqrt{8\beta - \alpha^2}$ and $r = \frac{\alpha}{4\beta}(\alpha \pm \sqrt{\alpha^2 - 8\beta})$. Note that $r = 0$ and 1 correspond to the trivial cases $\alpha = 0$ and $\beta = 0$, respectively, and so they are not considered here separately.

As it is very difficult to integrate equations (2)-(4) and obtain the general solutions by direct integration, we correlate these integrals with appropriate Hamiltonians. In the following we briefly give the method of obtain the Hamiltonian from the known time independent integral.

2.2. Hamiltonian description

To explore the Hamiltonian description of (2), let us assume the existence of a Hamiltonian

$$I(x, \dot{x}) = H(x, p) = p\dot{x} - L(x, \dot{x}), \tag{5}$$

where L is the Lagrangian and p is the canonically conjugate momentum. Then

$$\frac{\partial I}{\partial \dot{x}} = \frac{\partial H}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x} + p - \frac{\partial L}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x}. \tag{6}$$

From (6) we identify

$$p = \int \frac{I_{\dot{x}}}{\dot{x}} d\dot{x} + f(x), \tag{7}$$

where $f(x)$ is an arbitrary function of x and the Lagrangian $L = p\dot{x} - I(x,\dot{x})$. Here, without loss of generality, we take $f(x) = 0$. Substituting the integrals (2)-(4) into (7) and integrating the resultant integrals we can obtain the expression for the canonical momentum p. Substituting back the latter into the equation (5) and simplifying the resultant equation we arrive at the following Lagrangian,

$$L = \begin{cases} -\log(\dot{x} + \frac{\alpha}{4}x^2), & \alpha^2 = 8\beta \\ \frac{1}{(2-r)}(\dot{x} + \frac{(r-1)}{2r}\alpha x^2)^{(2-r)}, & \alpha^2 > 8\beta \\ \frac{1}{\omega}\Big(tan^{-1}\Big[\frac{4\dot{x}+\alpha x^2}{\omega x^2}\Big](\frac{4\dot{x}}{x}) - \alpha tan^{-1}\Big[\frac{\alpha\dot{x}+2\beta x^2}{\omega\dot{x}}\Big]\Big) & \\ \qquad -\frac{1}{2}\log(2\dot{x}^2 + \alpha x^2\dot{x} + \beta x^4), & \alpha^2 < 8\beta \end{cases} \tag{8}$$

and Hamiltonian

$$H = \begin{cases} \log(-\frac{4\alpha}{p}) - \frac{\alpha}{4}px^2, & \alpha^2 = 8\beta \\ \dfrac{(r-1)}{(r-2)}(p)^{\frac{r-2}{r-1}} - \dfrac{(r-1)}{2r}\alpha x^2 p, & \alpha^2 > 8\beta \\ \dfrac{1}{2}\log[x^4 sec^2(\dfrac{\omega}{4}x^2 p)] - \dfrac{\alpha}{4}x^2 p, & \alpha^2 < 4\beta \end{cases} \tag{9}$$

where the canonically conjugate momentum

$$p = \begin{cases} -\frac{1}{(\dot{x}+\frac{\alpha}{4}x^2)} & \alpha^2 = 8\beta \\ \Big(\dot{x} + \dfrac{(r-1)}{2r}\alpha x^2\Big)^{1-r} & \alpha^2 > 8\beta \\ \dfrac{4}{\omega x^2} tan^{-1}\Big[\dfrac{4\dot{x}+\alpha x^2}{2\omega x^2}\Big], & \alpha^2 < 8\beta \end{cases} \tag{10}$$

respectively for the MEE (1).

One can easily check that the canonical equations of motion for the above Hamiltonians are nothing but the equation of motion (1) of the MEE in the appropriate parametric regimes.

3. General solutions

In this section, we consider each of the above three cases separately, namely, (i) $\alpha^2 = 8\beta$, (ii) $\alpha^2 > 8\beta$ and (iii) $\alpha^2 < 8\beta$, and obtain their respective general solutions using suitable canonical transformations.

3.1. Case 1: $\alpha^2 = 8\beta$

To derive the general solution for this parametric choice first we consider the Hamiltonian given in (9) for $\alpha^2 = 8\beta$ as

$$H = \log(-\frac{4\alpha}{p}) - \frac{\alpha}{4}px^2. \tag{11}$$

Introducing the canonical transformation

$$x = \frac{4P}{\alpha U}, \qquad p = -\frac{\alpha U^2}{8}, \tag{12}$$

the Hamiltonian (11) can be recast into the standard form

$$H = \frac{1}{2}P^2 + \log(\frac{32}{U^2}) \equiv \hat{E}, \tag{13}$$

where $\hat{E}$ is a constant. From the corresponding canonical equations $\dot{U} = P$, $\dot{P} = \frac{2}{U}$, equation (13) can be rewritten as

$$E = \frac{1}{2}\dot{U}^2 - 2\log(U), \quad E = \hat{E} - \log(32). \tag{14}$$

Rewriting the above equation we get

$$\frac{dU}{\sqrt{2E + 4\log(U)}} = dt. \tag{15}$$

Integrating the equation (15) we obtain

$$U = e^{-\frac{1}{2}\left(E + 2\mathrm{erf}^{-1}(z)^2\right)}, \tag{16}$$

where $z = \dfrac{2e^{\frac{E}{2}}(t_0 + it)}{\sqrt{\pi}}$ and t_0 is the second arbitrary integration constant and *erf* is the error function [31].

Substituting the equation (16) into equation (12) we get the general solution for the equation (1), with the parametric choice $\beta = \frac{\alpha^2}{8}$, in the form (after some modifications)

$$x(t) = \frac{8}{\alpha}\mathrm{erf}^{-1}(\bar{z})e^{\frac{1}{2}\left(E + 2\mathrm{erf}^{-1}(\bar{z})^2\right)}, \quad \bar{z} = \frac{2e^{\frac{E}{2}}(t_0 - t)}{\sqrt{\pi}}. \tag{17}$$

In Fig. 1, we have plotted the solution of MEE (1) given by the expression (17) for the parametric choice $\alpha^2 = 8\beta$ for different initial conditions.

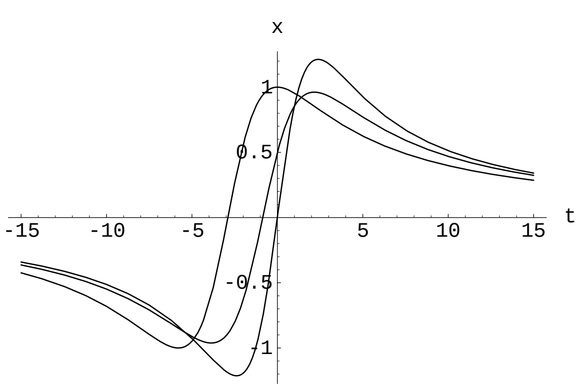

Figure 1. Solution plot for case $\alpha^2 = 8\beta$ for different initial conditions.

3.2. Case 2: $\alpha^2 > 8\beta$

Now let us consider the Hamiltonian for the case $\alpha^2 > 8\beta$ from (9), that is,

$$H = \frac{(r-1)}{(r-2)}(p)^{\frac{r-2}{r-1}} - \frac{(r-1)}{2r}\alpha x^2 p. \tag{18}$$

Interestingly, here we identify a canonical transformation for the Hamiltonian (18) in the form

$$x = -a\frac{P}{U}, \qquad p = \frac{U^2}{2a}, \tag{19}$$

where $a = \frac{2r}{\alpha(1-r)}$. It is straightforward to check that when U and P are canonical so do x and p (and vice versa) so that the Hamiltonian H in Eq. (18) can be rewritten as

$$H = \frac{1}{2}P^2 - kU^m \equiv E, \tag{20}$$

where $m = \dfrac{2(r-2)}{(r-1)}$ and $k = \dfrac{(1-r)}{(2a)^{\frac{m}{2}}(r-2)}$. From the canonical equations then we have $\dot{U} = P$ and $\dot{P} = mkU^{m-1}$.

To obtain the general solution, we introduce a transformation $X = U^m$ in equation (20) so that the latter can be brought to a quadrature of the form

$$t - t_0 = \int \frac{X^{\frac{1-m}{m}}dX}{m\sqrt{2E+2kX}}, \quad m \neq 0, 2, 4. \tag{21}$$

Now fixing $\frac{1-m}{m} = n$, the above integral leads us to the following expression [32], that is,

$$t - t_0 = \int \frac{X^n dX}{m\sqrt{2E+2kX}} = \frac{1}{m}\Big(\frac{X^n\sqrt{2E+2kX}}{2k(n+\frac{1}{2})} - \frac{2En}{2k(n+\frac{1}{2})}\int \frac{X^{n-1}dX}{\sqrt{2E+2kX}}\Big). \tag{22}$$

On the other hand fixing $\frac{1-m}{m} = -n$ in (21), we end up with the following expression [32]

$$\begin{aligned} t - t_0 &= \int \frac{dX}{mX^n\sqrt{2E+2kX}} \\ &= \frac{1}{m}\Big(-\frac{\sqrt{2E+2kX}}{2EX^{n-1}(n-1)} - \frac{k(n-\frac{3}{2})}{E(n-1)}\int \frac{dX}{X^{n-1}\sqrt{2E+2kX}}\Big). \end{aligned} \tag{23}$$

For integer values of n the integrals (22) and (23) can be integrated explicitly by repeated use of these formulas. When n is a noninteger value the final term in the integrals (22) and (23) can be integrated in terms of logarithmic function or beta function [32] (since the value of $n-1$ in the final integral lies between 0 to 1). We also note here that for the special choices $n = 0$ or $-\frac{3}{2}$ (which corresponds to the case $\beta = \frac{\alpha^2}{9}$) and $n = -\frac{2}{3}$ or $-\frac{5}{6}$ (which corresponds to the case $\beta = -\alpha^2$) respectively in equations (22) and (23), one can get the respective solutions reported in the literature [1–5, 13–24].

3.3. Case 3: $\alpha^2 < 8\beta$

Finaly, let us focus our attention in the regime $\alpha^2 < 8\beta$. In this parametric regime the Hamiltonian takes the form (vide equation (9))

$$H = \frac{1}{2}\log[x^4 sec^2(\frac{\omega}{4}x^2 p)] - \frac{\alpha}{4}x^2 p. \tag{24}$$

For the present case, let us choose the canonical transformation in the form

$$x = \frac{U}{P}, \qquad p = \frac{P^2}{2}, \tag{25}$$

and transform the Hamiltonian to

$$H = \frac{1}{2}\log[\frac{U^4}{P^4}sec^2(\frac{\omega}{8}U^2)] - \frac{\alpha}{8}U^2 \equiv E. \tag{26}$$

Now making use of the canonical equations, $\dot{U} = -\frac{2}{P}$ and $\dot{P} = \frac{1}{4U}((\alpha - \omega\tan(\frac{\omega}{8}U^2))U^2 - 8)$, equation (26) can be rewritten as

$$E = \frac{1}{2}\log[\frac{\dot{U}^4U^4}{16}sec^2(\frac{\omega}{8}U^2)] - \frac{\alpha}{8}U^2. \tag{27}$$

Introducing now the transformation $V = \frac{U^2}{2}$ in (27) we arrive at

$$E = \frac{1}{2}\log[\frac{\dot{V}^4}{16}sec^2(\frac{\omega}{4}V)] - \frac{\alpha}{4}V. \tag{28}$$

Integrating the equation (28) we get

$$t - t_0 = \frac{e^{-\frac{E}{2}}}{2}\int\sqrt{sec(\frac{\omega}{4}V)e^{-\frac{\alpha}{8}V}}\, dV, \tag{29}$$

where t_0 is the second integration constant. By substituting $W = e^V$ in (29) we obtain

$$t - t_0 = \frac{e^{-\frac{E}{2}}}{\sqrt{2}}\int\frac{W^{q_1-1}dW}{\sqrt{1+W^{q_2}}} = \frac{e^{-\frac{E}{2}}W^{q_1}}{\sqrt{2}q_1}F(\frac{1}{2},\frac{q_1}{q_2};1+\frac{q_1}{q_2};-W^{q_2}) \tag{30}$$

where $q_1 = \frac{-\alpha+i\omega}{8}$, $q_2 = \frac{i\omega}{2}$ and $F(\alpha,\beta;\gamma;z)$ is the hypergeometric function [31, 32].

4. Connection to two other nonlinear models

In this section, we show that the MEE (1) is intimately connected with two other well known nonlinear models, namely, the force-free Duffing oscillator type equation and two dimensional Lotka-Volterra (LV) equation.

4.1. Force-free Duffing oscillator type equation

The MEE equation (1) can be transformed to the following oscillator equation

$$w'' + (\alpha w + \gamma)w' + \beta w^3 + \frac{\alpha\gamma}{3}w^2 + \frac{2\gamma^2}{9}w = 0, \quad ('= \frac{d}{d\tau}) \tag{31}$$

through the invertible point transformation $x = we^{\frac{\gamma}{3}\tau}, t = -\frac{3}{\gamma}e^{-\frac{\gamma}{3}\tau}$, where γ is an arbitrary parameter. Equation (31) includes force-free Duffing oscillator (in the specific case $\alpha = 0$) [23, 25, 26] and quadratic oscillator with $\beta = 0$ [23].

One can deduce the general solution of the equation (31) from the general solution of MEE. For example, in the parametric choice $\beta = \frac{\alpha^2}{8}$ the general solution of the equation (31) can be derived from (17) in the form

$$w(\tau) = \frac{8}{\alpha}\mathrm{erf}^{-1}(\hat{z})e^{\frac{1}{2}\left(E-\frac{2}{3}\gamma\tau+2\mathrm{erf}^{-1}(\hat{z})^2\right)}, \tag{32}$$

where now $\hat{z} = \dfrac{6e^{\frac{E}{2}}(\tau_0 + e^{-\frac{\gamma}{3}\tau})}{\gamma\sqrt{\pi}}$ and E and τ_0 are arbitrary integration constants. Similarly, one can fix the general solution for the equation (31) in the other parametric regimes also, that is, $\alpha^2 > 8\beta$ and $\alpha^2 < 8\beta$.

4.2. Two dimensional Lotka-Volterra equation

Let us consider the two dimensional Lotka-Volterra equation of the form

$$\dot{x} = x(a_1 + a_2x + a_3y), \quad \dot{y} = y(b_1 + b_2x + b_3y), \tag{33}$$

where a_i's and b_i's are six real parameters. Equation (33) models two species in competition in ecology and is being analyzed for the past three decades or so in mathematical biology [27–29]. Interestingly, equation (33) can also be transformed to (31) as follows. Rewriting the first equation in (33) for the variable y and substituting it into the second equation in equation (33) we get the following second order ODE for x, namely,

$$\begin{aligned}\ddot{x} &- (1 + \frac{b_3}{a_3})\frac{\dot{x}^2}{x} + ((2a_2\frac{b_3}{a_3} - a_2 - b_2)x + (2a_1\frac{b_3}{a_3} - b_1))\dot{x} \\ &+(b_2a_2 - \frac{b_3}{a_3}a_2^2)x^3 + (a_2b_1 + b_2a_1 - 2a_1a_2\frac{b_3}{a_3})x^2 + (a_1b_1 - \frac{b_3}{a_3}a_1^2)x = 0. \end{aligned}\tag{34}$$

Let us choose the parameters in (34) in the form $b_3 = -a_3$ and $b_1 = a_1$ so that equation (34) can be brought to the form

$$\ddot{x} - ((3a_2 + b_2)x + 3a_1)\,\dot{x} + a_2(a_2 + b_2)x^3 + a_1(3a_2 + b_2)x^2 + 2a_1^2x = 0. \tag{35}$$

The associated LV equation takes the form

$$\dot{x} = x(a_1 + a_2x + a_3y), \quad \dot{y} = y(a_1 + b_2x - a_3y). \tag{36}$$

Now comparing the equations (31) and (35) we obtain $\alpha = -(3a_2 + b_2)$, $\beta = a_2(a_2 + b_2)$ and $\gamma = -3a_1$. Choosing $a_2 = b_2$ ($\beta = \frac{\alpha^2}{8}$) the general solution for the LV equation (35) can be obtained from (32) and using this in the first equation in (36) we arrive the general solution for the LV equation (36) in the form

$$x(t) = -\frac{2}{a_2}\mathrm{erf}^{-1}(z)e^{\frac{1}{2}\left(E+2a_1t+2\mathrm{erf}^{-1}(z)^2\right)}, \quad y(t) = -\frac{e^{\frac{1}{2}\left(E+2a_1t+2\mathrm{erf}^{-1}(z)^2\right)}}{a_3\mathrm{erf}^{-1}(z)}, \tag{37}$$

where $z = \dfrac{2e^{\frac{E}{2}}(t_0 - e^{a_1t})}{a_1\sqrt{\pi}}$ and E and t_0 are arbitrary integration constants. Finally, we mention that the general solution for the equation (35) for other parametric choices can be obtained from equation (31) in a similar manner. Again (to our knowledge) the complete integrability of the LV system (36) in these regimes is new to the literature.

5. Conclusions

In this paper, we have shown that the MEE (1) is integrable either explicitly or by quadratures for any value of α and β. We have also obtained the time independent

Hamiltonians for the equation (1). We have transformed the Hamiltonians into simpler forms, with appropriate canonical transformations, and deduced the general solution by direct integration so that our approach helps to understand the dynamics of equation (1) in phase-space clearly. Further, we have demonstrated that the complete integrability of equation (1) also helps one to understand the dynamics of two other nonlinear models, namely, the generalized oscillator equation and the two dimensional LV equation. As a consequence the solutions which we have explored for the equation (1) also provide solutions for these models as well.

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Appendix A. Method of deriving integrals of motions

In the following we briefly explain the generalized extended or modified Prelle-Singer (PS) procedure for second order ODEs [22, 23, 30] which is used to identify the integrals of motions (2)-(4).

To begin with, let us rewrite equation (1) in the form

$$\ddot{x} = -(\alpha x\dot{x} + \beta x^3) \equiv \phi(x, \dot{x}). \tag{A.1}$$

Further, we assume that the ODE (A.1) admits a first integral $I(t, x, \dot{x}) = C$, with C constant on the solutions, so that the total differential becomes

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0, \tag{A.2}$$

where each subscript denotes partial differentiation with respect to that variable. Rewriting equation (A.1) in the form $\phi dt - d\dot{x} = 0$ and adding a null term $S(t, x, \dot{x})\dot{x}$ $dt - S(t, x, \dot{x})dx$ to the latter, we obtain that on the solutions the 1-form

$$(\phi + S\dot{x})dt - Sdx - d\dot{x} = 0, \quad \phi = -(\alpha x\dot{x} + \beta x^3). \tag{A.3}$$

Hence, on the solutions, the 1-forms (A.2) and (A.3) must be proportional. Multiplying (A.3) by the factor $R(t, x, \dot{x})$ which acts as the integrating factors for (A.3), we have on the solutions that

$$dI = R(\phi + S\dot{x})dt - RSdx - Rd\dot{x} = 0. \tag{A.4}$$

Comparing Eq. (A.2) with (A.4) we have, on the solutions, the relations

$$I_t = R(\phi + \dot{x}S), \quad I_x = -RS, \quad I_{\dot{x}} = -R. \tag{A.5}$$

Then the compatibility conditions, $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, between the Eqs. (A.5), provide us

$$S_t + \dot{x}S_x + \phi S_{\dot{x}} \;\; = \; -\phi_x + \phi_{\dot{x}}S + S^2, \tag{A.6}$$

$$R_t + \dot{x}R_x + \phi R_{\dot{x}} \;=\; -(\phi_{\dot{x}} + S)R, \tag{A.7}$$
$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \tag{A.8}$$

Solving equations (A.6)-(A.8) one can obtain expressions for S and R. It may be noted that any set of special solutions (S, R) is sufficient for our purpose. Once these forms are determined the integral of motion $I(t, x, \dot{x})$ can be deduced from the relation

$$I = r_1 - r_2 - \int \left[R + \frac{d}{d\dot{x}}(r_1 - r_2)\right] d\dot{x}, \tag{A.9}$$

where

$$r_1 = \int R(\phi + \dot{x}S)dt, \quad r_2 = \int (RS + \frac{d}{dx}r_1)dx.$$

Equation (A.9) can be derived straightforwardly by integrating the equation (A.5).

As our motivation is to explore time independent integral of motion for the equation (A.1) let us choose $I_t = 0$. In this case one can easily fix the null form S from the first equation in (A.5) as

$$S = \frac{-\phi}{\dot{x}} = \frac{(\alpha x\dot{x} + \beta x^3)}{\dot{x}}. \tag{A.10}$$

Substituting this form of S into (A.7) we get

$$\dot{x}R_x - (\alpha x\dot{x} + \beta x^3)R_{\dot{x}} = -\frac{\beta x^3}{\dot{x}}R. \tag{A.11}$$

Equation (A.11) is a first order linear partial differential equation with variable coefficients. As we noted earlier any particular solution is sufficient to construct an integral of motion (along with the function S). To seek a particular solution for R one can make a suitable ansatz instead of looking for the general solution. We assume R to be of the form,

$$R = \frac{\dot{x}}{(A(x) + B(x)\dot{x} + C(x)\dot{x}^2)^r}, \tag{A.12}$$

where A, B and C are functions of their arguments, and r is a constant which are all to be determined. *We demand the above form of ansatz (A.12), which is very important to derive the Hamiltonian structure associated with the given equation*, due to the following reason. To deduce the first integral I we assume a rational form for I, that is, $I = \frac{f(x,\dot{x})}{g(x,\dot{x})}$, where f and g are arbitrary functions of x and $\dot{x}$ and are independent of t, from which we get $I_x = \frac{f_x g - f g_x}{g^2}$ and $I_{\dot{x}} = \frac{f_{\dot{x}} g - f g_{\dot{x}}}{g^2}$. From (A.5) one can see that $R = I_{\dot{x}} = \frac{f_{\dot{x}} g - f g_{\dot{x}}}{g^2}$, $S = \frac{I_x}{I_{\dot{x}}} = \frac{f_x g - f g_x}{f_{\dot{x}} g - f g_{\dot{x}}}$ and $RS = I_x$, so that the denominator of the function S should be the numerator of the function R. Since the denominater of S is $\dot{x}$ (vide Eq. (A.10)) we fixed the numerator of R as $\dot{x}$. To seek a suitable function in the denominator initially one can consider an arbitrary form $R = \frac{\dot{x}}{h(x,\dot{x})}$. However, it is difficult to proceed with this choice of h. So let us assume that $h(x, \dot{x})$ is a function which is polynomial in $\dot{x}$. To begin with let us consider the case where h is quadratic in $\dot{x}$, that is, $h = A(x) + B(x)\dot{x} + C(x)\dot{x}^2$, which is a generalized version of the form considered in Ref. [23], where only the linear form in $\dot{x}$ was investigated (that is $C(x) = 0$). Since R is in rational form while taking differentiation or integration the form of the denominator remains same but the power of the denominator decreases or increases by a unit order from that of the initial one. So instead of considering h to be of the form $h = A(x) + B(x)\dot{x} + C(x)\dot{x}^2$, one may consider a more general form $h = (A(x) + B(x)\dot{x} + C(x)\dot{x}^2)^r$, where r is a constant to be determined. The parameter r plays an important role, as we see below.

Substituting (A.12) into (A.11) and solving the resultant equations, we arrive at the relation

$$r\Big[\dot{x}(A_x + B_x\dot{x} + C_x\dot{x}^2) - (\alpha x\dot{x} + \beta x^3)(B + 2C\dot{x})\Big] = -\alpha x(A + B\dot{x} + C\dot{x}^2). \qquad \text{(A.13)}$$

Solving equation (A.13), we can fix the forms of A, B, C and r and substituting them into equation (A.12) we can get the integrating factor R. Doing so, we find

$$R = \begin{cases} \dfrac{\dot{x}}{(\dot{x} + \frac{(r-1)}{2r}\alpha x^2)^r}, & \alpha^2 \geq 8\beta \\ \dfrac{\dot{x}}{(2\dot{x}^2 + \alpha x^2\dot{x} + \beta x^4)}, & \alpha^2 < 8\beta \\ \dot{x}, & \alpha = 0 \end{cases} \qquad \text{(A.14)}$$

where $r = \frac{\alpha}{4\beta}\Big[\alpha \pm \sqrt{\alpha^2 - 8\beta}\Big]$. One can easily check the functions S and R given in (A.10) and (A.14), respectively, satisfy (A.8) also. Finally, substituting R and S into the form (A.9) for the integral we get the integrals of motion (2)-(4).

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