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Shape changing collisions of optical solitons, universal logic gates and partially coherent solitons in coupled nonlinear Schrödinger equations

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Abstract. Coupled nonlinear Schrödinger equations (CNLS) very often represent wave propagation in optical media such as multicore fibers, photorefractive materials and so on. We consider specifically the pulse propagation in integrable CNLS equations (generalized Manakov systems). We point out that these systems possess novel exact soliton type pulses which are shape changing under collision leading to an intensity redistribution. The shape changes correspond to linear fractional transformations allowing for the possibility of construction of logic gates and Turing equivalent all optical computers in homogeneous bulk media as shown by Steiglitz recently. Special cases of such solitons correspond to the recently much discussed partially coherent stationary solitons (PCS). In this paper, we review critically the recent developments regarding the above properties with particular reference to 2-CNLS.

Keywords. Optical solitons; integrable systems; nonlinear optics.

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1. Introduction

The recent advances in using the possibility of ultrashort optical pulses in long distance communication via optical fibers [1] and the observation of self trapping of optical beams [2] have motivated an intense study of both temporal as well as spatial optical solitons. From a theoretical point of view, in the context of intense optical pulse propagation, the governing equation for such wave propagation through single mode optical fibers is represented by the scalar nonlinear Schrödinger family of equations [3,4]. Here the formation of optical solitons is due to the interplay between the spreading of the pulse and the nonlinear response of the medium (Kerr effect) which leads to an intensity dependent phase change (known as self phase modulation(SPM)) of the incident pulse. In the case of birefringent fiber and multimode fibers, in addition to SPM one has to consider cross phase modulation (XPM) which leads to a phase dependence of each mode on the intensity of the co-propagating modes. Then the resulting propagation equation is a set of coupled nonlinear Schrödinger (CNLS) equations [4], which are nonintegrable in general. The CNLS equations are ubiquitous in the sense that they have diverse applications such as in the theory of soliton wavelength division multiplexing [5], multi-channel bit parallel-wavelength optical fiber networks [6], and so on. They even occur in the study of launching and propagation of solitons along the three spines of an alpha-helix in protein [7].

The notion of optical spatial solitons is also receiving a great deal of attention recently. Here the optical spatial solitons are formed due to an exact balance between the diffraction and the self-focussing due to an optical nonlinearity. Very recently, it has been noted that in the context of beam propagation in a Kerr-like photorefractive medium, which generally exhibits very strong nonlinear effects with extremely low optical powers,the governing equations are a set of N-CNLS equations. In recent years, there is an increasing interest in studying soliton propagation through such photorefractive media after the observation of so-called partially incoherent solitons through excitation by partially coherent light [8] and also with an ordinary incandescent light bulb [9]. Following these observations, various theoretical approaches like coherent density approach [10,11], diffractionless ray optics limit approach [12], etc., have been developed. It has been observed both experimentally and theoretically that the N-CNLS equations support a class of partially coherent stationary solitons (PCS) [13].These PCS solutions have been interpreted as multisoliton complexes, which are nonlinear superpositions of several fundamental solitons [14]. Further these PCS are found to be of variable shape [13].

Though a large number of investigations exist in the literature on the study of soliton propagation in CNLS equations, exact results are scarce, except for a special parametric choice of the two coupled nonlinear Schrödinger equations, namely the Manakov model [15]. In a recent work Radhakrishnan, Lakshmanan and Hietarinta [16] have obtained the explicit 2-soliton solution of the integrable Manakov system and identified a novel shape changing collision property in it. In a very recent study the present authors [17] have extended the analysis to the case of integrable 3-CNLS and arbitrary N-CNLS systems, where a similar property has been identified. In addition, it has been pointed out that the PCS solutions available in the literature are special cases of these shape changing solitons. Making use of the above shape changing collision property of the 2-CNLS (Manakov) system as corresponding to a linear fractional transformation, Jakubowski, Steiglitz and Squier [18] and later Steiglitz [19] have shown that universal logic gates and an all optical computer equivalent to a Turing machine can be constructed at least in a mathematical sense.

Our aim in this paper is to critically review the basic integrability properties of the integrable N-CNLS systems, which are generalized Manakov systems, and point out the recent developments in this connection. To start with in §2, we quickly point out how CNLS equations arise in connection with the study of intense optical pulse propagation in multimode fibers and photorefractive materials. In §3, we point out the integrability properties of the 2- and N-CNLS systems. In §4, we deduce the explicit two-soliton solution of the above systems and point out in §5 how shape changing collision of solitons takes place. By treating the shape change as equivalent to a bilinear or linear fractional transformation, we point out in §6 the procedure of Steiglitz [19] to construct universal logic gates and all optical computers in bulk homogeneous media without interconnecting discrete components. Then in §7, we show how the various stationary PCS solutions follow by particularising the general soliton solutions of the above N-CNLS equations. Finally in §8 we summarize our discussions.

2. CNLS equations as governing equations for intense light propagation in multimode fibers and photorefractive materials

As a prelude to the CNLS equations, let us consider the intense electromagnetic wave propagation in a birefringent fiber (see for example, ref. [4]). Due to birefringence, a single mode fiber can support two distinct modes of polarization which are orthogonal to each other. These two modes can be viewed as the ordinary ray (O-ray) for which the refractive index of the medium is constant along every direction of the incident ray and the other as the extraordinary ray (E-ray) whose refractive index for the medium varies with the direction of the incident ray. Then the nonlinear phase variation of a particular mode not only depends on its own intensity but also on that of the co-propagating mode.

The study of such a system starts straight away from the Maxwell's equations for electromagnetic wave propagation in a dielectric medium,

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\mu_0\frac{\partial^2\vec{P}}{\partial t^2}, \tag{1}$$

where $\vec{E}(\vec{r},t)$ is the electric field, μ_0 represents the free space permeability, c is the velocity of light and $\vec{P}$ is the induced polarization which can be separated into two parts: $\vec{P} = \vec{P}_L + \vec{P}_{NL}$, where $\vec{P}_L(\vec{r},t)$ and $\vec{P}_{NL}(\vec{r},t)$ represent the linear and nonlinear parts of the induced polarization. It is well known that they can be expressed in terms of the electric field as [4]

$$\vec{P}_L(\vec{r},t) = \varepsilon_0\int_{-\infty}^{\infty}\chi^{(1)}(t-t')\vec{E}(\vec{r},t')\mathrm{d}t', \tag{2a}$$

$$\vec{P}_{NL}(\vec{r},t) = \varepsilon_0\int\int_{-\infty}^{\infty}\int\chi^{(3)}(t-t_1,t-t_2,t-t_3)\vdots \vec{E}(\vec{r},t_1)\vec{E}(\vec{r},t_2)\vec{E}(\vec{r},t_3)\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3, \tag{2b}$$

where ε_0 is the free space permittivity and $\chi^{(j)}$ is the jth order susceptibility tensor of rank $(j+1)$.

Considering wave propagation along elliptically birefringent optical fibers, the electric-field $\vec{E}(\vec{r},t)$ can be written in the quasi-monochromatic approximation as

$$\vec{E}(\vec{r},t) = \frac{1}{2}\left[\hat{e}_1E_1(z,t) + \hat{e}_2E_2(z,t)\right]\mathrm{e}^{-i\omega_0 t} + \text{ c.c}, \tag{3}$$

where z is the direction of propagation, t represents the retarded time and c.c stands for the complex conjugate and the orthonormal polarization vectors $\hat{e}_1$ and $\hat{e}_2$ can be expressed in terms of the unit polarization vectors $\hat{x}$ and $\hat{y}$ along the x and y directions respectively as

$$\hat{e}_1 = \frac{\hat{x}+ir\hat{y}}{(1+r^2)^{1/2}}, \tag{4a}$$

$$\hat{e}_2 = \frac{r\hat{x}-i\hat{y}}{(1+r^2)^{1/2}}, \tag{4b}$$

in which the parameter r is a measure of the extent of ellipticity. In eq. (3) E_1 and E_2 are the complex amplitudes of the two polarization components at frequency ω_0. Considering the medium to be isotropic, the nonlinear polarization $\vec{P}_{NL}(\vec{r},t)$ can be obtained by substituting eqs (3) and (4) in eq. (2b).

Under a slowly varying approximation, E_1 and E_2 can be written as

$$E_j(z,t) = F_j(x,y)Q_j(z,t)\mathrm{e}^{iK_{0j}z}, \quad j = 1,2, \tag{5}$$

where $F_j(x,y)$ and K_{0j} , $j = 1,2$, are the fiber mode distributions in the transverse directions and the propagation constants for the two modes, respectively. Then the resulting evolution equations for $Q_j(z,t)$ can be deduced as

$$\begin{aligned} &iQ_{1z} + \frac{i}{v_{g1}}Q_{1t} - \frac{k''}{2}Q_{1tt} + \mu(|Q_1|^2 + B|Q_2|^2)Q_1 = 0, \\ &iQ_{2z} + \frac{i}{v_{g2}}Q_{2t} - \frac{k''}{2}Q_{2tt} + \mu(|Q_2|^2 + B|Q_1|^2)Q_2 = 0, \end{aligned} \tag{6}$$

where v_{g1} and v_{g2} are the group velocities of the two co-propagating waves respectively, $k'' = \left[\frac{\partial^2 K}{\partial\omega^2}\right]_{\omega=\omega_0}$ accounts for the group velocity dispersion, μ is the nonlinearity coefficient and $B = \frac{2+2\sin^2\vartheta}{2+\cos^2\vartheta}$ is the XPM coupling parameter (ϑ - birefringence ellipticity angle which varies between 0 and $\frac{\pi}{2}$). Here we have considered the fiber to be lossless and neglected the third order dispersion term. Further, we have also treated the fiber as strongly birefringent and hence the four-wave mixing terms also can be neglected.

Introducing now the transformation $T = t - \frac{z}{2}\left(\frac{1}{v_{g1}} + \frac{1}{v_{g2}}\right)$, $Z = z$, eqs (6) become

$$\begin{aligned} &iQ_{1Z} + \frac{i}{2}\hat{\rho}Q_{1T} - \frac{k''}{2}Q_{1TT} + \mu(|Q_1|^2 + B|Q_2|^2)Q_1 = 0, \\ &iQ_{2Z} - \frac{i}{2}\hat{\rho}Q_{2T} - \frac{k''}{2}Q_{2TT} + \mu(|Q_2|^2 + B|Q_1|^2)Q_2 = 0, \end{aligned} \tag{7}$$

where $\hat{\rho} = \left(\frac{1}{v_{g1}} - \frac{1}{v_{g2}}\right)$. Then, by using the transformation $\tilde{q_j} = \left(\frac{T_0^2}{k''}\right)^{\frac{1}{2}} Q_j$, $j = 1,2$, $z' = \frac{|k''|Z}{T_0^2}$, $t' = \frac{T}{T_0}$ and redefining z' and t' as z and t respectively, in the anomalous dispersion regime (k'' is negative), we end up with the following set of equations,

$$\begin{aligned} &i\tilde{q}_{1z} + i\rho\tilde{q}_{1t} + \frac{1}{2}\tilde{q}_{1tt} + \mu(|\tilde{q}_1|^2 + B|\tilde{q}_2|^2)\tilde{q}_1 = 0, \\ &i\tilde{q}_{2z} - i\rho\tilde{q}_{2t} + \frac{1}{2}\tilde{q}_{2tt} + \mu(|\tilde{q}_2|^2 + B|\tilde{q}_1|^2)\tilde{q}_2 = 0, \end{aligned} \tag{8}$$

where $\rho = \frac{T_0\hat{\rho}}{2|k''|}$ and T_0 is a measure of the pulse width.

Equations (8) can be further simplified with the transformation

$$\tilde{q}_1 = q_1 e^{i\left(\frac{\rho^2 z}{2} - \rho t\right)}, \tilde{q}_2 = q_2 e^{i\left(\frac{\rho^2 z}{2} + \rho t\right)} \tag{9}$$

and then redefining z as $2z$, we obtain

$$iq_{1z} + q_{1tt} + 2\mu(|q_1|^2 + B|q_2|^2)q_1 = 0,$$
$$iq_{2z} + q_{2tt} + 2\mu(|q_2|^2 + B|q_1|^2)q_2 = 0. \qquad (10)$$

Equation (10) is the 2-CNLS equation in the standard form. It is in general nonintegrable and fails to satisfy the Painlevé property unless $B = 1$ [20,21]. In the later case, we have the celebrated Manakov system of equations [15]

$$iq_{1z} + q_{1tt} + 2\mu(|q_1|^2 + |q_2|^2)q_1 = 0,$$
$$iq_{2z} + q_{2tt} + 2\mu(|q_1|^2 + |q_2|^2)q_2 = 0, \qquad (11)$$

which is a completely integrable soliton system (see §3 below).

Besides the above, there exists other situations also where the above type of CNLS equations arise. For example, in the case of propagation of two optical fields which are having different frequencies, the governing equations have a similar form [4] as eq. (10). Similarly let us consider the simultaneous propagation of N-optical fields (beams) with different wavelengths in a single mode fiber. This kind of simultaneous propagation of multiple beams in a single mode fiber is known as wavelength division multiplexed (WDM) transmission. Here also, by extending the above analysis, one can find that the governing equations are related to a set of N-CNLS equations [4,6],

$$iq_{jz} + q_{jtt} + 2\mu(|q_j|^2 + B\sum_{p=1(\neq j)}^{N}|q_p|^2)q_j = 0, \;\; j = 1,2,\ldots N, \qquad (12)$$

where $q_j(z,t)$ is the slowly varying amplitude of the jth wave. In addition, it is also found that a similar set of N-CNLS equations represents the soliton propagation through an optical fiber array, with q_j representing the envelope in the jth core. System (12) is also found to be nonintegrable, except for the special case $B = 1$, when the Painlevé property is satisfied [20,21]. Also Lax pair exists for this case [22] and the corresponding integrable equations can be written as

$$iq_{jz} + q_{jtt} + 2\mu\sum_{p=1}^{N}|q_p|^2 q_j = 0, \;\; j = 1,2,\ldots N. \qquad (13)$$

Another important medium in which the above type of CNLS equations arise is the photorefractive medium. Let us consider a beam propagating along the z axis, which is allowed to diffract along the x direction only in a photorefractive (PR) medium. The biasing electric field is also applied externally along the x direction . The space-charge field $\vec{E}_{sc} = E_{sc}\,\hat{x}$ induced in the PR medium causes a change in the extraordinary ray index of refraction n_e and the perturbed extraordinary refractive index n'_e is given by [23]

$$n'_e = n_e^2 - n_e^4 r_{33} E_{sc}, \qquad (14)$$

where n_e is the unperturbed refractive index and r_{33} is the electro-optic coefficient. Considering the electric field component $\vec{E}$ of the optical beam, it is found to satisfy the Helmholtz equation

$$\nabla^2\vec{E} + (k_0 n'_e)^2\vec{E} = 0, \qquad (15)$$

where $k_0 = \frac{2\pi}{\lambda_0}$ and λ_0 is the free-space wavelength of the light wave. In the slowly varying envelope, approximation $\overrightarrow{E}$ can be written as

$$\vec{E} = \hat{x}Q(x,z)e^{ikz}, \tag{16}$$

where $k = k_0 n_e$. Substituting eq. (16) in eq. (15) one obtains the following paraxial wave equation of diffraction,

$$iQ_z + \frac{1}{2k}Q_{xx} - \frac{k_0}{2}\left(n_e^3 r_{33} E_{sc}\right)Q = 0. \tag{17}$$

For a strong external biasing condition, starting from the transport model of Kukhtarev *et al* [24], E_{sc} can be obtained as

$$E_{sc} = \frac{E_0\left(I_\infty + I_d\right)}{\left(I + I_d\right)}, \tag{18}$$

where I is the power density of the optical beam, I_d is the so-called dark-irradiance, I_∞ is the power density as $x \to \pm\infty$, which is a constant, and E_0 is the induced space-charge field in that regime ($x \to \pm\infty$).

Incorporation of eq. (18) in eq. (17) and introduction of the transformation, $t = \frac{x}{x_0}$, $\zeta = \frac{z}{2kx_0^2}$ and $\tilde{q} = (\frac{2\eta_0 I_d}{n_e})^{\frac{1}{2}}Q$, where $\eta_0 = (\frac{\mu_0}{\varepsilon_0})^{1/2}$ and x_0 is an arbitrary spatial width, yields the following normalized equation,

$$i\tilde{q}_\zeta + \tilde{q}_{tt} - \frac{2\mu\,(1+\rho)\,\tilde{q}}{(1+|\tilde{q}|^2)} = 0, \tag{19}$$

where $\rho = \frac{I_\infty}{I_d}$ and $\mu = \left(\frac{(k_0 x_0)^2 n_e^4 r_{33}}{2}\right)E_0$.

In the limit $|q|^2 << 1$, which corresponds to the low-amplitude case, with the transformation $\tilde{q} = qe^{-2i\mu\zeta}$ and with the redefinition of ζ as z, the above equation becomes the nonlinear Schrödinger equation,

$$iq_z + q_{tt} + 2\mu|q|^2 q = 0. \tag{20}$$

In the case of incoherent beam propagation in a biased photorefractive crystal, which is a noninstantaneous nonlinear media, the diffraction behaviour of that incoherent beam is to be treated somewhat differently rather than as in the previous case. Along the lines of ref. [10] the diffraction behaviour of an incoherent beam can be effectively described by the sum of the intensity contributions from all its coherent components. Then under the assumptions and approximations which we have considered in the above one component case, the governing equation of N-self-trapped mutually incoherent wave packets in such a media is given by the N-CNLS equations (13). In this context, q_j represents the jth component of the beam, z and t represent the normalized co-ordinates along the direction of propagation and the transverse co-ordinates respectively, and $\sum_{p=1}^N |q_p|^2$ represents the change in the refractive index profile created by all the incoherent components of the light beam and 2μ represents the strength of nonlinearity, as before.

3. Lax pairs and integrability of CNLS equations

One of the salient features of the integrable systems is the existence of Lax pairs [25]. Manakov had shown that the integrable 2-CNLS eq. (11) admits the following Lax pair [15],

$$L = \begin{pmatrix} -i\lambda & q_1 & q_2 \\ -q_1^* & i\lambda & 0 \\ -q_2^* & 0 & i\lambda \end{pmatrix}, \quad B = \begin{pmatrix} -2i\lambda^2 + i\mu(|q_1|^2+|q_2|^2) & 2\lambda q_1 + iq_{1t} & 2\lambda q_2 + iq_{2t} \\ -2\lambda q_1^* + iq_{1t}^* & 2i\lambda^2 - i\mu|q_1|^2 & -i\mu q_1^* q_2 \\ -2\lambda q_2^* + iq_{2t}^* & -i\mu q_1 q_2^* & 2i\lambda^2 - i\mu|q_2|^2 \end{pmatrix}, \tag{21}$$

where λ is the constant spectral parameter. It can also be verified that the compatability condition

$$L_z - B_t + [L,B] = 0 \tag{22}$$

leads to the original Manakov system (11). In his work Manakov [15] had obtained the one-soliton solution of eq. (11) and presented an analysis of the asymptotic properties of the two-soliton solution by using the Inverse Scattering Transform (IST) technique. The proof of the Liouville type complete integrability of the Manakov system (11) requires the existence of an infinite number of involutive integrals of motion and a canonical transformation to action-angle variables so that the Hamiltonian for the Manakov system can be expressed as a function of action variables alone. Consequently the resulting Hamilton's equations of motion can be trivially integrated and the system becomes completely integrable in the Liouville sense. This can be proved in the following way using the standard procedure of AKNS type systems [25].

Now let us consider the linear eigenvalue problem and its 'z-evolution' associated with the Manakov system,

$$V_t = LV, \tag{23a}$$

$$V_z = BV, \tag{23b}$$

where

$$V = (v_1, v_2, v_3)^T. \tag{23c}$$

In its component form the linear eigenvalue problem (23a) reads as

$$v_{1t} = -i\lambda v_1 + q_1 v_2 + q_2 v_3, \tag{24a}$$

$$v_{2t} = i\lambda v_2 - q_1^* v_1, \tag{24b}$$

$$v_{3t} = i\lambda v_3 - q_2^* v_1. \tag{24c}$$

Now eq. (24a) can be rewritten as

$$ln(v_1 e^{i\lambda t}) = \int_{-\infty}^{t} \mathrm{d}t' \left[q_1 \Gamma^{(1)} + q_2 \Gamma^{(2)}\right], \tag{25a}$$

where $\Gamma^{(1)} = \frac{v_2}{v_1}$ and $\Gamma^{(2)} = \frac{v_3}{v_1}$. From the analytical properties of the function $v_1 e^{i\lambda t}$, we can introduce the scattering coefficient $a(\lambda)$ as

$$a(\lambda) = v_1 e^{i\lambda t}|_{t\to\infty}. \tag{25b}$$

Further, from the analytical properties of $a(\lambda)$ and the fact that it is independent of z as may be deduced from eq. (23b), one can expand $\ln[a(\lambda)]$ as

$$\ln[a(\lambda)] = \sum_{n=1}^{\infty} \frac{c_n}{\lambda^n}. \tag{26}$$

On the other hand, expanding $\Gamma^{(1)}$ and $\Gamma^{(2)}$ as power series in $\frac{1}{\lambda}$ as

$$\Gamma^{(a)} = \sum_{n=1}^{\infty} \Gamma_n^{(a)} \lambda^{-n}, \quad a = 1,2 \tag{27}$$

and using (26) and (27) in eq. (25a), one can obtain an infinite number of conserved quantities, as

$$c_n = \int_{-\infty}^{\infty} \mathrm{d}t (q_1 \Gamma_n^{(1)} + q_2 \Gamma_n^{(2)}), \quad n \geq 1. \tag{28}$$

The quantities $\Gamma_n^{(a)}$ can be easily shown to satisfy the following set of recursion equations by writing down the coupled set of Riccati equations satisfied by $\Gamma^{(1)}$ and $\Gamma^{(2)}$ as deduced from the linear eigenvalue problem (24). Its form reads

$$2i\Gamma_{n+1}^{(1)} = \Gamma_{nt}^{(1)} + q_1 \sum_{i=1}^{n-1} \Gamma_{n-i}^{(1)} \Gamma_i^{(1)} + q_2 \sum_{i=1}^{n-1} \Gamma_{n-i}^{(1)} \Gamma_i^{(2)}, \tag{29a}$$

$$2i\Gamma_{n+1}^{(2)} = \Gamma_{nt}^{(2)} + q_2 \sum_{i=1}^{n-1} \Gamma_{n-i}^{(2)} \Gamma_i^{(2)} + q_1 \sum_{i=1}^{n-1} \Gamma_{n-i}^{(2)} \Gamma_i^{(1)}, \tag{29b}$$

with $\Gamma_1^{(a)} = \frac{-iq_a^*}{2}$, $a = 1,2$. As a consequence, we can write the infinite number of conserved quantities straightaway. The first three of them for the Manakov system read as

$$c_1 = -\frac{i}{2} \int_{-\infty}^{+\infty} \mathrm{d}t (|q_1|^2 + |q_2|^2), \tag{30a}$$

$$c_2 = -\frac{i}{4} \int_{-\infty}^{+\infty} \mathrm{d}t [-i(q_1 q_{1t}^* + q_2 q_{2t}^*)] \tag{30b}$$

$$\text{and} \quad c_3 = -\frac{i}{8} \int_{-\infty}^{+\infty} \mathrm{d}t [(q_1 q_{1tt}^* + q_2 q_{2tt}^*) + (|q_1|^2 + |q_2|^2)^2]. \tag{30c}$$

The above conserved quantities can be identified as the number operator N, the total momentum P and the Hamiltonian (total energy) of the system, respectively. One can show that the fields $q(z)$ and $q^*(z)$ satisfy the canonical Poisson bracket relations

$$\{q_i(x), q_j^*(y)\} = \delta(x-y)\delta_{ij}, \tag{31a}$$

$$\{q_i(x), q_i^*(y)\} = 0, \ \ i,j = 1,2 \tag{31b}$$

with respect to which the conserved quantities c_n's are in involution.

Though the existence of Lax pairs accounts for integrablity, the proof of complete integrability of the system requires the existence of a canonical transformation to action and angle variables which allows the Hamiltonian of the system to be a function of the actions only.The existence of such a transformation to the Manakov system confirms its complete integrable nature in Liouville sense.

Further, the set of N-CNLS eq. (13) is also found to be integrable [20,21]. Then following a method similar to the above 2-CNLS equations and extending the procedure for (3×3) matrix linear eigenvalue problem to $(n\times n)$ matrix problem, the Lax pair associated with the N-CNLS eq. (13) can be written as

$$L = \begin{pmatrix} -i\lambda & q_1 & q_2 & \dots & q_N \\ -q_1^* & i\lambda & 0 & \dots & 0 \\ -q_2^* & 0 & i\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -q_N^* & 0 & 0 & \dots & i\lambda \end{pmatrix}, \tag{32a}$$

$$B = \begin{pmatrix} -2i\lambda^2 + i\mu\sum_{p=1}^{N}|q_p|^2 & 2\lambda q_1 + iq_{1t} & 2\lambda q_2 + iq_{2t} & \dots & 2\lambda q_N + iq_{Nt} \\ -2\lambda q_1^* + iq_{1t}^* & 2i\lambda^2 - i\mu|q_1|^2 & -i\mu q_1^* q_2 & \dots & -i\mu q_1^* q_N \\ -2\lambda q_2^* + iq_{2t}^* & -i\mu q_2^* q_1 & 2i\lambda^2 - i\mu|q_2|^2 & \dots & -i\mu q_2^* q_N \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -2\lambda q_N^* + iq_{Nt}^* & -i\mu q_N^* q_1 & -i\mu q_N^* q_2 & \dots & 2i\lambda^2 - i\mu|q_N|^2 \end{pmatrix}. \tag{32b}$$

Here also the conservation laws can be deduced along the lines of the 2-CNLS case. In order to write these conserved quantities one has to look for the coupled set of Riccati equations associated with the Lax operators and then obtain the relevant recurrence relations, which will be $(n-1)$ coupled first order differential equations in $\Gamma_n^{(a)}$'s.

4. Soliton solutions of the CNLS equations

Now in order to facilitate the understanding of the underlying dynamics of the above integrable CNLS equations it is essential to obtain the soliton solutions associated with these integrable systems. In this regard, by applying Hirota's technique [26], we point out that the most general bright one-soliton and two-soliton solutions for the Manakov system (11) can be easily obtained and novel properties deduced [16,27]. The procedure can be straightaway extended to obtain N-soliton solutions as well; however, we will confine ourselves to the study of one- and two-soliton solutions alone here.

Considering eq. (11) and by making the bilinear transformation $q_j = \frac{g^{(j)}}{f}$, $j = 1, 2$, where $g^{(j)}(z,t)$'s are complex functions while $f(z,t)$ is a real function, the following bilinear equations can be obtained,

$$(iD_z + D_t^2)(g^{(j)}.f) = 0, \tag{33a}$$

$$D_t^2(f.f) = 2\mu(g^{(1)}g^{(1)*} + g^{(2)}g^{(2)*}), \tag{33b}$$

where the Hirota's bilinear operators D_z and D_t are defined by

$$D_z^n D_t^m (a.b) = \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m a(z,t)b(z',t')\Big|_{(z=z',t=t')}. \tag{33c}$$

The above set of equations can be solved by introducing the following power series expansions to $g^{(j)}$'s and f:

$$g^{(j)} = \lambda g_1^{(j)} + \lambda^3 g_3^{(j)} + \cdots, \quad j = 1,2, \tag{34a}$$

$$f = 1 + \lambda^2 f_2 + \lambda^4 f_4 + \cdots, \tag{34b}$$

where λ is the formal expansion parameter. The resulting set of equations, after collecting the terms with the same power in λ, can be solved to obtain the forms of $g^{(j)}$ and f. For illustrative purposes we explain below the procedure to obtain the one- and two-soliton solutions.

A. *One-soliton solution of the 2-CNLS equation*

In order to get the one-soliton solution of the 2-CNLS equation, the power series expansions for $g^{(1)}$, $g^{(2)}$ and f are terminated as follows:

$$g^{(1)} = \lambda g_1^{(1)} \tag{35a}$$

$$g^{(2)} = \lambda g_1^{(2)} \tag{35b}$$

$$f = 1 + \lambda^2 f_2. \tag{35c}$$

Substituting eqs (35) in the bilinear eq. (33) we obtain the following differential equations at various powers of λ:

$$\lambda: \quad \hat{D}_1(g_1^{(j)}.1) = 0, \tag{36a}$$

$$\lambda^2: \quad \hat{D}_2(f_2.1 + 1.f_2) = 2\mu(g_1^{(1)}.g_1^{(1)*} + g_1^{(2)}g_1^{(2)*}), \tag{36b}$$

$$\lambda^3: \quad \hat{D}_1(g_1^{(j)}.f_2 + g_3^{(j)}.1) = 0, \tag{36c}$$

$$\lambda^4: \quad \hat{D}_2(f_2.f_2) = 0, \quad j = 1,2, \tag{36d}$$

where $\hat{D}_1 = (iD_z + D_t^2)$ and $\hat{D}_2 = D_t^2$. The solution which is consistent with the above system is

$$g^{(1)} = \alpha_1^{(1)} \mathrm{e}^{\eta_1}, \tag{37a}$$

$$g^{(2)} = \alpha_1^{(2)} \mathrm{e}^{\eta_1}, \tag{37b}$$

$$f = 1 + \mathrm{e}^{\eta_1 + \eta_1^* + R}, \quad e^R = \frac{\mu(|\alpha_1^{(1)}|^2 + |\alpha_1^{(2)}|^2)}{(k_1 + k_1^*)^2}, \tag{37c}$$

where $\eta_1 = k_1(t + ik_1 z)$, $\alpha_1^{(1)}$, $\alpha_1^{(2)}$ and k_1 are complex parameters. Then the resulting bright one-soliton solution is obtained as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \alpha_1^{(1)} \\ \alpha_1^{(2)} \end{pmatrix} \frac{e^{\eta_1}}{1+e^{\eta_1+\eta_1^*+R}} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \frac{k_{1R}e^{i\eta_{1I}}}{\cosh(\eta_{1R}+\frac{R}{2})}. \tag{38}$$

Here $\sqrt{\mu}(A_1, A_2) = \sqrt{\mu}(\alpha_1^{(1)}, \alpha_1^{(2)})/(\mu(|\alpha_1^{(1)}|^2 + |\alpha_1^{(2)}|^2))^{1/2}$ represents the unit polarization vector, $k_{1R}A_j$, $j = 1, 2$ gives the amplitude of the jth mode and $2k_{1I}$ the soliton velocity.

B. *Two-soliton solution*

Here the series (34) is terminated as

$$g^{(1)} = \lambda g_1^{(1)} + \lambda^3 g_3^{(1)}, \tag{39a}$$
$$g^{(2)} = \lambda g_1^{(2)} + \lambda^3 g_3^{(2)}, \tag{39b}$$
$$f = 1 + \lambda^2 f_2 + \lambda^4 f_4, \tag{39c}$$

to obtain the bright two-soliton solution of eq. (11). Then the resulting partial differential equations at various powers of λ are as follows:

$$\lambda: \quad \hat{D}_1(g_1^{(j)}.1) = 0, \tag{40a}$$
$$\lambda^2: \quad \hat{D}_2(1.f_2 + f_2.1) = 2\mu(g_1^{(1)}g_1^{(1)*} + g_1^{(2)}g_1^{(2)*}), \tag{40b}$$
$$\lambda^3: \quad \hat{D}_1(g_1^{(j)}.f_2 + g_3^{(j)}.1) = 0, \tag{40c}$$
$$\lambda^4: \quad \hat{D}_2(1.f_4 + f_2.f_2 + f_4.1) = 2\mu(g_1^{(1)}g_3^{(1)*} + g_3^{(1)}g_1^{(1)*} + g_1^{(2)}g_3^{(2)*} + g_3^{(2)}g_1^{(2)*}), \tag{40d}$$
$$\lambda^5: \quad \hat{D}_1(g_1^{(j)}.f_4 + g_3^{(j)}.f_2) = 0, \tag{40e}$$
$$\lambda^6: \quad \hat{D}_2(f_2.f_4 + f_4.f_2) = 2\mu(g_3^{(1)}g_3^{(1)*} + g_3^{(2)}g_3^{(2)*}), \tag{40f}$$
$$\lambda^7: \quad \hat{D}_1(g_3^{(j)}.f_4) = 0, \tag{40g}$$
$$\lambda^8: \quad \hat{D}_2(f_4.f_4) = 0, \quad j = 1, 2. \tag{40h}$$

The solutions compatible with eqs (40) are

$$g_1^{(1)} = \alpha_1^{(1)}e^{\eta_1} + \alpha_2^{(1)}e^{\eta_2}, \tag{41a}$$
$$g_1^{(2)} = \alpha_1^{(2)}e^{\eta_1} + \alpha_2^{(2)}e^{\eta_2}, \tag{41b}$$
$$g_3^{(1)} = e^{\eta_1+\eta_1^*+\eta_2+\delta_1} + e^{\eta_1+\eta_2+\eta_2^*+\delta_2}, \tag{41c}$$
$$g_3^{(2)} = e^{\eta_1+\eta_1^*+\eta_2+\delta_1'} + e^{\eta_1+\eta_2+\eta_2^*+\delta_2'}, \tag{41d}$$
$$f_2 = e^{\eta_1+\eta_1^*+R_1} + e^{\eta_1+\eta_2^*+\delta_0} + e^{\eta_1^*+\eta_2+\delta_0^*} + e^{\eta_2+\eta_2^*+R_2}, \tag{41e}$$
$$f_4 = e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_3}, \tag{41f}$$

where

$$\eta_i = k_i(t+ik_iz),\ \ e^{\delta_0} = \frac{\kappa_{12}}{k_1+k_2^*},\ \ e^{R_1} = \frac{\kappa_{11}}{k_1+k_1^*},\ \ e^{R_2} = \frac{\kappa_{22}}{k_2+k_2^*},$$
$$e^{\delta_1} = \frac{k_1-k_2}{(k_1+k_1^*)(k_1^*+k_2)}(\alpha_1^{(1)}\kappa_{21}-\alpha_2^{(1)}\kappa_{11}),$$
$$e^{\delta_2} = \frac{k_2-k_1}{(k_2+k_2^*)(k_1+k_2^*)}(\alpha_2^{(1)}\kappa_{12}-\alpha_1^{(1)}\kappa_{22}),$$
$$e^{\delta_1'} = \frac{k_1-k_2}{(k_1+k_1^*)(k_1^*+k_2)}(\alpha_1^{(2)}\kappa_{21}-\alpha_2^{(2)}\kappa_{11}),$$
$$e^{\delta_2'} = \frac{k_2-k_1}{(k_2+k_2^*)(k_1+k_2^*)}(\alpha_2^{(2)}\kappa_{12}-\alpha_1^{(2)}\kappa_{22}),$$
$$e^{R_3} = \frac{|k_1-k_2|^2}{(k_1+k_1^*)(k_2+k_2^*)|k_1+k_2^*|^2}(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21})$$
$$\text{and } \kappa_{ij} = \frac{\mu(\alpha_i^{(1)}\alpha_j^{(1)*}+\alpha_i^{(2)}\alpha_j^{(2)*})}{k_i+k_j^*},\ i,j=1,2. \tag{41g}$$

Then we can write the final form of the bright two-soliton solution of eq. (11) as

$$q_1 = \frac{\alpha_1^{(1)}e^{\eta_1}+\alpha_2^{(1)}e^{\eta_2}+e^{\eta_1+\eta_1^*+\eta_2+\delta_1}+e^{\eta_1+\eta_2+\eta_2^*+\delta_2}}{D}, \tag{42a}$$
$$q_2 = \frac{\alpha_1^{(2)}e^{\eta_1}+\alpha_2^{(2)}e^{\eta_2}+e^{\eta_1+\eta_1^*+\eta_2+\delta_1'}+e^{\eta_1+\eta_2+\eta_2^*+\delta_2'}}{D}, \tag{42b}$$

where

$$D = 1+e^{\eta_1+\eta_1^*+R_1}+e^{\eta_1+\eta_2^*+\delta_0}+e^{\eta_1^*+\eta_2+\delta_0^*} +e^{\eta_2+\eta_2^*+R_2}+e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_3}. \tag{42c}$$

The above most general bright two-soliton solution with six arbitrary complex parameters k_1, k_2, $\alpha_1^{(j)}$ and $\alpha_2^{(j)}$, $j=1,2$ corresponds to a shape changing (inelastic) collision of two bright solitons which will be explained in the following sections.

C. *One- and two-soliton solutions of the N-CNLS equation*

By extending the above procedure further to the integrable N-CNLS eq. (13) one can obtain the one-soliton and two-soliton solutions [17] of this system as well. After making the bilinear transformation $q_j = g^{(j)}/f,\ j=1,2,\ldots,N$ in eq. (13), one can get a set of bilinear equations of the form (40) but now with $j=1,2,3,\ldots,N$. Then by expanding $g^{(j)}$s and f in power series up to N terms and following the procedure mentioned above, the one-soliton and two-soliton solutions of eq. (13) can be obtained.

(a) *One-soliton solution:*

$$\left(q_1,q_2,\ldots,q_N\right)^T = \frac{k_{1R}e^{i\eta_{1I}}}{\cosh\left(\eta_{1R}+\frac{R}{2}\right)}\left(A_1,A_2,\ldots,A_N\right)^T, \tag{43}$$

where $\eta_1 = k_1(t + ik_1 z)$, $A_j = \alpha_1^{(j)}/\Delta$, $\Delta = (\mu(\sum_{j=1}^{N}|\alpha_1^{(j)}|^2))^{1/2}$, $e^R = \Delta^2/(k_1+k_1^*)^2$, $\alpha_1^{(j)}$ and k_1, $j=1,2\ldots,N$, are $(N+1)$ arbitrary complex parameters.

(b) *Two-soliton solution:*

$$\begin{aligned}(q_1,q_2,\ldots,q_N)^T = \Big(&\alpha_1^{(1)}e^{\eta_1} + \alpha_2^{(1)}e^{\eta_2} + e^{\eta_1+\eta_1^*+\eta_2+\delta_{11}} \\ &+e^{\eta_1+\eta_2+\eta_2^*+\delta_{21}},\ \alpha_1^{(2)}e^{\eta_1} + \alpha_2^{(2)}e^{\eta_2} \\ &+e^{\eta_1+\eta_1^*+\eta_2+\delta_{12}} + e^{\eta_1+\eta_2+\eta_2^*+\delta_{22}},\ldots, \\ &\alpha_1^{(N)}e^{\eta_1} + \alpha_2^{(N)}e^{\eta_2} + e^{\eta_1+\eta_1^*+\eta_2+\delta_{1N}} \\ &+e^{\eta_1+\eta_2+\eta_2^*+\delta_{2N}}\Big)^T/D, \end{aligned} \tag{44}$$

where $D = 1 + e^{\eta_1+\eta_1^*+R_1} + e^{\eta_1+\eta_2^*+\delta_0} + e^{\eta_1^*+\eta_2+\delta_0^*} + e^{\eta_2+\eta_2^*+R_2} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_3}$, $e^{R_1} = \kappa_{11}/(k_1+k_1^*)$, $e^{R_2} = \kappa_{22}/(k_2+k_2^*)$, $e^{\delta_0} = \kappa_{12}/(k_1+k_2^*)$, $e^{\delta_{1m}} = ((k_1-k_2)(\alpha_1^{(m)}\kappa_{21} - \alpha_2^{(m)}\kappa_{11}))/((k_1+k_1^*)(k_1^*+k_2))$, $e^{\delta_{2m}} = ((k_2-k_1)(\alpha_2^{(m)}\kappa_{12} - \alpha_1^{(m)}\kappa_{22}))/((k_2+k_2^*)(k_1+k_2^*))$, $e^{R_3} = (|k_1-k_2|^2(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}))/((k_1+k_1^*)(k_2+k_2^*)|k_1+k_2^*|^2)$ and $\kappa_{il} = \mu\sum_{n=1}^{N}\alpha_i^{(n)}\alpha_l^{(n)*}/(k_i+k_l^*)$, where $i,\ l = 1,2$ and $m = 1,2,\ldots,N$. As mentioned earlier, higher order solitons can also be constructed in a similar way with more labour; however, we will not consider them in this article.

5. Shape changing collisions in coupled nonlinear Schrödinger equations

The novel collision properties associated with the CNLS equations can be revealed by analysing the asymptotic forms of the two-soliton solutions [16,27]. In this connection, let us first consider the two-soliton solution (42) of the Manakov system, which is an integrable 2-CNLS system. Without loss of generality we assume that $k_{jR} > 0$ and $k_{1I} > k_{2I}$, $k_j = k_{jR} + ik_{jI}$, $j = 1,2$, which corresponds to a head-on collision of the solitons. One can easily check that asymptotically the two-soliton solution becomes two well separated solitons S_1 and S_2. For the above parametric choice, the variables η_{jR}'s for the two solitons $(\eta_j = \eta_{jR} + i\eta_{jI})$ behave asymptotically as (i) $\eta_{1R} \sim 0$, $\eta_{2R} \to \pm\infty$ as $z \to \pm\infty$ and (ii) $\eta_{2R} \sim 0$, $\eta_{1R} \to \mp\infty$ as $z \to \pm\infty$. This leads to the following asymptotic forms for the two-soliton solution.

A. *Before collision (limit $z \to -\infty$)*

In the limit $z \to -\infty$, the solution (42) can be easily seen to take the following forms.
(a) S_1 $(\eta_{1R} \sim 0,\ \eta_{2R} \to -\infty)$:

$$\begin{aligned}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &\to \begin{pmatrix} \alpha_1^{(1)} \\ \alpha_1^{(2)} \end{pmatrix} \frac{e^{\eta_1}}{1+e^{\eta_1+\eta_1^*+R_1}} \\ &= \begin{pmatrix} A_1^{1-} \\ A_2^{1-} \end{pmatrix} k_{1R}e^{i\eta_{1I}}\operatorname{sech}\left(\eta_{1R} + \frac{R_1}{2}\right), \end{aligned} \tag{45}$$

where $(A_1^{1-}, A_2^{1-}) = [\mu(\alpha_1^{(1)}\alpha_1^{(1)*} + \alpha_1^{(2)}\alpha_1^{(2)*})]^{-\frac{1}{2}}(\alpha_1^{(1)}, \alpha_1^{(2)})$. In A_i^{1-}, $i = 1, 2$, superscript 1– denotes S_1 at the limit $z \to -\infty$ and subscripts 1 and 2 refer to the modes q_1 and q_2, respectively.
(b) S_2 ($\eta_{2R} \sim 0$, $\eta_{1R} \to \infty$):

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to \begin{pmatrix} e^{\delta_1 - R_1} \\ e^{\delta_1' - R_1} \end{pmatrix} \frac{e^{\eta_2}}{1 + e^{\eta_2 + \eta_2^* + R_3 - R_1}}$$
$$= \begin{pmatrix} A_1^{2-} \\ A_2^{2-} \end{pmatrix} k_{2R} e^{i\eta_{2I}} \operatorname{sech}\left(\eta_{2R} + \frac{(R_3 - R_1)}{2}\right), \tag{46}$$

where $(A_1^{2-}, A_2^{2-}) = (\frac{a_1}{a_1^*})\ c\ [\mu(\alpha_2^{(1)}\alpha_2^{(1)*} + \alpha_2^{(2)}\alpha_2^{(2)*})]^{-\frac{1}{2}} \left[(\alpha_1^{(1)}, \alpha_1^{(2)})\kappa_{11}^{-1} - (\alpha_2^{(1)}, \alpha_2^{(2)})\kappa_{21}^{-1}\right]$ in which $a_1 = (k_1 + k_2^*)\left[(k_1 - k_2)(\alpha_1^{(1)*}\alpha_2^{(1)} + \alpha_1^{(2)*}\alpha_2^{(2)})\right]^{\frac{1}{2}}$ and $c = \left[\frac{1}{|\kappa_{12}|^2} - \frac{1}{\kappa_{11}\kappa_{22}}\right]^{-\frac{1}{2}}$. Other quantities have been defined earlier under eq. (41).

B. *After collision (limit $z \to \infty$)*

Similarly, for $z \to \infty$, we have the following forms.
(a) S_1 ($\eta_{1R} \sim 0$, $\eta_{2R} \to \infty$):

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to \begin{pmatrix} e^{\delta_2 - R_2} \\ e^{\delta_2' - R_2} \end{pmatrix} \frac{e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + R_3 - R_2}}$$
$$= \begin{pmatrix} A_1^{1+} \\ A_2^{1+} \end{pmatrix} k_{1R} e^{i\eta_{1I}} \operatorname{sech}\left(\eta_{1R} + \frac{(R_3 - R_2)}{2}\right), \tag{47}$$

where $(A_1^{1+}, A_2^{1+}) = (\frac{a_2}{a_2^*})\ c\ [\mu(\alpha_1^{(1)}\alpha_1^{(1)*} + \alpha_1^{(2)}\alpha_1^{(2)*})]^{-\frac{1}{2}} \left[(\alpha_1^{(1)}, \alpha_1^{(2)})\kappa_{12}^{-1} - (\alpha_2^{(1)}, \alpha_2^{(2)})\kappa_{22}^{-1}\right]$ in which $a_2 = (k_2 + k_1^*)\left[(k_1 - k_2)(\alpha_1^{(1)}\alpha_2^{(1)*} + \alpha_1^{(2)}\alpha_2^{(2)*})\right]^{\frac{1}{2}}$.
(b) S_2 ($\eta_{2R} \sim 0$, $\eta_{1R} \to -\infty$):

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to \begin{pmatrix} \alpha_2^{(1)} \\ \alpha_2^{(2)} \end{pmatrix} \frac{e^{\eta_2}}{1 + e^{\eta_2 + \eta_2^* + R_2}}$$
$$= \begin{pmatrix} A_1^{2+} \\ A_2^{2+} \end{pmatrix} k_{2R} e^{i\eta_{2I}} \operatorname{sech}\left(\eta_{2R} + \frac{R_2}{2}\right), \tag{48}$$

where $(A_1^{2+}, A_2^{2+}) = [\mu(\alpha_2^{(1)}\alpha_2^{(1)*} + \alpha_2^{(2)}\alpha_2^{(2)*})]^{-\frac{1}{2}}(\alpha_2^{(1)}, \alpha_2^{(2)})$.

The asymptotic forms of the solitons S_1 and S_2 after interaction, given respectively by eqs (47) and (48), can be related to the forms before interaction, given by eqs (45) and (46) respectively, by introducing a transition matrix T_j^l such that

$$A_j^{l+} = A_j^{l-} T_j^l, \quad j, l = 1, 2, \tag{49a}$$

where the superscripts $l\pm$ represent the solitons designated as S_1 and S_2 at $z \to \pm\infty$. Here

$$|T_j^1|^2 = |1-\lambda_2(\alpha_2^{(j)}/\alpha_1^{(j)})|^2/|1-\lambda_1\lambda_2|, \tag{49b}$$

$$|T_j^2|^2 = |1-\lambda_1\lambda_2|/|1-\lambda_1(\alpha_1^{(j)}/\alpha_2^{(j)})|^2, \quad j=1,2, \tag{49c}$$

$$\lambda_1 = \kappa_{21}/\kappa_{11} \quad \text{and} \quad \lambda_2 = \kappa_{12}/\kappa_{22}. \tag{49d}$$

Besides the above changes in the amplitudes, there is a phase shift of the soliton positions as indicated below.

The expressions (49) clearly show that there is an intensity redistribution among the two modes of the solitons S_1 and S_2 and that the transition matrices act as a measure of this redistributed intensity. However, we can also check easily that in the special case in which the parameters $\alpha_i^{(j)}, i, j = 1,2$, satisfy the relation $\alpha_1^{(1)}/\alpha_2^{(1)} = \alpha_1^{(2)}/\alpha_2^{(2)}$, the transition matrices take the value $T_i^j = 1$. In this case, we have the standard pure shape preserving, elastic collision of solitons that is familiar in the literature. However for all other values of $\alpha_i^{(j)}, i, j = 1,2$, such that relation $\alpha_1^{(1)}/\alpha_2^{(1)} \neq \alpha_1^{(2)}/\alpha_2^{(2)}$, the amplitudes of the solitons do undergo changes and we have shape changing (inelastic) collisions.

(a) *Intensity (amplitude) redistribution during the shape changing collision process*

The amplitude change of individual modes of S_1 and S_2 during the collision process shows some interesting features. Actually, from eqs (45) to (48) we observe that the initial amplitudes of the two modes of the two solitons $(A_1^{1-}k_{1R}, A_2^{1-}k_{1R})$ and $(A_1^{2-}k_{2R}, A_2^{2-}k_{2R})$ undergo a redistribution among them and the solitons emerge with amplitudes $(A_1^{1+}k_{1R}, A_2^{1+}k_{1R})$ and $(A_1^{2+}k_{2R}, A_2^{2+}k_{2R})$, respectively, where $A_i^{l\pm}, i, l = 1,2$, are given above. The changes in the amplitudes due to collision are essentially given by the transition matrices T_i^j, see eqs (49). It can be easily checked that for a suitable choice of parameters it is even possible to make one of the T_i^j's vanish so that one of the modes after collision (or before collision) has zero intensity (see figure 1).

Another noticeable observation of this interaction process is that even though there is a redistribution of intensity among the modes the total intensity of the individual solitons is conserved, that is, $|A_1^{n\pm}|^2 + |A_2^{n\pm}|^2 = \frac{1}{\mu}$, $n = 1,2$, see eqs (45)–(48).

(b) *Phase shift of solitons during the collision process*

Another important quantity which is altered during the collision is the phase (which is related to the position of the soliton). For the soliton S_1 the initial phase $\frac{R_1}{2}$ becomes $\frac{R_3-R_2}{2}$ after collision and the corresponding phase shift is given by

$$\begin{aligned}\Phi^1 &= \frac{R_3-R_1-R_2}{2} \\ &= \frac{1}{2}\log\left[\frac{|k_1-k_2|^2(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21})}{|k_1+k_2^*|^2\kappa_{11}\kappa_{22}}\right].\end{aligned} \tag{50}$$

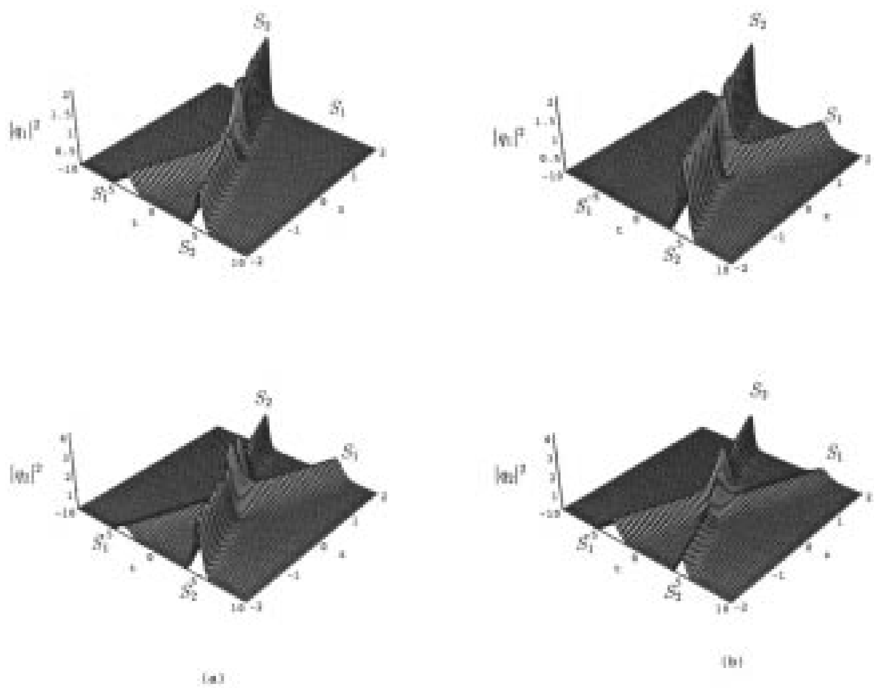


Figure 1. Two distinct possibilities of the shape changing collision in the Manakov system.

Similarly the phase shift for the soliton S_2 is given by

$$\Phi^2 = -\frac{(R_3 - R_1 - R_2)}{2} = -\Phi^1. \tag{51}$$

Note that the phase shift depends very much on the soliton parameters $\alpha_i^{(j)}$'s through their occurrence in the expressions for κ_{ij}'s and so on the amplitudes of the modes.

(c) *Change in the relative separation distance of the solitons*

As a consequence of the above phase shift, the relative separation distances $x_{12}^{\pm}$ (position of S_2 (at $t \to \pm\infty$)minus position of S_1 (at $t \to \pm\infty$)) also do vary during the collision process. In such a pair-wise collision, the change in the relative separation distance is found to be

$$\Delta x_{12} = x_{12}^{-} - x_{12}^{+} = \frac{(k_{1R} + k_{2R})}{2k_{1R}k_{2R}}\Phi^1. \tag{52}$$

Again Δx_{12} depends upon the parameters $\alpha_i^{(j)}$'s and so on the amplitudes of the modes.

(d) *Role of α_i^j's in the collision process*

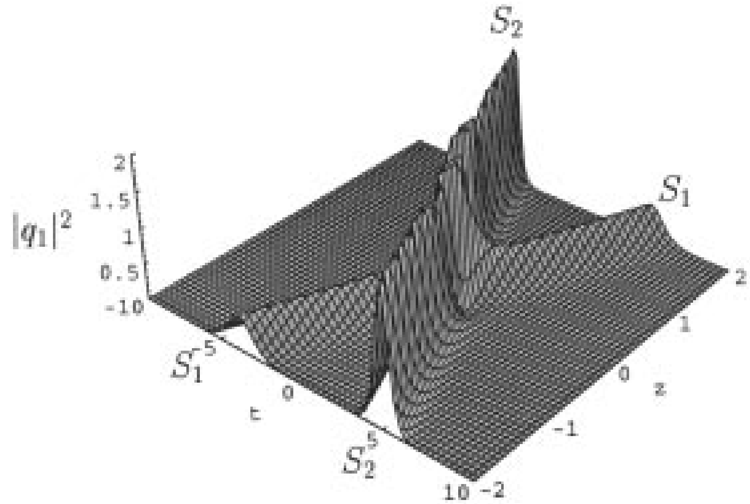
It can be straightaway seen that the above mentioned three quantities, intensity (amplitude) redistribution, phase shift and relative separation distances, characterising the CNLS soliton collisions, are nontrivially dependent on the complex parameters α_i^j's, which in turn determine the quantities $A_i^{j\pm}$ defining the amplitudes of the modes. These parameters play a pivotal role in the shape changing collision process. In particular, the change in the amplitudes of the two modes of S_1 and S_2 can be varied dramatically by changing $\alpha_i^{(j)}$'s and even the amplitudes before and after interaction can be made equal, a case corresponding to elastic collision, for the particular choice $\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}} = \frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}$ as the transition matrices T_i^j, $i,j = 1,2$, are equal to one in this special case. For all other choices, the amplitudes undergo changes due to collision and under suitable circumstances the amplitude of one of the modes (either before or after collision) can even vanish, showing in a dramatic way the shape changing nature of the collisions.

It can also be observed from eqs (50)–(52) that not only the amplitudes of the solitons but also the phases and hence the relative separation distances between them depend on the complex parameters $\alpha_i^{(j)}$'s. As a result, their variation during collision is also determined predominantly by $\alpha_i^{(j)}$'s.

Now let us look at the possible ways by which such shape changing collision can occur in the Manakov system. We can identify two distinct types of interactions for each of the solitons. The first possibility is an enhancement of intensity in any one of the modes of either one of the solitons (say S_1) and suppression in the remaining mode of the corresponding soliton with commensurate changes in the other soliton. The other possibility is an interaction which allows one of the modes of either one of the solitons (say S_1) to get suppressed while allowing the other mode of the corresponding soliton to get enhanced (with corresponding changes in S_2). In either of the cases the intensity may be completely or partially suppressed (enhanced), as determined by the transition matrices T_i^j's.

For illustrative purposes, we have shown the head-on collision of two solitons for the parametric values, $k_1 = 1+i$, $k_2 = 2-i$, $\alpha_1^{(1)} = \alpha_1^{(2)} = \alpha_2^{(2)} = 1$, and $\alpha_2^{(1)} = \frac{(39+i80)}{89}$ in figure 1a. Here initially the time profiles of the two solitons are evenly split between the two components q_1 and q_2. At the large positive z end the profile of the S_1 soliton is almost completely suppressed in the q_1 component while it is suitably enhanced in the q_2 component. Also there is a rearrangement of the amplitudes in the second soliton S_2 in both the modes also. In figure 1b, we have shown the reverse possibility for $k_1 = 1+i$, $k_2 = 2-i$, $\alpha_1^{(1)} = 0.02+0.1i$, $\alpha_1^{(2)} = \alpha_2^{(1)} = \alpha_2^{(2)} = 1$. Finally, figure 2 shows the possibility of elastic collision for the same k_1, k_2 values as in figure 1 but with $\alpha_i^{(j)} = 1$, $i,j = 1,2$.

The above analysis can be analogously extended to study the collision properties of the 3-CNLS equations and N-CNLS equations. For example, for the 3-CNLS system, we find that the above kind of shape changing collisions of the Manakov system occurs in this case also but with many possibilities of intensity exchange among the three modes. During the inelastic (shape changing) interaction among the two one-solitons S_1 and S_2 of the 3-CNLS, the soliton $S_1(S_2)$ has the following six possible combinations to exchange the intensity among its modes: $(q_1,q_2,q_3) \to (q_1^a,q_2^b,q_3^c)_i$, $[a,b,c = S$ (suppression), E (enhancement)], with $i=1$, $a=E$, $b=S$, $c=S$; $i=2$, $a=S$, $b=E$, $c=S$; $i=3$, $a=S$, $b=S$, $c=E$; $i=4$, $a=S$, $b=E$, $c=E$; $i=5$, $a=E$, $b=S$, $c=E$ and $i=6$, $a=E$, $b=E$, $c=S$. In figure 3 we have shown two such possibilities in which figure 3a is plotted for the parametric values $k_1 = 1+i$, $k_2 = 2-i$, $\alpha_2^{(1)} = \alpha_2^{(2)} = (39+i80)/89$, $\alpha_1^{(1)} = \alpha_1^{(2)} = \alpha_1^{(3)} = \alpha_2^{(3)} = 1$ and $\mu = 1$ and in figure 3b they are chosen as $\alpha_1^{(1)} = 0.02+0.1i$, $\alpha_1^{(2)} = 0.1i$, $\alpha_1^{(3)} = \alpha_2^{(1)} = \alpha_2^{(2)} = \alpha_2^{(3)} = 1$ with the same k_1, k_2 and μ values as in figure 3a. Generalizing the above analysis for the 2-CNLS and 3-CNLS equations, to N-CNLS equations, one can verify that the shape changing interaction can lead to intensity redistribution among the modes of each of the solitons of the N-CNLS system in 2^N-2 ways. The details of these analysis will be published elsewhere [28].



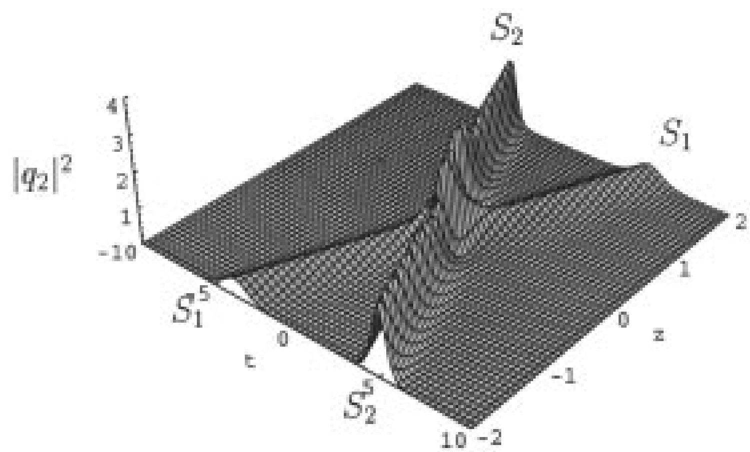


Figure 2. Elastic collision of two solitons in the Manakov system for a specific choice of the parameters.

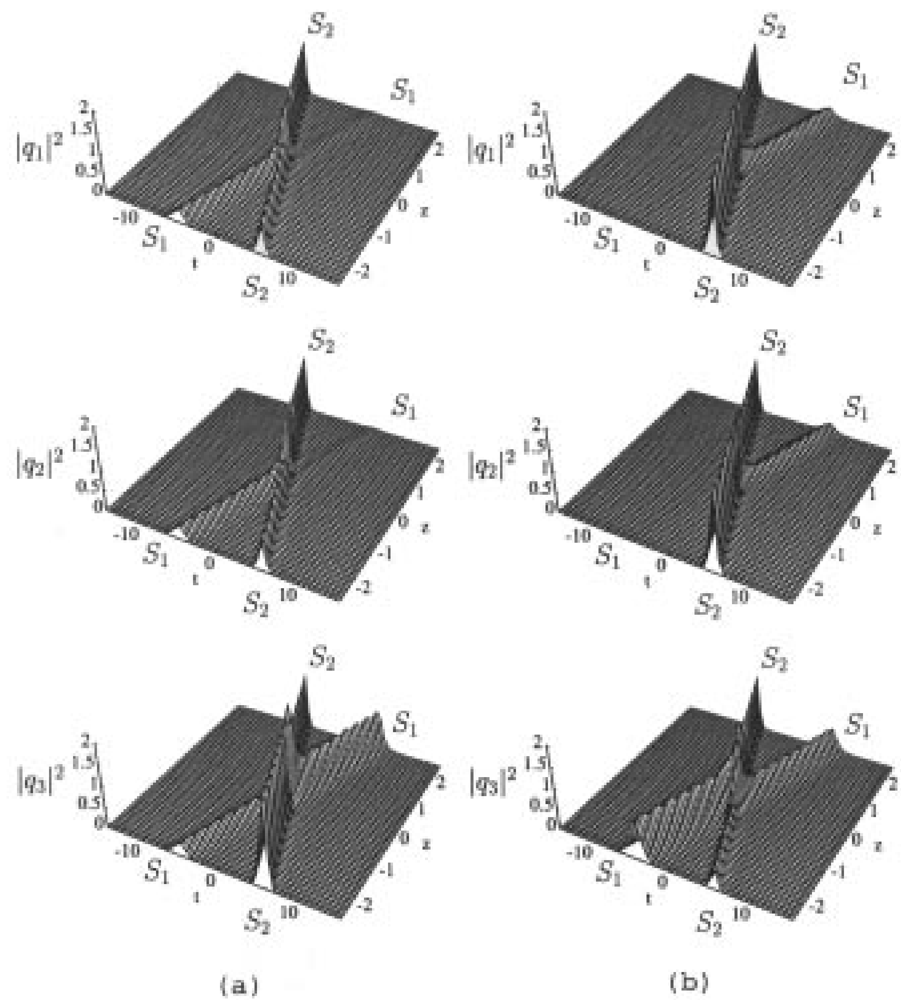


Figure 3. Intensity profiles of the three modes of the two-soliton solution in a waveguide described by the CNLS eq. (13) with $n = 3$ showing two different dramatic scenarios of the shape changing collision.

6. Shape changing collisions and construction of logic gates

The shape changing collisions of solitons of CNLS equations discussed in the previous section can also be characterized by introducing a complex parameter ρ as studied by Jakubowski, Steiglitz and Squier in ref. [18], which is the ratio of the two modes of a particular soliton of the two-soliton solution evaluated asymptotically as in the previous section, that is

$$\rho_j^{\pm} = \frac{q_1^j(t \to \pm\infty)}{q_2^j(t \to \pm\infty)} = \frac{A_1^{j\pm}}{A_2^{j\pm}}, \quad j = 1,2. \tag{53}$$

Obviously during collision the complex parameter ρ changes and it may be used to define the change in the soliton state. Thus the state of the soliton can be essentially parametrized by two complex parameters ρ and k: during collision ρ changes in general, while k does not. The schematic representation of the two-soliton collision in the Manakov system with constant parameters k_1 and k_2 is shown below [18].

In figure 4, ρ_1 and ρ_L represent the variable states of the solitons S_1 and S_2 before collision, respectively. The final variable states are denoted by ρ_2 and ρ_R, respectively. In ref. [18] Jakubowski, Steiglitz and Squier have pointed out that the transition in the soliton states due to collision discussed in the previous section, eqs (49), can be represented equivalently through a linear fractional transformation (LFT) for the change in the variable ρ as

$$\rho_2 = \frac{a(\rho_1)\rho_L + b(\rho_1)}{c(\rho_1)\rho_L + d(\rho_1)}, \quad (ad - bc) \neq 0 \tag{54a}$$

where

$$\begin{aligned} a &= (1-g)/\rho_1^* + \rho_1, \quad b = g\rho_1/\rho_1^*, \quad c = g, \\ d &= (1-g)\rho_1 + 1/\rho_1^*, \quad g(k_1,k_2) = \frac{k_1 + k_1^*}{k_2 + k_1^*} \end{aligned} \tag{54b}$$

and

$$\rho_R = \frac{a'(\rho_L)\rho_1 + b'(\rho_L)}{c'(\rho_L)\rho_1 + d'(\rho_L)}, \quad (a'd' - b'c') \neq 0, \tag{55a}$$

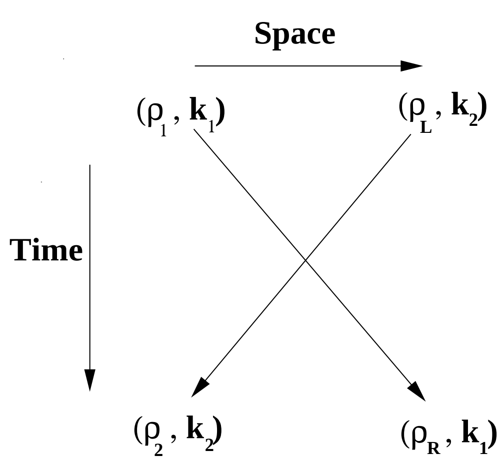


Figure 4. Schematic representation of the two-soliton collision.

where

$$a' = (1-h^*)/\rho_L^* + \rho_L, \quad b' = h^*\rho_L/\rho_L^*, \quad c' = h^*,$$
$$d' = (1-h^*)\rho_L + \frac{1}{\rho_L^*}, \;\; h^* = h^*(k_1,k_2) = g(k_2,k_1). \tag{55b}$$

In writing the above set of equations it is assumed that k_{1R} and $k_{2R} > 0$ and $k_{1I} > k_{2I}$. The LFT possesses many interesting properties [29] which include the following:

1. Existence of inverse transformations and group property,
2. Existence of one or two fixed points,
3. Existence of implicit forms,

and so on. Jakubowski *et al* [18] have in particular pointed out that when viewed as an operator every soliton has an inverse with the same value of the parameter k that will undo the effect of the operator on state. This property can then be used profitably to design logic gates as shown below.

Now let us treat the right moving soliton states as corresponding to data (particles) and left moving solitons as operators (or vice versa), such that

$$\rho_R = T_{\rho_L}(\rho_1) \tag{56a}$$

and

$$\rho_2 = T_{\rho_1}(\rho_L) \tag{56b}$$

Then it is easy to see that for every operator T_{ρ_L} or T_{ρ_1}, there exists an inverse $T_{\rho_L}^{-1}$ or $T_{\rho_1}^{-1}$ such that the successive operations by the operator and its inverse restore the original data.

Example:

$$\text{Let} \;\; \rho_L = 0. \tag{57}$$

Then using the LFT(55), we have

$$\rho_R = T_0(\rho_1) = (1-h^*)\rho_1 \tag{58}$$

Let a second operator corresponding to $\rho_L' = \infty$ operate on ρ_R (see figure 5). The new state ρ_R' is then

$$\rho_R' = \frac{1}{(1-h^*)}\rho_R = \frac{1}{(1-h^*)}(1-h^*)\rho_1 = \rho_1 \tag{59}$$

so that the initial state ρ_1 is restored. Thus the operator (states) with ρ values 0 and ∞ are inverses to each other. Similarly for every operator corresponding to the state ρ an inverse can be obtained. We will now see briefly the consequences of this for computation.

The existence of an inverse for any given operator so that the data is restored on successive operation by the operator and its inverse allows one to assign a logical, binary, 0 and 1 or TRUE and FALSE states in terms of the complex 0 and 1 states of the parameter ρ. In particular, one can use the actuator as 0 and its inverse as ∞ (see figure 5).

Using the above facts, very recently Steiglitz [19] has shown that one can construct the various logic gates such as COPY, FANOUT, NOT, ONE and finally the universal NAND gate also so as to deduce a Turing equivalent machine purely based on optical soliton interactions which do not use any interconnecting discrete components in bulk nonlinear media like photorefractive materials. We briefly summarize below the construction of Steiglitz [19].

For obtaining the above gates let us rotate the figure of the scattering process suitably so as to treat data as solitons travelling vertically downwards and the operators as solitons travelling horizontally (figure 6). Then the various logic gates can be constructed as follows.

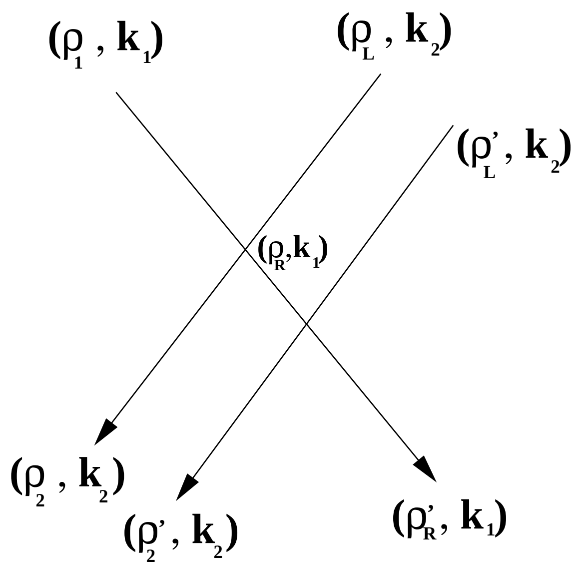


Figure 5. State restoring property of shape changing solitons under collision. When the operator $T_{\rho'_L}$ is the inverse of T_{ρ_L}, then $\rho'_R = \rho_1$.

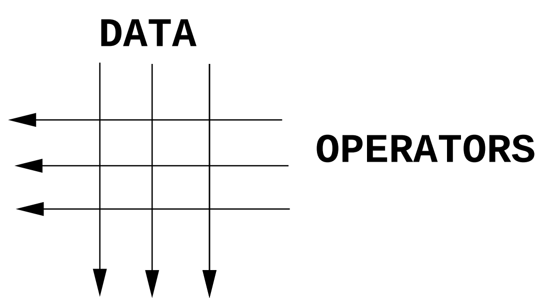


Figure 6. Representation of the left moving operators and down moving data [19].

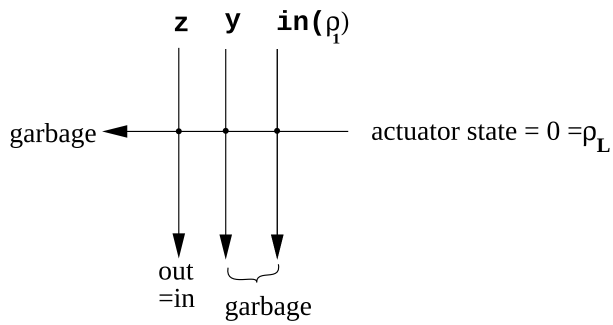


Figure 7. COPY gate [19].

A. *The COPY gate and FANOUT gate*

Let us consider the collision of three down moving solitons with a horizontal soliton as in figure 7.

Let the input state be represented by ρ_1, while the actuator state is taken as $\rho_L = 0$. The other two down moving solitons are in the arbitrary states z and y. Then using the transformation eqs (54) and (55), we can easily check that the horizontally moving soliton after each of the first two collisions becomes

$$\rho_2 = T_{\rho_1}(0), \tag{60}$$

$$\rho_2' = T_y(\rho_2) = T_y\left[T_{\rho_1}(0)\right]. \tag{61}$$

Finally the vertical down moving soliton z after collision becomes

$$\begin{aligned} \text{output} = \rho_R' &= T_{\rho_2}'(z), \\ &= T_{T_y\left[T_{\rho_1}(0)\right]}(z). \end{aligned} \tag{62}$$

Now if we assign for the input state the logical values 0 or 1 corresponding to $\rho_1 = 0$ or $\rho_1 = \infty$ respectively and demand that in = out, we obtain two complex equations for the two complex arbitrary parameters y and z,

$$0 = T_{T_y\left[T_0(0)\right]}(z), \tag{63}$$

$$\infty = T_{T_y[T_\infty(0)]}(z). \tag{64}$$

These equations take the explicit form

$$\{(1-h^*)[(1-g)yy^*+1][(1-g^*)yy^*+1]+gg^*yy^*\}z+h^*g[(1-g^*)yy^*+1]y=0, \tag{65}$$

$$h^*z[g^*(1-g)]y^*+(1-h^*)[g(1-g^*)]y+(1-g)(1-g^*)yy^*=0. \tag{66}$$

Solving the eqs (65) and (66) one can obtain a set of solution (z_c, y_c), if it exists, which then copies the input at one site and places it at the output site, giving rise to the COPY gate.

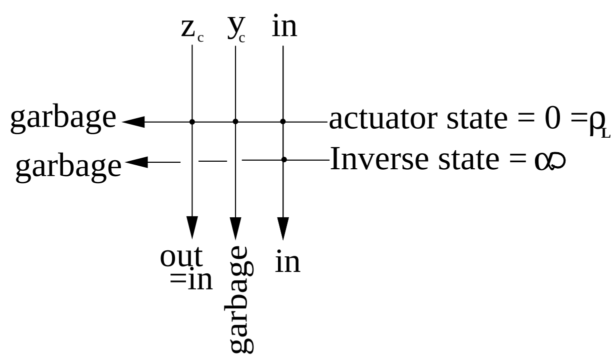


Figure 8. The FANOUT gate [19].

In the above collision process, several unutilized solitons emerge after scattering, which are named as 'garbage' solitons. However, one can use them also profitably to generate copies of inputs as outputs in a 'time gated' manner by colliding appropriate 'inverse' solitons with them. Thus one can obtain the FANOUT gate (figure 8).

B. *The NOT and ONE gates*

As we have obtained the COPY gate by requiring the state out = in for both the logical values 0 and 1 (or the ρ values 0 and ∞), we can require that when the input state is 0 the output is 1 and when the input is 1 the output is 0. This requirement again gives two complex equations for the two unknowns z and y:

$$\infty = T_{T_y[T_0(0)]}(z), \tag{67}$$

$$0 = T_{T_y[T_\infty(0)]}(z). \tag{68}$$

The resultant solution gives the set (z_n, y_n), if it exists, giving rise to the NOT gate.

Similarly if we require that for both the inputs 0 and 1, the output should be 1, the resultant two complex equations solve for (z_1, y_1) giving rise to the ONE gate.

C. *The NAND gate*

The existence of FANOUT, NOT and ONE gates are sufficient to design a NAND gate.

The importance of the NAND gate is that it is universal [19]. Using suitable interconnects and fanouts, it can lead to other logical functions. One can also implement 'wiring' through the light-light collisions, implying that one can implement any logic using the Manakov model.

Using two outputs as equivalent to a given input as in the FANOUT gate (figure 8), one can form a z converter and y converter such that the following values are chosen:

Value of input to the converter	Value of z converter	Value of y converter
0	z_1	y_1
1	z_n	y_n

In the above (z_1,y_1) are the values of (z,y) in the ONE gate, while (z_n,y_n) are the values of (z,y) in the NOT gate.

Thus using one output from a given gate as the input of a FANOUT gate to produce the z- and y-converter to the required value as above, while the other output as the second input of the present gate, one can have a standard three collision arrangement as discussed earlier. Then the following outputs result for the gate depicted in figure 9.

Left input	Right input	value of z	value of y	Output
0	0	z_1	y_1	1
0	1	z_1	y_1	1
1	0	z_n	y_n	1
1	1	z_n	y_n	0

As a result, one finds that in this two input-one output arrangement one essentially obtains the universal NAND gate binary operations. Steiglitz [19] further shows that the gates that are constructed as above can also be wired such that any outputs can be fed into any inputs. Memory can also be introduced suitably. Thus an all optical computer equivalent to a Turing machine using soliton interactions is possible.

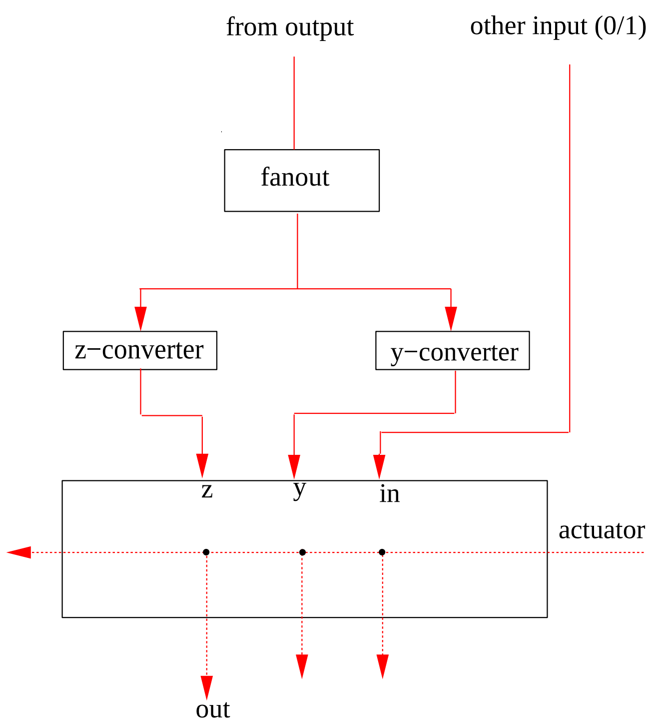


Figure 9. The NAND gate as designed by Steiglitz [19].

7. Partially coherent solitons in CNLS equations

We have shown in §2 that propagation of N-self trapped mutually incoherent wavepackets in Kerr-like photorefractive media is governed by the N-CNLS equations (13). Recently [13,14] it has been shown that these systems support a kind of stationary solutions known as partially coherent stationary solitons (PCS). Further, it has also been observed that these PCS are of variable shapes.

In reference [13] the explicit forms of PCS for $N = 2,3$ and 4 are given. First let us consider the $N = 2$ case, which corresponds to the Manakov system (11). We have given the explicit 2-soliton expression of eq. (11) in eqs (42). Now let us look for a special case of the two-soliton solution (42) of eq. (11) with the choice of the parameters $\alpha_2^{(1)} = \alpha_1^{(2)} = 0$, $\alpha_1^{(1)} = e^{\eta_{10}}$, $\alpha_2^{(2)} = -e^{\eta_{20}}$ and $k_{nI} = 0$, where $\eta_{i0}, i = 1,2$ are real parameters.

Then the solution(42) reduces to the following form

$$q_1 = \left(e^{\eta_1} + \frac{\mu(k_1-k_2)e^{\eta_1+\eta_2+\eta_2^*}}{4k_2^2(k_1+k_2)}\right) \Bigg/ \tilde{D}, \tag{69a}$$

$$q_2 = \left(-e^{\eta_2} + \frac{\mu(k_1-k_2)e^{\eta_1+\eta_1^*+\eta_2}}{4k_1^2(k_1+k_2)}\right) \Bigg/ \tilde{D}, \tag{69b}$$

where

$$\tilde{D} = 1 + \mu\left[\frac{e^{\eta_1+\eta_1^*}}{4k_1^2} + \frac{e^{\eta_2+\eta_2^*}}{4k_2^2}\right] + \frac{\mu^2(k_1-k_2)^2 e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}}{16k_1^2k_2^2(k_1+k_2)^2}. \tag{69c}$$

In the above set of equations η_{10} and η_{20} are absorbed into η_1 and η_2 by rewriting η_j's as $\eta_j = k_j(t + ik_jz) + \eta_{j0}, j = 1,2$. The above solution can be easily rewritten as

$$q_1 = 2k_1\sqrt{\frac{k_1+k_2}{k_1-k_2}}\cosh(k_2\bar{t}_2)e^{ik_1^2z}/D_1, \tag{70a}$$

$$q_2 = 2k_2\sqrt{\frac{k_1+k_2}{k_1-k_2}}\sinh(k_1\bar{t}_1)e^{ik_2^2z}/D_1, \tag{70b}$$

$$D_1 = \sqrt{\mu}\left\{\cosh(k_1\bar{t}_1 + k_2\bar{t}_2) + \left(\frac{k_1+k_2}{k_1-k_2}\right)\cosh(k_1\bar{t}_1 - k_2\bar{t}_2)\right\}, \tag{70c}$$

$$\bar{t}_1 = t - t_1 = t + \frac{\eta_{10}}{k_1} + \frac{1}{2k_1}\log\left[\frac{\mu(k_1-k_2)}{4k_1^2(k_1+k_2)}\right], \tag{70d}$$

$$\bar{t}_2 = t - t_2 = t + \frac{\eta_{20}}{k_2} + \frac{1}{2k_2}\log\left[\frac{\mu(k_1-k_2)}{4k_2^2(k_1+k_2)}\right]. \tag{70e}$$

These solutions are exactly the same PCS solutions given in ref. [13] corresponding to the $N = 2$ case. Here we have shown that the 2-PCS solution is merely a special stationary state of the shape changing 2-soliton solution (42) of the 2-CNLS system.

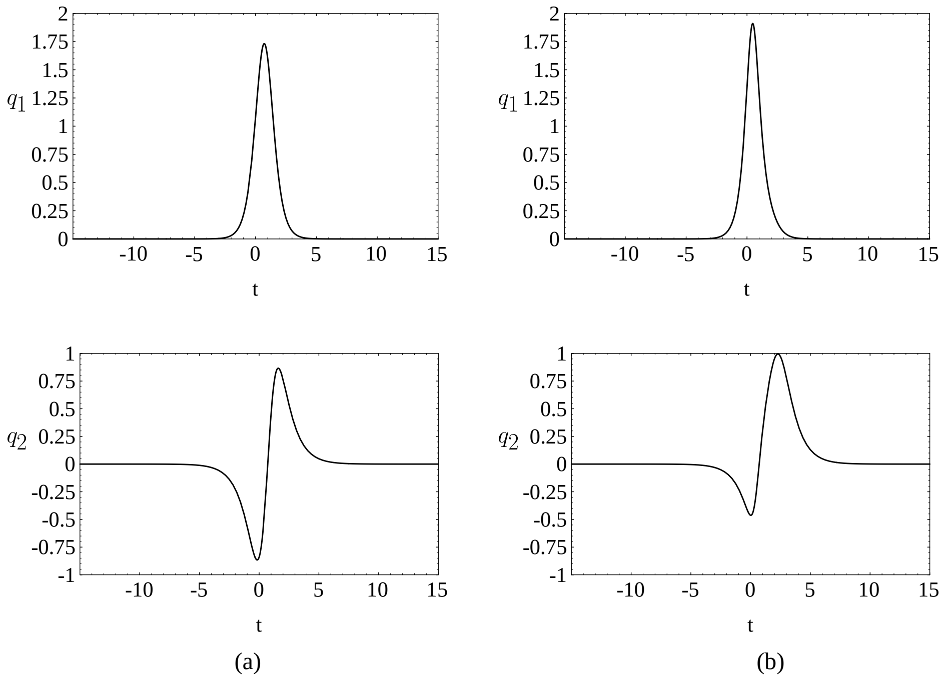


Figure 10. Typical PCS forms for the Manakov system for $z = 0$, see eqs (70): (**a**) symmetric case ($\Delta t_{12} = 0$), (**b**) asymmetric case ($\Delta t_{12} = 1.0$).

Further, one can consider the relative separation distance

$$\Delta t_{12} = t_2 - t_1 = \frac{\eta_{10}}{k_1} - \frac{\eta_{20}}{k_2} + \frac{1}{2k_1}\log\left[\frac{\mu(k_1-k_2)}{4k_1^2(k_1+k_2)}\right] - \frac{1}{2k_2}\log\left[\frac{\mu(k_1-k_2)}{4k_2^2(k_1+k_2)}\right] \tag{70f}$$

and identify symmetric ($\Delta t_{12} = 0$) and asymmetric ($\Delta t_{12} \neq 0$) PCS solutions. Note that the shapes of the PCS depend upon the value of Δt_{12}. In figure 10, we plot typical PCS forms.

Now let us look at the $N = 3$ case. The PCS associated with this case can be obtained as follows. It is also observed that PCS are formed only when the number of incoherent components is equal to the number of solitons created in the system. So in order to obtain the 3-PCS solution we have to consider the three-soliton solution of the 3-CNLS equations. Instead of writing down the full 3-soliton solution of the $N = 3$ case explicitly and choosing the special parametric values, we make the following simplified procedure.

Starting from the bilinear eqs (33) and terminating the series for $g^{(j)}$ and f as

$$g^{(j)} = \lambda g_1^{(j)} + \lambda^3 g_3^{(j)} + \lambda^5 g_5^{(j)} \tag{71a}$$

and

$$f = 1 + \lambda^2 f_2 + \lambda^4 f_4 + \lambda^6 f_6. \tag{71b}$$

At various powers of λ we obtain the following set of equations.

$$\lambda^1 : \hat{D}_1(g_1^{(j)}.1) = 0, \tag{72a}$$

$$\lambda^2 : \hat{D}_2(1.f_2 + f_2.1) = 2\mu \sum_{j=1}^{3} g_1^{(j)}.g_1^{(j)*}, \tag{72b}$$

$$\lambda^3 : \hat{D}_1(g_1^{(j)}.f_2 + g_3^{(j)}.1) = 0, \tag{72c}$$

$$\lambda^4 : \hat{D}_2(1.f_4 + f_2.f_2 + f_4.1) = 2\mu \sum_{j=1}^{3} (g_1^{(j)}.g_3^{(j)*} + g_3^{(j)}.g_1^{(j)*}), \tag{72d}$$

$$\lambda^5 : \hat{D}_1(g_1^{(j)}.f_4 + g_3^{(j)}.f_2 + g_5^{(j)}.1) = 0, \tag{72e}$$

$$\lambda^6 : \hat{D}_2(1.f_6 + f_2.f_4 + f_4.f_2 + f_6.1) = 2\mu \sum_{j=1}^{3} (g_1^{(j)}.g_5^{(j)*} + g_3^{(j)}.g_3^{(j)*} + g_5^{(j)}.g_1^{(j)*}), \tag{72f}$$

$$\lambda^7 : \hat{D}_1(g_1^{(j)}.f_6 + g_3^{(j)}.f_4 + g_5^{(j)}.f_2) = 0, \tag{72g}$$

$$\lambda^8 : \hat{D}_2(f_2.f_6 + f_4.f_4 + f_6.f_2) = 2\mu \sum_{j=1}^{3} (g_3^{(j)}.g_5^{(j)*} + g_5^{(j)}.g_3^{(j)*}), \tag{72h}$$

$$\lambda^9 : \hat{D}_1(g_3^{(j)}.f_6 + g_5^{(j)}.f_4) = 0, \tag{72i}$$

$$\lambda^{10} : \hat{D}_2(f_4.f_6 + f_6.f_4) = 2\mu \sum_{j=1}^{3} (g_5^{(j)}.g_5^{(j)*}), \tag{72j}$$

$$\lambda^{11} : \hat{D}_1(g_5^{(j)}.f_6) = 0, \tag{72k}$$

$$\lambda^{12} : \hat{D}_2(f_6.f_6) = 0, \quad j = 1,2,3. \tag{72l}$$

Solving (72a), we obtain

$$g_1^{(j)} = \alpha_1^{(j)} e^{\eta_1} + \alpha_2^{(j)} e^{\eta_2} + \alpha_3^{(j)} e^{\eta_3} \tag{73a}$$

where $\eta_n = k_n(t + ik_n z)$, j, $n = 1,2,3$ in which $\alpha_i^{(n)}$ and k_n are complex parameters. Here as a special case, we look for a stationary solution with $k_{nI} = 0$, $\alpha_1^{(1)} = e^{\eta_{10}}$, $\alpha_2^{(2)} = -e^{\eta_{20}}$, $\alpha_3^{(3)} = e^{\eta_{30}}$ and $\alpha_1^{(2)} = \alpha_1^{(3)} = \alpha_2^{(1)} = \alpha_2^{(3)} = \alpha_3^{(1)} = \alpha_3^{(2)} = 0$, in order to gain insight into the physics of the problem. Then, solving the remaining set of equations recursively, one can obtain the following special stationary case of the three-soliton solution,

$$q_1 = \left[e^{\eta_1} + \frac{\mu(k_1 - k_2)e^{\eta_1+\eta_2+\eta_2^*}}{4k_2^2(k_1+k_2)} + \frac{\mu(k_1 - k_3)e^{\eta_1+\eta_3+\eta_3^*}}{4k_3^2(k_1+k_3)} + \frac{\mu^2(k_2-k_1)(k_3-k_1)(k_3-k_2)^2 e^{\eta_3+\eta_3^*+\eta_2+\eta_2^*+\eta_1}}{16k_2^2k_3^2(k_2+k_1)(k_3+k_1)(k_3+k_2)^2} \right] \Big/ \tilde{D}, \tag{74a}$$

$$q_2 = \left[-e^{\eta_2} + \frac{\mu(k_1 - k_2)e^{\eta_1+\eta_1^*+\eta_2}}{4k_1^2(k_1+k_2)} + \frac{\mu(k_3 - k_2)e^{\eta_3+\eta_3^*+\eta_2}}{4k_3^2(k_3+k_2)} \right.$$

$$+\frac{\mu^2(k_2-k_1)(k_3-k_2)(k_3-k_1)^2 e^{\eta_3+\eta_3^*+\eta_1+\eta_1^*+\eta_2}}{16k_1^2k_3^2(k_2+k_1)(k_3+k_2)(k_3+k_1)^2}\Bigg]\Bigg/\tilde{D}, \qquad (74b)$$

$$q_3=\Bigg[e^{\eta_3}+\frac{\mu(k_3-k_1)e^{\eta_1+\eta_1^*+\eta_3}}{4k_1^2(k_1+k_3)}+\frac{\mu(k_3-k_2)e^{\eta_2+\eta_2^*+\eta_3}}{4k_2^2(k_3+k_2)}$$
$$+\frac{\mu^2(k_3-k_1)(k_3-k_2)(k_2-k_1)^2 e^{\eta_2+\eta_2^*+\eta_1+\eta_1^*+\eta_3}}{16k_1^2k_2^2(k_3+k_1)(k_3+k_2)(k_2+k_1)^2}\Bigg]\Bigg/\tilde{D}. \qquad (74c)$$

Here,

$$\tilde{D}=1+\mu\left[\frac{e^{\eta_1+\eta_1^*}}{4k_1^2}+\frac{e^{\eta_2+\eta_2^*}}{4k_2^2}+\frac{e^{\eta_3+\eta_3^*}}{4k_3^2}\right]+\frac{\mu^2(k_1-k_2)^2e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}}{16k_1^2k_2^2(k_1+k_2)^2}$$
$$+\frac{\mu^2(k_1-k_3)^2e^{\eta_1+\eta_1^*+\eta_3+\eta_3^*}}{16k_1^2k_3^2(k_1+k_3)^2}+\frac{\mu^2(k_3-k_2)^2e^{\eta_2+\eta_2^*+\eta_3+\eta_3^*}}{16k_2^2k_3^2(k_2+k_3)^2}$$
$$+\left[\frac{\mu^3(k_2-k_1)^2(k_3-k_1)^2(k_3-k_2)^2e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\eta_3+\eta_3^*}}{64k_1^2k_2^2k_3^2(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}\right]. \qquad (74d)$$

Rewriting eq. (74),as in the case of $N=2$, here also one can verify that the above solution is exactly the same as the stationary PCS given in ref. [13] for $N=3$,

$$q_1=\frac{2k_1}{\sqrt{\mu}D_2}\left(\frac{(k_1+k_2)(k_1+k_3)}{(k_1-k_2)(k_1-k_3)}\right)^{1/2}$$
$$\times\left[\cosh\left[k_2\bar{t_2}+k_3\bar{t_3}\right]+\frac{(k_2+k_3)}{(k_2-k_3)}\cosh\left[k_2\bar{t_2}-k_3\bar{t_3}\right]\right]e^{ik_1^2z}, \qquad (75a)$$

$$q_2=\frac{2k_2}{\sqrt{\mu}D_2}\left(\frac{(k_2+k_3)(k_1+k_2)}{(k_1-k_2)(k_2-k_3)}\right)^{1/2}$$
$$\times\left[\sinh\left[k_1\bar{t_1}+k_3\bar{t_3}\right]+\frac{(k_1+k_3)}{(k_1-k_3)}\sinh\left[k_1\bar{t_1}-k_3\bar{t_3}\right]\right]e^{ik_2^2z}, \qquad (75b)$$

$$q_3=\frac{2k_3}{\sqrt{\mu}D_2}\left(\frac{(k_1+k_3)(k_2+k_3)}{(k_1-k_3)(k_2-k_3)}\right)^{1/2}$$
$$\times\left[\cosh\left[k_2\bar{t_2}+k_1\bar{t_1}\right]-\frac{(k_1+k_2)}{(k_1-k_2)}\cosh\left[k_1\bar{t_1}-k_2\bar{t_2}\right]\right]e^{ik_3^2z}. \qquad (75c)$$

The quantity D_2 is given by

$$D_2=\cosh\left[k_1\bar{t_1}+k_2\bar{t_2}+k_3\bar{t_3}\right]+\left(\frac{k_1+k_2}{k_1-k_2}\right)\left(\frac{k_1+k_3}{k_1-k_3}\right)\cosh\left[k_1\bar{t_1}-k_2\bar{t_2}-k_3\bar{t_3}\right]$$
$$+\left(\frac{k_1+k_2}{k_1-k_2}\right)\left(\frac{k_1+k_3}{k_1-k_3}\right)\cosh\left[k_1\bar{t_1}-k_2\bar{t_2}+k_3\bar{t_3}\right]$$
$$+\left(\frac{k_1+k_3}{k_1-k_3}\right)\left(\frac{k_2+k_3}{k_2-k_3}\right)\cosh\left[k_1\bar{t_1}+k_2\bar{t_2}+k_3\bar{t_3}\right], \qquad (75d)$$

where

$$\bar{t_1} = t - t_1 = t + \frac{\eta_{10}}{k_1} + \frac{1}{2k_1}\log\left[\frac{\mu(k_2-k_1)(k_3-k_1)}{4k_1^2(k_1+k_2)(k_1+k_3)}\right], \tag{75e}$$

$$\bar{t_2} = t - t_2 = t + \frac{\eta_{20}}{k_2} + \frac{1}{2k_2}\log\left[\frac{\mu(k_2-k_1)(k_3-k_2)}{4k_2^2(k_1+k_2)(k_2+k_3)}\right], \tag{75f}$$

$$\bar{t_3} = t - t_3 = t + \frac{\eta_{30}}{k_3} + \frac{1}{2k_3}\log\left[\frac{\mu(k_3-k_1)(k_3-k_2)}{4k_3^2(k_1+k_3)(k_2+k_3)}\right]. \tag{75g}$$

Thus from the above analysis we have shown that the 2-soliton and 3-soliton PCS are very special cases corresponding to specific parametric restrictions in the two-soliton solution of the $N=2$ case, and the three-soliton solution of the $N=3$ case, respectively. Extending this idea to $N=4$ (which we have verified explicitly) and then to arbitrary N, it is clear that the PCS which is formed due to a nonlinear superposition of N-fundamental solitons [13] is a special case of the N-soliton solution of the N-CNLS equations (13).

These PCS solutions are found to be of variable shape as pointed above. Further, they also change their shape during collision with another PCS [13,14]. The reason for the shape change of PCS arises naturally from the soliton interaction properties of eq. (13) discussed in §5. There it has been shown that during a pairwise interaction of two fundamental solitons of N-CNLS equation there is an energy sharing betweeen them resulting in a novel shape changing collision, depending on the transition matrix elements T_j^l, the phase shift Φ^l and a change in the relative separation distance Δx_{ij} given by eq. (52). Since, the PCS is a special case of the N-soliton solutions which are parametrized as above, it naturally possesses a variable shape.

The reason for the shape variation of PCS during collision with another PCS also follows from the nature of the fundamental bright soliton collision of the Manakov system and its generalization. The collision of two PCS each comprising m and n soliton complexes, respectively, such that $m+n=N$, is equivalent to the interaction of N-fundamental bright solitons (for suitable choice of parameters) represented by the special case of N-soliton solution of the N-CNLS system. More details of these results will be reported separately [28].

Our above analysis is not only of theoretical interest but also has considerable practical relevance in view of the various recently reported interesting experimental observations. The above discussed Manakov solitons have been recently observed in AlGaAs planar waveguides [30]. Further, the shape changing collision involving energy exchange as discussed above has been demonstrated experimentally [31]. Also, collision between PCS's of shape changing type as treated here were observed in a photorefractive strontium barium niobate crystal using screening solitons [32]. As mentioned earlier in the introduction, the partially incoherent solitons can be observed even with a light bulb [9] and also through excitation by partially coherent light [8]. We believe that our exact analytical results will give further impetus to the experimental investigations of these solitons.

8. Conclusion

We have pointed out in this paper that the study on CNLS equations has considerable physical relevance due to their appearance as governing equations for soliton propagation in different physical contexts like long distance optical communication systems and photorefractive media. From a theoretical point of view, first we have analysed the integrable nature of the generalized Manakov or integrable CNLS systems by giving their Lax pair along with the conserved quantities associated with them and then we have obtained the one- and two-soliton solutions explicitly. We have further shown that the bright solitons of the integrable CNLS equations exhibit a novel shape changing collision property which is of considerable physical interest. In particular, we have pointed following ref. [18] that state transformations of the solitons during this fascinating shape changing collision can be represented by a linear fractional (bilinear) transformation and illustrated how this property can be used advantageously in constructing various logic gates, including the NAND gate—a universal gate. Further, as another application we have also pointed out that the various partially coherent stationary solitons reported in the recent literature are special cases of the bright soliton solutions of the integrable CNLS equations. The above studies seem to clearly show that there is considerable relevance both from an experimental point of view as well as theoretical point of view to study the dynamics of CNLS systems.

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