Estimation of System Parameters and Predicting the Flow Function from Time Series of Continuous Dynamical Systems

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Abstract

We introduce a simple method to estimate the system parameters in continuous dynamical systems from the time series. In this method, we construct a modified system by introducing some constants (controlling constants) into the given (original) system. Then the system parameters and the controlling constants are determined by solving a set of nonlinear simultaneous algebraic equations obtained from the relation connecting original and modified systems. Finally, the method is extended to find the form of the evolution equation of the system itself. The major advantage of the method is that it needs only a minimal number of time series data and is applicable to dynamical systems of any dimension. The method also works extremely well even in the presence of noise in the time series. This method is illustrated for the case of Lorenz system.

Time series analysis (both vector and scalar) is of considerable relevance [1,2,3] to physical, chemical and biological systems as they very often exhibit temporal chaotic motions. The main objective of time series analysis is to study the detailed structure of the evolution equation of the underlying dynamical system. This includes the number of independent variables, the form of the flow functions and parameters involved in the system. In this Letter, the study is focussed on the last aspect, namely, estimating the system parameters from the time series when partial information about the system is known (the number of independent variables and the form of the flow functions), and then the study is extended to predict the form of the flow function itself when it

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is not known. A few methods have been proposed [1,2,3] in the literature for predicting the form of the flow functions. These include two point boundary value problem approach [4], Euler integration approach to odes [1] and modifications [5]. Also recent literature shows that the methods already proposed for estimating the system parameters are based on the concept of synchronization [6,7,8,9,10], Bayesian approach [11,12], least squares approach [13] and by successive integration method [14]. In this Letter, a very simple and efficient method for estimating the system parameters as well as the form of the flow functions of continuous dynamical systems from the vector time series is developed using the concept of controlling chaos [15,16,17], which can also in principle be extended to scalar time series. In our method we construct a modified system by inclusion of certain constants (controlling constants) in the given original system so that the evolution of the modified system is controlled to an equilibrium point. Then we find the dynamical relation between the original and modified systems and thereby determine the unknown system parameters and the controlling constants. After accomplishing this task, the method is extended to determine the form of the evolution equations (flow functions) itself for the system from which the time series was collected. This method is applicable to time series obtained from a continuous system of any dimensions and is also well suited for discrete dynamical systems as shown in ref. [18] earlier. The method can also be used for the time series which contains considerable amount of noise. Further this method can be used in the field of controlling chaos to find the exact values of controlling constants to make the given chaotically evolving system to be controlled to a required equilibrium point.

Consider an arbitrary N-dimensional continuous chaotic dynamical system (the original system),

$$\dot{x}_i = f_i(x_1, x_2, ..., x_N; \mathbf{p}), \tag{1}$$

where $i = 1, 2, 3, ...N$, and $\mathbf{p}$ denotes the system parameters of dimension M to be determined. Here we assume that the function f is sufficiently smooth. Let us construct a modified continuous dynamical system (the modified system) as

$$\dot{y}_i = f_i(y_1, y_2, ..., y_N; \mathbf{p}) + \kappa_i, \quad i = 1, 2, ..., N, \tag{2}$$

where κ_i's are constants (controlling constants). The crucial idea here is that the Jacobian matrix which determines the stability of the equilibrium point is the same for both the cases, namely the original and the modified systems. In fact, the inclusion of κ_i's in Eq. (2) makes the modified system to have an equilibrium point (either stable or unstable) which is effectively different from the equilibrium point (unstable) of the original system eventhough the Jacobian

matrix is not affected as stated above. From an examination of many maps and flows we have found that there is in general a possibility of making the modified system (2) to exhibit a stable equilibrium point by suitable inclusion of constants κ_i eventhough the original system (1) evolves chaotically.

Let $u_i(t)$ be the deviation, that is, $u_i(t) = y_i(t) - x_i(t)$, of the trajectory of the modified system from that of the original system due to the effect of controlling constants (κ_i's). After substituting the above relation in Eq. (2) and making use of Taylor expansion, we obtain

$$\dot{u}_i = \kappa_i + \sum_{j=1}^{N} u_j \frac{\partial f_i}{\partial x_j}\bigg|_{\mathbf{x}} + \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} u_j u_k \frac{\partial^2 f_i}{\partial x_j \partial x_k}\bigg|_{\mathbf{x}} + \cdots, \quad i = 1, 2, ..., N. \quad (3)$$

Here the noteworthy point is that the above equation contains $\mathbf{x}$ whose evolution is characterized by Eq. (1). Thus, while obtaining the solution to Eq. (3) one has to solve Eqs. (1) and (3) simultaneously.

Let us now consider the time evolution of the variables $x_i(t)$, $y_i(t)$ and $u_i(t)$, statisfying Eqs. (1)-(3), respectively, at discrete time intervals. For this purpose, we introduce the notation $x_i^{(j)} = x_i(j\Delta t)$, $y_i^{(j)} = y_i(j\Delta t)$, and $u_i^{(j)} = u_i(j\Delta t)$, where Δt is a small time interval and $j = 0, 1, 2, 3, ...$ With this notation, the relation between the original and modified systems at definite intervals can be written as

$$y_i^{(j)} = x_i^{(j)} + u_i^{(j)}, \quad j = 0, 1, 2, ... \quad (4)$$

After the transient time $k\Delta t$, let the modified system approaches an equilibrium point y_i^* and hence the above equation becomes,

$$x_i^{(j)} + u_i^{(j)} = y_i^*, \quad j = k + 1, k + 2, k + 3, ... \quad (5)$$

Let $z_i^{(0)}, z_i^{(1)}, ..., z_i^{(m-1)}$ be the m set of data points of the given chaotic time series sampled at the time interval Δt from the original system (1). Substituting this in the above equation, we get

$$z_i^{(n)} + u_i^{(n)} = y_i^*, \quad n = 0, 1, 2, ..., (m-1). \quad (6)$$

In the above relation, it is instructive to note that we need not bother about the transients because the time series data is collected only after sufficient transient time. So, the discrete time index (n) now can start from 0, that is

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from the first data of the time series. It may be noted that here, u_i's are functions of the parameters $\mathbf{p}$ and controlling constants κ_i, since the derivatives of u_i's are having specific functional relations with the same parameters and controlling constants through Eq. (3). Also, $u_i^{(n)}$'s are obtained by solving Eq. (3) numerically with time interval Δt (or submultiples of Δt). In general, for a given set of time series data collected from the system (1) for a particlular set of parameters, the right hand side of Eq. (6), that is, the equilibrium point, depends on the value of controlling constants (κ_i's) which can be varied by means of $u_i^{(0)}$. This freedom allows us to make any desired point in the region of the attractor of the system as equilibrium point (y_i^*) by choosing the values for $u_i^{(0)}$ accordingly. For example, one can easily have $z_i^{(0)}$ as an equilibrium point by setting $u_i^{(0)} = 0$ in Eq. (6). Similarly, an arbitrary point q_i in the attractor can be chosen as an equilibrium point if we start with $u_i^{(0)} = q_i - z_i^{(0)}$.

Now it is possible to obtain the required number ($M + N$, N for κ_i's and M for $\mathbf{p}$'s) of functional relations (implicit) between the unknown parameters and controlling constants through Eq. (6). Once we have the required number of functional relations, the next task is to solve them for the unknowns, that is for κ_i's and $\mathbf{p}$'s, using a suitable numerical technique, for example by the globally convergent Newton's method [19].

After estimating the values of the parameters, its accuracy can be checked as follows. Compare the equilibrium point (for example $y_i^* = z_i^{(0)}$ if we assume $u_i^{(0)} = 0$ in Eq. (6)) assumed to be exhibited by the modified system in the above procedure with the equilibrium point calculated from Eq. (2) with the values of parameters determined by the above method. The degree of closeness of these equilibrium points gives a measure of accuracy of the estimated parameters.

As an illustration to our method, let us consider the Lorenz system

$$\dot{x}_1 = \sigma(x_2 - x_1), \tag{7a}$$
$$\dot{x}_2 = \rho x_1 - x_2 - x_1 x_3, \tag{7b}$$
$$\dot{x}_3 = x_1 x_2 - b x_3, \tag{7c}$$

where σ, ρ and b are the unknown system (control) parameters. Then the modified Lorenz system can be constructed as

$$\dot{y}_1 = \sigma(y_2 - y_1) + \kappa_1, \tag{8a}$$
$$\dot{y}_2 = \rho y_1 - y_2 - y_1 y_3 + \kappa_2, \tag{8b}$$
$$\dot{y}_3 = y_1 y_2 - b y_3 + \kappa_3, \tag{8c}$$

where κ_1, κ_2 and κ_3 are constants to be determined which make (8) to exhibit equilibrium point solution while the original system (with $\kappa_1 = \kappa_2 = \kappa_3 = 0$)

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Table 1. Convergence of σ, ρ, b, κ_1, κ_2 and κ_3 in the Lorenz system

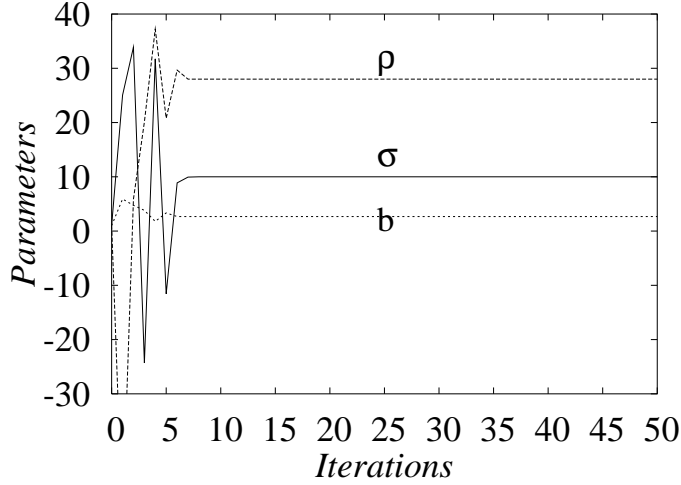| No.	σ	ρ	b	κ_1	κ_2	κ_3
0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
1	25.06611000	-53.43927900	5.90702312	-1.09685478	-1.64477291	5.66012411
2	33.88597350	5.87195581	4.74703140	-1.70082489	-2.33269811	7.14465601
3	-24.31336050	20.11146620	3.81209093	-3.00648132	-3.72360435	9.50270473
4	31.73494710	37.21100300	1.76763806	-7.61414582	-9.53379493	19.84796380
5	-11.56754590	20.78649710	3.33689508	-15.06851440	-19.96489090	38.24238610
6	8.87479048	29.65454090	2.67975225	-15.31294170	-19.87467490	38.27824720
7	9.95266093	27.99241480	2.66686356	-15.26076520	-20.11294050	38.27375090
8	10.00000000	28.00000000	2.66666667	-15.25561980	-20.11136290	38.27364540
9	10.00000000	28.00000000	2.66666667	-15.25561980	-20.11136290	38.27364540
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	10.00000000	28.00000000	2.66666667	-15.25561980	-20.11136290	38.27364540

Fig. 1. Convergence of the parameters σ, ρ and b towards their exact values in the Lorenz system.

exhibits chaotic solution. Then the equation (3) for the deviations, $u_i = y_i - x_i$, $i = 1, 2, 3$, becomes

$$\dot{u}_1 = \sigma(u_2 - u_1) + \kappa_1, \tag{9a}$$
$$\dot{u}_2 = (\rho - x_3)u_1 - u_2 - (x_1 + u_1)u_3 + \kappa_2, \tag{9b}$$
$$\dot{u}_3 = (x_2 + u_2)u_1 + x_1 u_2 - bu_3 + \kappa_3. \tag{9c}$$

It may be noted that the presence of x_1, x_2 and x_3 in (9) indicates that one has to solve Eqs. (7) & (9) simultaneously.

Let $(z_1^0, z_2^0, z_3^0), (z_1^1, z_2^1, z_3^1), ..., (z_1^{(m-1)}, z_2^{(m-1)}, z_3^{(m-1)})$ be the time series data obtained from the Lorenz system at some arbitrary time interval with the sampling rate Δt for an unknown set of system parameters. After assuming $z_i^{(0)}$ as the equilibrium point for the modified Lorenz system (that is by assigning $u_i^{(0)} = 0$ in Eq. (6), so that $y_i^* = z_i^{(0)}$) and substituting three data points $z_i^{(0)}$, $z_i^{(1)}$ and $z_i^{(2)}$ (one can in fact take any three successive data points but the first data point has to be used as initial condition for x_i) in Eq. (6), we get

$$z_1^{(1)} + u_1^{(1)} = z_1^{(0)}, \quad z_2^{(1)} + u_2^{(1)} = z_2^{(0)}, \quad z_3^{(1)} + u_3^{(1)} = z_3^{(0)},$$
$$z_1^{(2)} + u_1^{(2)} = z_1^{(0)}, \quad z_2^{(2)} + u_2^{(2)} = z_2^{(0)}, \quad z_3^{(2)} + u_3^{(2)} = z_3^{(0)}. \tag{10}$$

Note that the time derivatives of u_i's have specific functional relations with the unknown parameters (σ, ρ and b) and controlling constants (κ_1, κ_2 and κ_3) through Eq. (9), and hence u_i's are also functions of the same parameters and controlling constants. Now we have six (implicit) functional relations for
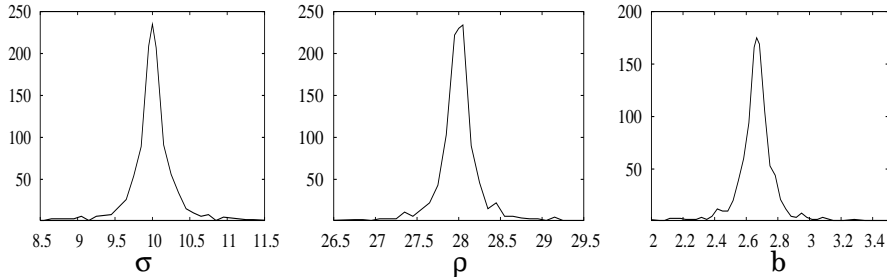
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Fig. 2. The distribution of the values of the system parameters estimated from the time series which contains random dynamical noise in the Lorenz system.

the six unknowns namely σ, ρ, b, κ_1, κ_2 and κ_3 through Eq. (10), in terms of the equilibrium point (here $z_i^{(0)}$) and the two sets of data points $z_i^{(1)}$ and $z_i^{(2)}$, $i = 1, 2, 3$. To solve for these unknowns, we have to obtain values of $u_1^{(1)}$, $u_2^{(1)}$, $u_3^{(1)}$, $u_1^{(2)}$, $u_2^{(2)}$ and $u_3^{(2)}$ by solving Eqs. (7)&(9) simultaneously so that Eq. (10) is satisfied. This can be done, for example, by the globally convergent Newton's method [19] provided an initial guess is given to all the unknown parameters. In this example, while obtaining the value of $u_i^{(1)}$ initial conditions should be taken as $x_i^{(0)} = z_i^{(0)}$ and $u_i^{(0)} = 0$, respectively. Further, the evaluation of $u_i^{(2)}$ needs to reset $x_i^{(1)} = z_i^{(1)}$ in the numerical algorithm used above. Similar procedure has to be followed if any other set of successive data is used in place of $z_i^{(0)}$, $z_i^{(1)}$ and $z_i^{(2)}$ in Eq. (10). For illustration purpose, we have used the numerically generated time series of the Lorenz system for the system parameters $\sigma = 10.0$, $\rho = 28.0$ and $b = 8/3$ and solved the Eq. (10) by globally convergent Newton's method with an initial guess 1.0 to all the unknowns κ_1, κ_2, κ_3, σ, ρ and b. The convergence of the system parameters and controlling constants (κ's) is shown in the Table I, which shows that the estimated values are indeed the exact values of the parameters at which the time series data of the Lorenz system is generated. We also note that the convergence is very rapid, which is further confirmed in Fig. (1). Note that if we take a different set of data points from the time series as the equilibirium point, the corresponding controlling contstants (κ_i's) will be different eventhough the system parameters are unaltered.

In order to show that our method is robust, a time series which contains *random dynamical noise* of strength 10^{-3} was additionaly introduced in the above Lorenz model. In this case, the distribution of the values of the system parameters estimated using 1000 data points is shown in fig. (2). Also in the case of *random observational noise* of same strength, the distribution of the values of the parameters is shown in fig. (3). The peaks about the true values of the parameters in the above two figures indicate that our method works extremely well even in the presence of dynamical noise or in the presence observational noise in the time series.
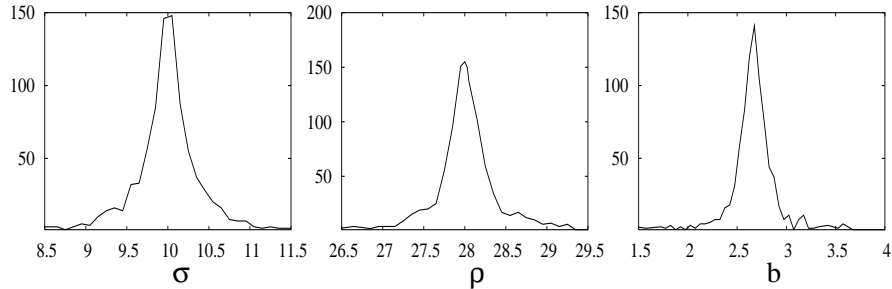
Fig. 3. The distribution of the values of the system parameters estimated from the time series which contains random observational noise in the Lorenz system.

We have also carried out a similar analysis for the autonomous Chua's Circuit [16] in its dimensionless form,

$$\dot{x}_1 = \alpha(x_2 - x_1 - h(x_1)), \tag{11a}$$
$$\dot{x}_2 = x_1 - x_2 + x_3, \tag{11b}$$
$$\dot{x}_3 = -\beta x_2, \tag{11c}$$

where

$$h(x) = bx + 0.5(a - b)(|x + 1| - |x - 1|),$$

α and β are parameters to be determined, and for the Rössler system,

$$\dot{x}_1 = -(x_2 + x_3), \tag{12a}$$
$$\dot{x}_2 = x_1 + ax_2, \tag{12b}$$
$$\dot{x}_3 = b + (x_1 - c)x_3, \tag{12c}$$

where a, b and c are the parameters to be determined. In both the cases, we are able to obtain the correct values of the parameters as in the case of the Lorenz system.

Next, we wish to point out that it is also possible to extend the analysis to predict the flow function of the system itself in principle, by assuming a polynomial form (here as an illustration) for the functions $f_i(x_1, x_2, ...x_N; \mathbf{p})$ in the right hand side of Eq. (1), with unknown coefficients, and solving sufficient number of Eqs. (6) to determine them. One can as well choose other types basis functions and our procedure is equally applicable here also. To illustrate this idea, let us consider the same time series data used in the above example of Lorenz system and assume that the form of the underlying dynamical equations is unknown. Let us assume a general quadratic form for the flow function (dynamical equations) as

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$$\dot{x}_1 = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2 + c_5 x_2 x_3 + c_6 x_1 x_3, \tag{13a}$$
$$\dot{x}_2 = c_7 x_1 + c_8 x_2 + c_9 x_3 + c_{10} x_1 x_2 + c_{11} x_2 x_3 + c_{12} x_1 x_3, \tag{13b}$$
$$\dot{x}_3 = c_{13} x_1 + c_{14} x_2 + c_{15} x_3 + c_{16} x_1 x_2 + c_{17} x_2 x_3 + c_{18} x_1 x_3, \tag{13c}$$

where c_i's, $i = 1, 2, ...18$, are parameters to be determined from the time series data. Again following the method suggested above, we write the form of the modified system as

$$\dot{y}_1 = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_1 y_2 + c_5 y_2 y_3 + c_6 y_1 y_3 + \kappa_1, \tag{14a}$$
$$\dot{y}_2 = c_7 y_1 + c_8 y_2 + c_9 y_3 + c_{10} y_1 y_2 + c_{11} y_2 y_3 + c_{12} y_1 y_3 + \kappa_2, \tag{14b}$$
$$\dot{y}_3 = c_{13} y_1 + c_{14} y_2 + c_{15} y_3 + c_{16} y_1 y_2 + c_{17} y_2 y_3 + c_{18} y_1 y_3 + \kappa_3, \tag{14c}$$

where κ_1, κ_2 and κ_3 are again the controlling constants to be determined so that the above modified dynamical system exhibits a stable equilibrium point. For this system, Eq. (3) becomes

$$\dot{u}_1 = \kappa_1 + c_1 u_1 + c_2 u_2 + c_3 u_3 + (c_4 x_2 + c_6 x_3 + c_4 u_2) u_1$$
$$+ (c_4 x_1 + c_5 x_3 + c_5 u_3) u_2 + (c_5 x_2 + c_6 x_1 + c_6 u_1) u_3, \tag{15a}$$
$$\dot{u}_2 = \kappa_2 + c_7 u_1 + c_8 u_2 + c_9 u_3 + (c_{10} x_2 + c_{12} x_3 + c_{10} u_2) u_1$$
$$+ (c_{10} x_1 + c_{11} x_3 + c_{11} u_3) u_2 + (c_{11} x_2 + c_{12} x_1 + c_{12} u_1) u_3, \tag{15b}$$
$$\dot{u}_3 = \kappa_3 + c_{13} u_1 + c_{14} u_2 + c_{15} u_3 + (c_{16} x_2 + c_{18} x_3 + c_{16} u_2) u_1$$
$$+ (c_{16} x_1 + c_{17} x_3 + c_{17} u_3) u_2 + (c_{17} x_2 + c_{18} x_1 + c_{18} u_1) u_3. \tag{15c}$$

In order to determine the values of the parameters and controlling constants $(c_i, i = 1, 2, ...18, \kappa_1, \kappa_2$ and $\kappa_3)$, we have to solve 21 algebraic equations which are actually constructed by making use of eight successive time series data in Eq. (6) with assumption that the first one is the equilibrium point. Then the required equations will be

$$z_i^{(j)} + u_i^{(j)} = z_i^{(0)}, \quad i = 1, 2, 3 \text{ and } j = 1, 2, ..., 7. \tag{16}$$

Again we follow exactly the steps mentioned earlier for the Lorenz system and obtain the values of the unknown parameters. The distribution of the values of the parameters (c_i's, $i = 1, 2, ..., 18$) obtained by solving Eq. (16) using 3000 data of time series is shown is Fig. (4). And the values are found to be distributed around $\{c_i\}_1^{18} = \{-10, 10, 0, 0, 0, 0, 28, -1, 0, 0, 0, -1, 0, 0, -2.67, 1, 0, 0\}$. Substituting these values in Eq. (13) we obtain the flow function (evolution equation) of the form

$$\dot{x}_1 = c_1 x_1 + c_2 x_2, \tag{17a}$$
$$\dot{x}_2 = c_7 x_1 + c_8 x_2 + c_{12} x_1 x_3, \tag{17b}$$
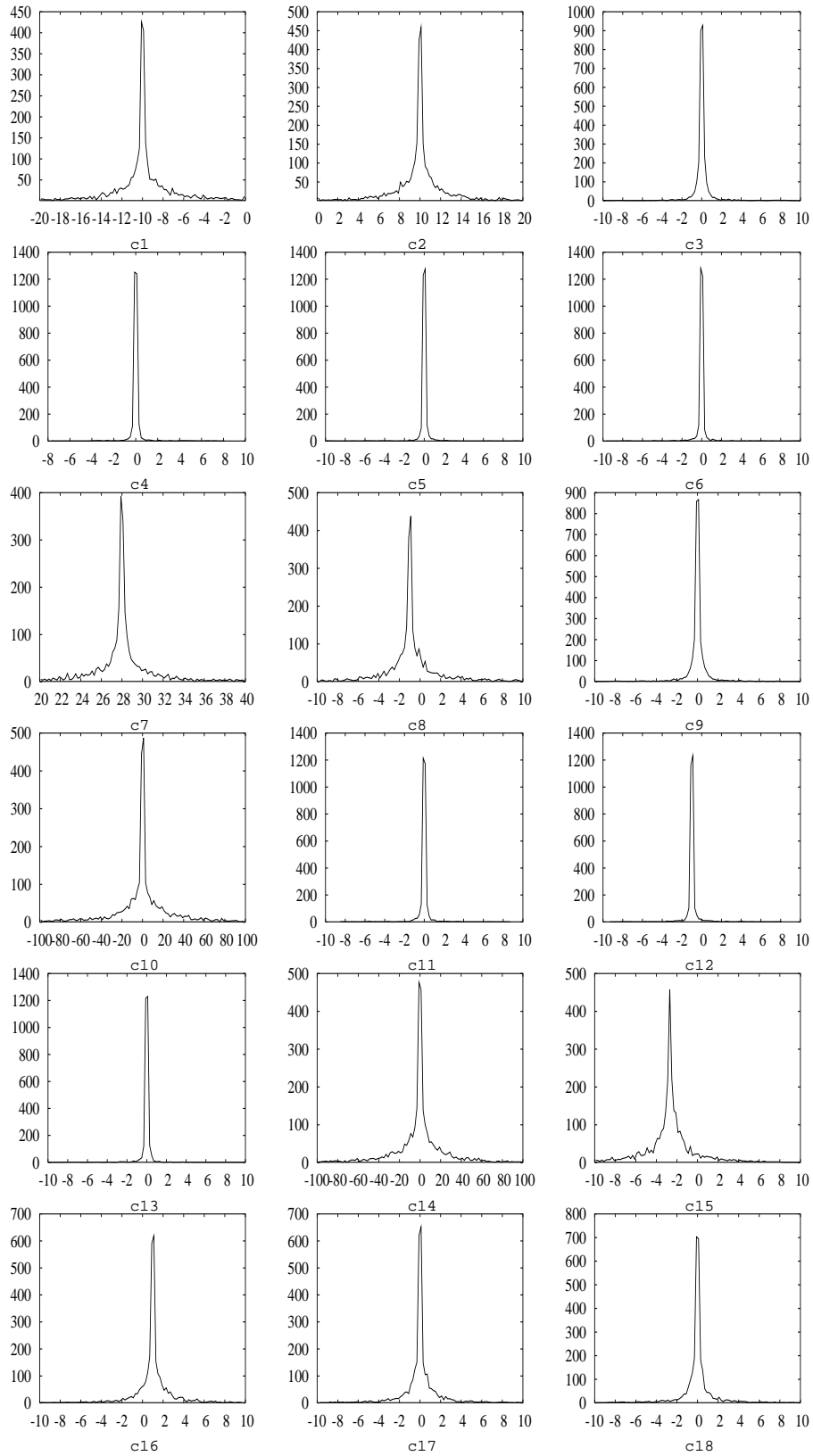$$\dot{x}_3 = c_{15} x_3 + c_{16} x_1 x_2, \tag{17c}$$

Fig. 4. The distribution of the values of the system parameters of the flow functions given by Eq. (13), estimated from the time series of Lorenz system

with $\sigma = -c_1 = c_2 = 10.0$, $\rho = c_7 = 28.0$, $b = -c_{15} = 2.67$ and the remaining constants $c_{16} = 1.0$ and $c_8 = c_{12} = -1$, which is nothing but the original Lorenz system.

Finally, we would like to point out that the procedure outlined above also gives a method to obtain the values of the controlling constants (κ_i) for a chaotic system to be controlled to a desired equilibrium point if the form of evolution equation is known.

To conclude, we have developed a very simple as well as useful method for estimating the unknown system parameters of the continuous dynamical systems of any dimensions from the vector time series, if partial information is known, namely the form of the dynamical equations . It has also been extended to obtain the form of the system equation itself at least in the case of polynomial forms. Both the methods have been illustrated by means of vector time series collected from the Lorenz system, while further confirmation has been made with other systems including the Chua's Circuit and Rössler systems. We have also checked that the method is robust againt dynamical and observational noise and that the procedure exhibits a rapid convergence.

Acknowledgements

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