A Simple and Unified Approach to Identify Integrable Nonlinear Oscillators and Systems

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Abstract

In this paper, we consider a generalized second order nonlinear ordinary differential equation of the form $\ddot{x} + (k_1 x^q + k_2)\dot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x = 0$, where k_i's, $i = 1, 2, 3, 4$, λ_1 and q are arbitrary parameters, which includes several physically important nonlinear oscillators such as the simple harmonic oscillator, anharmonic oscillator, force-free Helmholtz oscillator, force-free Duffing and Duffing-van der Pol oscillators, modified Emden type equation and its hierarchy, generalized Duffing-van der Pol oscillator equation hierarchy and so on and investigate the integrability properties of this rather general equation. We identify several new integrable cases for arbitrary value of the exponent q, $q \in R$. The $q = 1$ and $q = 2$ cases are analyzed in detail and the results are generalized to arbitrary q. Our results show that many classical integrable nonlinear oscillators can be derived as sub-cases of our results and significantly enlarge the list of integrable equations that exist in the contemporary literature. To explore the above underlying results we use the recently introduced generalized extended Prelle-Singer procedure applicable to second order ODEs. As an added advantage of the method we not only identify integrable regimes but also construct integrating factors, integrals of motion and general solutions for the integrable cases, wherever possible, and bring out the mathematical structures associated with each of the integrable cases.

I. INTRODUCTION

A. Overview of the problem

In a recent paper[1] we have shown that the force-free Duffing-van der Pol (DVP) oscillator,

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 = 0, \tag{1}$$

is integrable for the parametric restriction $\alpha = \frac{4}{\beta}$ and $\gamma = -\frac{3}{\beta^2}$. In Eq. (1) over dot denotes differentiation with respect to t and α, β and γ are arbitrary parameters. Under the transformation

$$w = -xe^{\frac{1}{\beta}t}, \qquad z = e^{-\frac{2}{\beta}t}, \tag{2}$$

Eq. (1) with restriction $\alpha = \frac{4}{\beta}$ and $\gamma = -\frac{3}{\beta^2}$ was shown to be transformable to the form

$$w'' - \frac{\beta^2}{2}w^2 w' = 0, \tag{3}$$

which can then be integrated[1].

In a parallel direction, while performing the invariance analysis of a similar kind of problem, we find that not only the Eq. (1) but also its generalized version,

$$\ddot{x} + (\frac{4}{\beta} + \beta x^2)\dot{x} + \frac{3}{\beta^2}x + x^3 + \delta x^5 = 0, \quad \delta = \text{arbitrary parameter}, \tag{4}$$

is invariant under the same set of Lie point symmetries[2]. As a consequence one can use the same transformation (2) to integrate the Eq. (4). The transformation (2) modifies Eq. (4) to the form

$$w'' - \frac{\beta^2}{2}w^2 w' + \delta w^5 = 0 \tag{5}$$

which is not so simple to integrate straightforwardly. However, we observe that this equation coincides with the second equation in the so called modified Emden equation (MEE) hierarchy, investigated by Feix et al.[3],

$$\ddot{x} + x^l\dot{x} + gx^{2l+1} = 0, \quad l = 1, 2, \ldots, n, \tag{6}$$

where g is an arbitrary parameter.

In fact Feix et al.[3] have shown that through a direct transformation to a third order equation the above Eq. (6) can be integrated to obtain the general solution for the specific choice of the parameter g, namely, for $g = \frac{1}{(l+2)^2}$. For this choice of g, the general solution of (6) can be written as

$$x(t) = \left(\frac{(2+3l+l^2)(t+I_1)^l}{l(t+I_1)^{l+1} + (2+3l+l^2)I_2} \right)^{\frac{1}{l}}, \quad I_1, I_2: \text{ arbitrary constants.} \tag{7}$$

Consequently Eq. (4) can be integrated under the specific parametric choice $\delta = \frac{1}{16}$, and it belongs to the $l = 2$ case of the MEE hierarchy (6) with $g = \frac{1}{16}$. Now the question arises as to whether there exist other new integrable second order nonlinear differential equations which are linear in $\dot{x}$ and containing fifth and other powers of nonlinearity. As far as our knowledge goes only few equations in this class have been shown to be integrable. For example, Smith[4] had investigated a class of nonlinear equations coming under the category

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{8}$$

with $f(x) = (n+2)bx^n - 2a$ and $g(x) = x(c + (bx^n - a)^2)$ where a, b, c and n are arbitrary parameters. He had shown that the Eq. (8) with this specific forms of f and g admits explicit oscillatory solutions. However, one can also expect that there should be a number of integrable equations which also admit solutions which are both oscillatory and non-oscillatory types in the class

$$\ddot{x} + (k_1 x^q + k_2)\dot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x = 0, \qquad q \in R, \tag{9}$$

where k_i's, $i = 1, 2, 3, 4$ and λ_1 are arbitrary parameters. When $q = 1$, Eq. (9) becomes the generalized MEE

$$\ddot{x} + (k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0, \tag{10}$$

and for $q = 2$ it becomes

$$\ddot{x} + (k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x = 0. \tag{11}$$

We note that Eq. (4) is a special case of (11).

Needless to say Eq. (9) is a unified model for several ground breaking physical systems which includes simple harmonic oscillator, anharmonic oscillator, force-free Helmholtz oscillator, force-free Duffing oscillator, MEE hierarchy, generalized DVP hierarchy and so on.

As noted earlier there exists no rigorous mathematical analysis in the literature for the second order nonlinear differential equations which contain fifth or higher degree nonlinearity in x and linear in $\dot{x}$ and the results are very scarce on integrability or exact solutions. Our motivation to analyze this problem is not only to explore new integrable cases/systems of Eq. (9) but also to synthesize all earlier results under one approach.

Having described the problem and motivation now we can start analyzing the integrability properties of Eq. (9). To identify the integrable regimes we employ the recently introduced extended Prelle-Singer procedure applicable to second order ODEs[5−11]. Through this method we not only identify integrable regimes but also construct integrating factors, integrals of motion and general solution for the integrable cases, wherever possible.

B. Results

We unearth several new integrable equations for any real value of the exponent q in Eq. (9). In the following we summarize the results for the case $q =$ *arbitrary* only and discuss in detail the $q = 1$, $q = 2$ and $q =$ *arbitrary* cases separately in the following sections.

For the choice $q =$ *arbitrary* we find that the following equations are completely integrable (after suitable reparametrizations), all of which appear to be new to the literature:

$$\ddot{x} + (k_1 x^q + (q+2)k_2)\dot{x} + k_1 k_2 x^{q+1} + (q+1)k_2^2 x = 0 \tag{12}$$

$$\ddot{x} + ((q+2)k_1 x^q + k_2)\dot{x} + k_1^2 x^{2q+1} + k_1 k_2 x^{q+1} + \lambda_1 x = 0 \tag{13}$$

$$\ddot{x} + (q+4)k_2\dot{x} + k_4 x^{q+1} + 2(q+2)k_2^2 x = 0 \tag{14}$$

$$\ddot{x} + ((q+1)k_1 x^q + k_2)\dot{x} + \frac{(r-1)}{r^2}[(q+1)k_1^2 x^{2q+1} + (q+2)k_1 k_2 x^{q+1} + k_2^2 x] = 0, \quad r \neq 0 \tag{15}$$

$$\ddot{x} + ((q+1)k_1 x^q + (q+2)k_2)\dot{x} + (q+1)[\frac{(r-1)}{r^2}k_1^2 x^{2q+1} + k_1 k_2 x^{q+1} + k_2^2 x] = 0, \quad r \neq 0, \tag{16}$$

where $k_1,\ k_2,\ k_4,\ \lambda_1$ and r are arbitrary parameters. We stress that the above results are true for any arbitrary values of q. We discuss the special cases, namely, $q = 1$ and $q = 2$ separately in detail in sections 3 and 4 in order to put the results of q arbitrary case in proper perspective.

We show that the Eq. (12) is nothing but a generalization of the Duffing-van der Pol oscillator Eq. (1). In a recent work[1,9] three of the present authors have established the integrability of Eq. (12) with $q = 2$. However, in this work we show that the generalized Eq. (12) itself is integrable. Eq. (13) is nothing but the generalized MEE among which the hierarchy of Eq. (6), studied by Feix et al.[3], can be identified as a sub-case. In fact the general solution constructed by Feix et al., Eq. (7), can be derived straightforwardly as a sub-case. Eq. (13) also contains the family of equations studied by Smith[4]. In particular the latter author have derived general solution for the case $k_2^2 < 4\lambda_1$, which turns out to be an oscillatory one. However, in this work, we show that even for arbitrary values of k_2 and λ_1 one can construct the general solution. Interestingly, the system (14) generalizes several physically important nonlinear oscillators. For example, in the case $q = 1$ and 2, Eq. (14) provides us the force-free Helmholtz and Duffing oscillators, respectively, whose nonlinear dynamics is well documented in the literature[12–16]. Here, we present certain integrable generalizations of these nonlinear oscillators. Eq. (15) admits a conservative Hamiltonian for all values of the parameters r, k_1 and k_2 and any integer value of q. We also provide the explicit form of the Hamiltonian for all values of q. As a result we conclude that it is a Liouville integrable system. As far as Eq. (16) is concerned we construct a time dependent integral of motion and transform the latter to time independent Hamiltonian one and thereby ensuring its Liouville integrability.

The plan of the paper is us follows. In the following section we briefly describe the extended Prelle-Singer procedure applicable to second order ODEs. In Sec. III, we consider the case $q = 1$ in (9) and identify the integrable parametric choices of this equation through the extended PS procedure. To do so first we identify the integrable cases where the system admits time independent integrals and construct explicit conservative Hamiltonians for the respective parametric choices. We then identify the cases which admit explicit time dependent integrals of motion. To establish the complete integrability of these cases we use our own procedure and transform the time dependent integrals of motion into time independent integrals of motion and integrate the latter and derive the general solution. In Sec. IV, we repeat the procedure for the case $q = 2$ in Eq. (9) and identify the integrable systems. In Sec. V, we consider the case $q =$ arbitrary in (9) and unearth several new integrable equations and their associated mathematical structures. Finally, we present our conclusions in Sec. VI.

II. GENERALIZED EXTENDED PRELLE-SINGER (PS) PROCEDURE

In this section we briefly recall the generalized extended or modified PS procedure before applying it to the specific problem in hand. Sometime ago, Prelle and Singer[5] have proposed a procedure for solving first order ODEs that admit solutions in terms of elementary functions if such solutions exist. The attractiveness of the PS method is that if the given system of first order ODEs has a solution in terms of elementary functions then the method guarantees that this solution will be found. Very recently Duarte et al.[7,8] have modified the technique developed by Prelle and Singer[5,6] and applied it to second order ODEs. Their approach was based on the conjecture that if an elementary solution exists for the given second order ODE then there exists at least one elementary first integral $I(t, x, \dot{x})$ whose derivatives are all rational functions of t, x and $\dot{x}$. For a class of systems these authors have deduced first integrals and in some cases for the first time through their procedure[7]. Recently the present authors have generalized the theory of Duarte et al.[7] in different directions and shown that for the second order ODEs one can isolate even two independent integrals of motion[9−11] and obtain general solutions explicitly without any integration. This theory has also been illustrated for a class of problems[1,9−11]. The authors have also generalized the theory successfully to higher order ODEs[10,17]. For example, in the case of third order ODEs the theory has been appropriately generalized to yield three independent integrals of motion unambiguously so that the general solution follows immediately from these integrals of motion[17].

We stress that the PS procedure has many advantages over other methods. To name a few, we cite: (1) For a given problem if the solution exists it has been conjectured that the PS method guarantees to provide first integrals. (2) The PS method not only gives the first integrals but also the underlying integrating factors, that is, multiplying the equation with these functions we can rewrite the equation as a perfect differentiable function which upon integration gives the first integrals directly. (3) The PS method can be used to solve nonlinear as well as linear second order ODEs. (4) As the PS method is based on the equations of motion rather than on Lagrangian or Hamiltonian description, the analysis is applicable to deal with both Hamiltonian and non-Hamiltonian systems.

A. PS method

Let us rewrite Eq. (9) in the form

$$\ddot{x} = -((k_1 x^q + k_2)\dot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x) \equiv \phi(x, \dot{x}). \tag{17}$$

Further, we assume that the ODE (17) admits a first integral $I(t, x, \dot{x}) = C$, with C constant on the solutions, so that the total differential becomes

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0, \tag{18}$$

where each subscript denotes partial differentiation with respect to that variable. Rewriting Eq. (17) in the form $\phi dt - d\dot{x} = 0$ and adding a null term $S(t, x, \dot{x})\dot{x}\ dt - S(t, x, \dot{x})dx$ to the latter, we obtain that on the solutions the 1-form

$$\left(\phi + S\dot{x}\right)dt - Sdx - d\dot{x} = 0. \tag{19}$$

Hence, on the solutions, the 1-forms (18) and (19) must be proportional. Multiplying (19) by the factor $R(t, x, \dot{x})$ which acts as the integrating factors for (19), we have on the solutions that

$$dI = R(\phi + S\dot{x})dt - RSdx - Rd\dot{x} = 0. \tag{20}$$

Comparing Eq. (18) with (20) we have, on the solutions, the relations

$$I_t = R(\phi + \dot{x}S), \quad I_x = -RS, \quad I_{\dot{x}} = -R. \tag{21}$$

Then the compatibility conditions, $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, between the Eqs. (21), provide us

$$S_t + \dot{x}S_x + \phi S_{\dot{x}} = -\phi_x + \phi_{\dot{x}}S + S^2, \tag{22}$$

$$R_t + \dot{x}R_x + \phi R_{\dot{x}} = -(\phi_{\dot{x}} + S)R, \tag{23}$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \tag{24}$$

Solving Eqs. (22)-(24) one can obtain expressions for S and R. It may be noted that any set of special solutions (S, R) is sufficient for our purpose. Once these forms are determined the integral of motion $I(t, x, \dot{x})$ can be deduced from the relation

$$I = r_1 - r_2 - \int \left[R + \frac{d}{d\dot{x}}(r_1 - r_2)\right] d\dot{x}, \tag{25}$$

where

$$r_1 = \int R(\phi + \dot{x}S)dt, \quad r_2 = \int (RS + \frac{d}{dx}r_1)dx.$$

Equation (25) can be derived straightforwardly by integrating the Eq. (21).

The crux of the problem lies in finding the explicit solutions satisfying all the three determining Eqs. (22)-(24), since once a particular solution is known the integral of motion can be readily constructed. The difficulties in constructing admissible set of solutions (S, R) satisfying all the three Eqs. (22)-(24) and possible ways of obtaining the solutions have been discussed in detail in Ref. 9.

III. APPLICATION OF PS PROCEDURE TO EQ. (10)

Let us first consider the case $q = 1$ in Eq. (9) or equivalently (10)

$$\ddot{x} + (k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0. \tag{10}$$

Eq. (10) itself includes several physically important models. For example, choosing $k_i = 0,\ i = 1, ...4$, we get the simple harmonic oscillator equation and the choice $k_1,\ k_2 = 0$ gives us the anharmonic oscillator equation. When $k_1,\ k_4 = 0$ Eq. (10) becomes the force-free Duffing oscillator equation[12]. The choice $k_2,\ k_4,\ \lambda_1 = 0$ provides us the MEE[18]. In the limit $k_3 = \frac{k_1^2}{9},\ k_4 = \frac{k_1 k_2}{3}$, Eq. (10) becomes MEE with linear term which is another linearizable equation which we have studied extensively in Refs. 9 and 19. The restriction $k_1,\ k_3 = 0$ leads us to the force-free Helmhotz oscillator[12,13]. In the following we investigate whether the system (10) admits any other integrable case besides the above.

We solve Eq. (10) through the extended PS procedure in the following way. For a given second order equation, (10), the first integral I should be either a time independent or time dependent one. In the former case, it is a conservative system and we have $I_t = 0$ and in the later case we have $I_t \neq 0$. So let us first consider the case $I_t = 0$ and determine the null forms and the corresponding integrating factors and from these we construct the integrals of motion and then we do extend the analysis for the case $I_t \neq 0$.

A. The case $I_t = 0$

1. Null forms

In this case one can easily fix the null form S from the first equation in (21) as

$$S = \frac{-\phi}{\dot{x}} = -\frac{((k_1x + k_2)\dot{x} + k_3x^3 + k_4x^2 + \lambda_1 x)}{\dot{x}}. \tag{26}$$

2. Integrating Factors

Substituting this form of S, given in (26), into (23) we get

$$\begin{aligned} R_t + \dot{x}R_x - ((k_1x + k_2)\dot{x} + k_3x^3 + k_4x^2 + \lambda_1 x)R_{\dot{x}} \\ = \left((k_1x + k_2) + \frac{((k_1x+k_2)\dot{x}+k_3x^3+k_4x^2+\lambda_1 x)}{\dot{x}}\right)R. \end{aligned} \tag{27}$$

Equation (27) is a first order linear partial differential equation with variable coefficients. As we noted earlier any particular solution is sufficient to construct an integral of motion (along with the function S). To seek a particular solution for R one can make a suitable ansatz instead of looking for the general solution. We assume R to be of the form,

$$R = \frac{\dot{x}}{(A(x) + B(x)\dot{x})^r}, \tag{28}$$

where A and B are functions of their arguments, and r is a constant which are all to be determined. We demand the above form of ansatz, (28), due to the following reason. To deduce the first integral I we assume a rational form for I, that is, $I = \frac{f(x,\dot{x})}{g(x,\dot{x})}$, where f and g are arbitrary functions of x and $\dot{x}$ and are independent of t. Since we already assumed that I is independent of t, we have, $I_x = \frac{f_x g - f g_x}{g^2}$ and $I_{\dot{x}} = \frac{f_{\dot{x}} g - f g_{\dot{x}}}{g^2}$. From (21) one can see that $R = I_{\dot{x}} = \frac{f_{\dot{x}} g - f g_{\dot{x}}}{g^2}$, $S = \frac{I_x}{I_{\dot{x}}} = \frac{f_x g - f g_x}{f_{\dot{x}} g - f g_{\dot{x}}}$ and $RS = I_x$, so that the denominator of the function S should be the numerator of the function R. Since the denominater of S is $\dot{x}$ (vide Eq. (26)) we fixed the numerator of R as $\dot{x}$. To seek a suitable function in the denominator initially one can consider an arbitrary form $R = \frac{\dot{x}}{h(x,\dot{x})}$. However, it is difficult to proceed with this choice of h. So let us assume that $h(x, \dot{x})$ is a function which is polynomial in $\dot{x}$. To begin with let us consider the case where h is linear in $\dot{x}$, that is, $h = A(x) + B(x)\dot{x}$. Since R is in rational form while taking differentiation or integration the form of the denominator remains same but the power of the denominator decreases or increases by a unit order from that of the initial one. So instead of considering h to be of the form $h = A(x) + B(x)\dot{x}$, one may consider a more general form $h = (A(x) + B(x)\dot{x})^r$, where r is a constant to be determined. Such a generalized form of h and so R leads to several new integrable cases as we see below.

Substituting (28) into (27) and solving the resultant equations, we arrive at the relation

$$r(\dot{x}(A_x + B_x\dot{x}) + \phi B) = (A + B\dot{x})\phi_{\dot{x}}. \tag{29}$$

Solving Eq. (29) with $\phi = -((k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x)$, we find nontrivial forms for the functions A and B for two choices, namely, (i) k_1, k_2 arbitrary and (ii) $k_1 =$ arbitrary, $k_2 = 0$ with restrictions on other parameters as given below. The respective forms of the functions and the restriction on the parameters are

(i) $\underline{k_1,\ k_2 : \text{arbitrary}}$

$$A(x) = \frac{(r-1)b_0}{r}(\frac{k_1}{2}x^2 + k_2 x), \quad B(x) = b_0 = \text{constant}, \quad r = \text{constant},$$

$$k_3 = \frac{b_0(r-1)}{2r^2}k_1^2, \quad k_4 = \frac{3b_0(r-1)}{2r^2}k_1 k_2, \quad \lambda_1 = \frac{b_0(r-1)k_2^2}{r^2}, \tag{30a}$$

(ii) $\underline{k_1 = \text{arbitrary},\ k_2 = 0}$

$$A(x) = \frac{(r-1)b_0}{2r}k_1 x^2 + \frac{r\lambda_1}{k_1}, \quad B(x) = b_0,$$

$$k_3 = \frac{b_0(r-1)}{2r^2}k_1^2, \quad k_4 = 0, \quad \lambda_1 = \text{arbitrary parameter (here)}. \tag{30b}$$

We note that the case (ii) cannot be derived from case (i) by taking $k_2 = 0$. For example, choosing $k_2 = 0$ in (30a) we get not only $k_4 = 0$ but also $\lambda_1 = 0$ whereas in the case (ii) we have the freedom $\lambda_1 =$ arbitrary, so the cases (30a) and (30b) are to be treated as separate. Making use of the forms of A and B from Eqs. (30a) and (30b) into (28), the integrating factor, 'R', for the two cases can be obtained as

(i) $\underline{k_1,\ k_2 : \text{arbitrary}}$

$$R = \frac{\dot{x}}{\left[\frac{(r-1)}{r}(\frac{k_1}{2}x^2 + k_2 x) + \dot{x}\right]^r}, \quad r \neq 0 \tag{31a}$$

(ii) $\underline{k_1 = \text{arbitrary},\ k_2 = 0}$

$$R = \frac{\dot{x}}{\left[\frac{(r-1)}{2r}k_1 x^2 + \frac{r\lambda_1}{k_1} + \dot{x}\right]^r}, \quad r \neq 0. \tag{31b}$$

We note that b_0 is a common parameter in the above and it is absorbed in the definition of 'R', see Eqs. (23) and (24). While deriving the above forms of R (Eqs. (31a) and (31b)) we assumed that $r \neq 0$ and for the choice $r = 0$ we obtain consistent solution only if both the parameters k_1 and k_2 become zero. Of course, this sub-case can be treated as a trivial one since when $k_1, k_2 = 0$ the damping term in Eq. (10) vanishes and the system becomes an integrable anharmonic oscillator. In this trivial case we have the integrating factor of the form:

(iii) $\underline{k_1,\ k_2 = 0}$

$$R = \dot{x}, \quad r = 0. \tag{31c}$$

Finally one has to check the compatibility of forms S and R with the third Eq. (24). We indeed verified that the sets

$$(i) \quad S = -\frac{((k_1 x + k_2)\dot{x} + \frac{(r-1)}{r^2}(\frac{k_1^3}{2}x^2 + \frac{3k_1k_2}{2}x^2 + k_2^2 x))}{\dot{x}},$$

$$R = \frac{\dot{x}}{(\frac{(r-1)}{r}(\frac{k_1}{2}x^2 + k_2 x) + \dot{x})^r}, \quad k_1, k_2 = \text{arbitrary}, \quad r \neq 0 \tag{32a}$$

$$(ii) \quad S = -\frac{(k_1 x\dot{x} + \frac{(r-1)}{2r^2}k_1^2 x^3 + \lambda_1 x)}{\dot{x}},$$

$$R = \frac{\dot{x}}{(\frac{(r-1)}{2r}k_1 x^2 + \frac{r\lambda_1}{k_1} + \dot{x})^r}, \quad k_1 = \text{arbitrary},\ k_2 = 0, \quad r \neq 0 \tag{32b}$$

and

$$(iii) \quad S = -\frac{(k_3 x^3 + k_4 x^2 + \lambda_1 x)}{\dot{x}}, \quad R = \dot{x}, \quad k_1,\ k_2 = 0, \tag{32c}$$

satisfy the Eq. (24) individually. As a consequence all the three pairs form compatible sets of solution for the Eqs. (22)-(24).

3. *Integrals of motion*

Having determined the explicit forms of S and R one can proceed to construct integrals of motion using the expressions (25). The parametric restrictions (30a) and (30b) fix the equation of motion (10) to the following specific forms,

$$(i) \quad \ddot{x} + (k_1 x + k_2)\dot{x} + \frac{(r-1)}{2r^2}\left(k_1^2 x^3 + 3k_1 k_2 x^2 + 2k_2^2 x\right) = 0, \;\; r \neq 0, \tag{33a}$$

$$(ii) \quad \ddot{x} + k_1 x\dot{x} + \frac{(r-1)k_1^2}{2r^2}x^3 + \lambda_1 x = 0, \;\; r \neq 0, \tag{33b}$$

$$(iii) \quad \ddot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0, \;\; r = 0. \tag{33c}$$

In the above $k_1,\ k_2,\ k_3,\ k_4,\ \lambda_1$ and r are arbitrary parameters.

We note that the transformation $x = y - \frac{k_2}{k_1}$ transforms equation (33a) to the form

$$\ddot{y} + k_1 y\dot{y} + \frac{(r-1)k_1^2}{2r^2}y^3 - \frac{(r-1)k_2^2}{2r^2}y = 0, \quad r \neq 0. \tag{34}$$

Eq. (34) is obtained from Eq. (33b) by fixing $\lambda_1 = -\frac{(r-1)k_2^2}{2r^2}$. So, hereafter, we consider Eq. (33a) as a special case of Eq. (33b) and so discuss only Eq. (33b) as the general one. It may be noted that Eq. (33b) includes several known integrable cases. For example, the choice $r = 3$ and $\lambda_1 = 0$ in Eq. (33b) yields the MEE[18]. On the other hand the choice $r = -1$ leads us to the equation $\ddot{x} + k_1 x\dot{x} - k_1^2 x^3 + \lambda_1 x = 0$ which can be solved in terms of Weierstrass elliptic function[20]. *The other choices of r lead to new integrable cases* as we see below.

Substituting the forms of S and R (vide Eqs. (32b) and (32c)) into the general form of the integral of motion (25) and evaluating the resultant integrals, we obtain the following time independent first integrals for the cases (33b) and (33c):

$$(iia) \quad I_1 = \left(\dot{x} + \frac{(r-1)}{2r}k_1 x^2 + \frac{r\lambda_1}{k_1}\right)^{-r} \tag{35a}$$
$$\times\left[\dot{x}(\dot{x} + \frac{k_1}{2}x^2 + \frac{r^2\lambda_1}{(r-1)k_1}) + \frac{(r-1)}{r^2}(\frac{k_1}{2}x^2 + \frac{r^2\lambda_1}{(r-1)k_1})^2\right], \;\; r \neq 0, 1, 2,$$

$$(iib) \quad I_1 = \frac{4k_1\dot{x}}{k_1^2 x^2 + 4k_1\dot{x} + 8\lambda_1} - \log(k_1^2 x^2 + 4k_1\dot{x} + 8\lambda_1), \;\; r = 2, \tag{35b}$$

$$(iic) \quad I_1 = \dot{x} + \frac{k_1}{2}x^2 - \frac{\lambda_1}{k_1}\log(k_1\dot{x} + \lambda_1), \;\; r = 1, \tag{35c}$$

$$(iii) \quad I_1 = \frac{\dot{x}^2}{2} + \frac{k_3}{4}x^4 + \frac{k_4}{3}x^3 + \frac{\lambda_1}{2}x^2, \;\; r = 0. \tag{35d}$$

Note that in Eq. (35a), r can take any real value, except $0, 1, 2$. In the above integrals I_1 given by Eqs. (35a) - (35c) correspond to the ODE (33b), while (35d) corresponds to the Eq. (33c).

Due to the fact that the integrals of motion (35) are time independent, one can look for a Hamiltonian description for the respective equations of motion. In fact, we obtain the explicit Hamiltonian forms for all the above cases.

4. Hamiltonian Description of (35)

Assuming the existence of a Hamiltonian

$$I(x,\dot{x}) = H(x,p) = p\dot{x} - L(x,\dot{x}), \tag{36}$$

where $L(x,\dot{x})$ is the Lagrangian and p is the canonically conjugate momentum, we have

$$\begin{aligned}
\frac{\partial I}{\partial \dot{x}} &= \frac{\partial H}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x} + p - \frac{\partial L}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x},\\
\frac{\partial I}{\partial x} &= \frac{\partial H}{\partial x} = \frac{\partial p}{\partial x}\dot{x} - \frac{\partial L}{\partial x}.
\end{aligned} \tag{37}$$

From (37) we identify

$$\begin{aligned}
p &= \int \frac{I_{\dot{x}}}{\dot{x}} d\dot{x},\\
L &= \int (p_x\dot{x} - I_x)dx + \int [p - \frac{d}{d\dot{x}} \int (p_x\dot{x} - I_x)dx] d\dot{x}.
\end{aligned} \tag{38}$$

Plugging the expressions (36) into (38) one can evaluate the canonically conjugate momentum and the associated Lagrangian as well as the Hamiltonian. They read as follows:

(a) <u>The canonical momenta</u> :

$$(iia,b) \quad p = \frac{1}{r-1}\left(\dot{x} + \frac{(r-1)}{r}\frac{k_1}{2}x^2 + \frac{r\lambda_1}{k_1}\right)^{1-r}, \quad r \neq 0,1 \tag{39a}$$

$$(iic) \quad p = \log(k_1\dot{x} + \lambda_1), \quad r = 1 \tag{39b}$$

$$(iii) \quad p = \dot{x}, \quad r = 0. \tag{39c}$$

(Note in the above $r = 2$ is included in Eq. (39b) itself).

(b) <u>The Lagrangian</u> :

$$(iia) \quad L = \frac{1}{(2-r)(r-1)}\left(\dot{x} + \frac{(r-1)}{r}\frac{k_1x^2}{2} + \frac{r\lambda_1}{k_1}\right)^{2-r}, \quad r \neq 0,1,2 \tag{40a}$$

$$(iib) \quad L = \log(4k_1\dot{x} + 8\lambda_1 + k_1^2x^2), \quad r = 2 \tag{40b}$$

$$(iic) \quad L = \frac{\lambda_1}{k_1}\log(k_1\dot{x} + \lambda_1) + \dot{x}(\log(k_1\dot{x} + \lambda_1) - 1) - \frac{1}{2}k_1x^2, \quad r = 1 \tag{40c}$$

$$(iii) \quad L = \frac{\dot{x}^2}{2} - \frac{k_3}{4}x^4 - \frac{k_4}{3}x^3 - \frac{\lambda_1}{2}x^2, \quad r = 0. \tag{40d}$$

(*c*) <u>The Hamiltonian</u> :

$$(iia)\quad H=\left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)}-p\left(\frac{(r-1)}{2r}k_1x^2+\frac{r\lambda_1}{k_1}\right)\right],\quad r\neq 0,1,2 \tag{41a}$$

$$(iib)\quad H=\frac{2\lambda_1}{k_1}p+\frac{k_1}{4}x^2p+\log\left(\frac{4k_1}{p}\right),\quad r=2 \tag{41b}$$

$$(iic)\quad H=\frac{1}{k_1}\left(e^p-\lambda_1p+\frac{k_1^2}{2}x^2-\lambda_1\right),\quad r=1 \tag{41c}$$

$$(iii)\quad H=\frac{p^2}{2}+\frac{k_3}{4}x^4+\frac{k_4}{3}x^3+\frac{\lambda_1}{2}x^2,\quad r=0. \tag{41d}$$

One can check that the Hamilton's equations of motion are indeed equivalent to the appropriate equation (10).

Since Eqs. (33b) and (33c) admit time independent Hamiltonians they can be classified as Liouville integrable systems. *The important fact we want to stress here is that for arbitrary values of* r*, including fractional values, the equation (33b) is integrable.*

5. *Canonical transformation for the Hamiltonian Eqs. (41)*

Interestingly, we also identifed suitable canonical transformation to standard particle in a potential description for the Hamiltonians (41). Now introducing the canonical transformations

$$x=\frac{2rP}{k_1U},\qquad p=-\frac{k_1U^2}{4r},\quad r\neq 0,1, \tag{42}$$

$$x=\frac{P}{k_1},\qquad p=-k_1U,\quad r=1 \tag{43}$$

the Hamiltonian H in Eq. (41) can be recast in the standard form (after rescaling)

$$H=\begin{cases}\frac{1}{2}P^2+\frac{(1-r)}{(r-2)}\left(\frac{(r-1)k_1U^2}{4r}\right)^{\frac{(r-2)}{r-1}}+\frac{(r-1)\lambda_1}{4}U^2, & r\neq 0,1,2\\[2ex] \frac{1}{2}P^2+\frac{\lambda_1}{4}U^2+\log\left(\frac{32}{U^2}\right), & r=2\\[2ex] \frac{1}{2}P^2+e^{-k_1U}+\lambda_1k_1U, & r=1\\[2ex] \frac{1}{2}P^2+\frac{k_3}{4}U^4+\frac{\lambda_1}{2}U^2, & r=0.\end{cases} \tag{44}$$

It is straightforward to check that when U and P are canonical so do x and p (and vice versa) and the corresponding equations of motion turn out to be

$$\ddot{U}-2\bigg(\frac{(r-1)k_1}{4r}\bigg)^{\frac{(2-r)}{(1-r)}}U^{\frac{(3-r)}{(1-r)}}+\frac{(r-1)\lambda_1}{2}U=0, \quad r\neq 0,1 \tag{45a}$$

$$\ddot{U}-k_1e^{-U}+k_1\lambda_1=0, \qquad r=1 \tag{45b}$$

$$\ddot{U}+k_3U^3+\lambda_1U=0, \qquad r=0. \tag{45c}$$

One may note that the equations of motion now become standard type anharmonic oscillator equations.

B. The case $I_t \neq 0$

In the previous sub-section we considered the case $I_t = 0$. As a consequence S turns out to be $\frac{-\phi}{\dot{x}}$. However in the case $I_t = 0$, the function S has to be determined from Eq. (22), that is,

$$\begin{aligned} S_t+\dot{x}S_x-((k_1x+k_2)\dot{x}+k_3x^3+k_4x^2+\lambda_1x)S_{\dot{x}} \\ =(k_1\dot{x}+3k_3x^2+2k_4x+\lambda_1)-(k_1x+k_2)S+S^2. \end{aligned} \tag{46}$$

Since it is too difficult to solve Eq. (46) for its general solution, we seek a particular solution for S, which is sufficient for our purpose. In particular, we seek a simple rational expression for S in the form

$$S=\frac{a(t,x)+b(t,x)\dot{x}}{c(t,x)+d(t,x)\dot{x}}, \tag{47}$$

where a, b, c and d are arbitrary functions of t and x which are to be determined. Of course, the analysis of this form alone does not exhaust all possible cases of interest. We hope to make a more exhaustive study of Eq. (46) separately. Substituting (47) into (46) and equating the coefficients of different powers of $\dot{x}$ to zero, we get

$$
\begin{aligned}
& db_x - bd_x - k_1 d^2 = 0, \\
& db_t - bd_t + cb_x - bc_x + a_x d - ad_x - 2k_1 cd - (3k_3x^2 + 2k_4x + \lambda_1)d^2 \\
& \qquad\qquad\qquad\qquad\qquad\qquad +(k_1x + k_2)bd - b^2 = 0, \\
& cb_t - bc_t + da_t - ad_t + ca_x - ac_x - k_1c^2 - 2(3k_3x^2 + 2k_4x + \lambda_1)cd \\
& \qquad\qquad\qquad\qquad\qquad\qquad +2(k_1x + k_2)ad - 2ab = 0, \\
& ca_t - ac_t - (k_3x^3 + k_4x^2 + \lambda_1 x)(bc - ad) - (3k_3x^2 + 2k_4x + \lambda_1)c^2 \\
& \qquad\qquad\qquad\qquad\qquad\qquad +(k_1x + k_2)ac - a^2 = 0. \qquad (48)
\end{aligned}
$$

The determining equation for the functions a, b, c and d have now turned out to be nonlinear. To solve these equations we further assume that the functions a, b, c and d are polynomials in x with coefficients which are arbitrary functions in t. Substituting these forms into Eqs. (48) we obtain another enlarged set of determining equations for the unknowns and solving the latter consistently we obtain nontrivial solutions for the functions a, b, c and d for four sets of parametric choices. We present the explicit forms of the associated null function S given by (47) and the parametric restrictions in Table I.

Now substituting the forms of S into Eq. (23) and solving the resultant equation we obtain the corresponding forms of R. To solve the determining equation for R we again seek the same form of ansatz (28) but with explicit t dependence on the coefficient functions, that is, $R = \frac{S_d}{(A(t,x)+B(t,x)\dot{x})^r}$, where S_d is the denominator of S. We report the resultant forms of R in Table I. Once S and R are determined then one has to verify the compatibility of this set (S, R) with the extra constraint Eq. (24). We find that the forms S and R given in Table I do satisfy the extra constraint equation and form a compatible solution. Now substituting S_i's and R_i's into Eq. (25) one can construct the associated integrals of motion. We report the integrals of motion (I) in Table I along with the forms S and R.

At this stage, we note that the first integral for the *case* (i) with $k_2, \lambda_1 = 0$ has been derived in Ref. 18 through Lie symmetry analysis. However, recently, we have derived[9] the first integral for arbitrary values of k_2 and λ_1. *The case* (ii) *is new to the literature.* The first integral for the case (iii) was reported recently in Refs. 9,12 and 13. *The first integral for the case* (iva) *is new to the literature.* The *case* $r = 0$ discussed as (ivb) is nothing but the force-free Duffing oscillator whose integrability has been discussed in Refs. 12 and 14.

TABLE I: Parametric restrictions, null forms (S), integrating factors (R) and time dependent integrals of motion (I) of $\ddot{x}+(k_1x+k_2)\dot{x}+k_3x^3+k_4x^2+\lambda_1x=0$ (identified with the assumed ansatz form of S and R)

Cases	Parametric restrictions	Null form (S)	Integrating factor (R)	Integrals of motion (I)
(i)	$k_3=\frac{k_1^2}{9},\ k_4=\frac{k_1k_2}{3}$ $(k_1,\ k_2,\ \lambda_1:\text{arbitrary})$	$\frac{(\frac{k_1}{3}x^2-\dot{x})}{x}$	$\frac{xe^{\mp\omega t}}{(\dot{x}-\frac{(k_2\pm\omega)}{2}x+\frac{k_1}{3}x^2)^2}$	$(a)\ I=e^{\mp\omega t}\left(\frac{3\dot{x}-\frac{3(-k_2\mp\omega)}{2}x+k_1x^2}{3\dot{x}-\frac{3(-k_2\pm\omega)}{2}x+k_1x^2}\right)$, $k_2,\lambda_1\neq0,\ \omega=(k_2^2-4\lambda_1)^{\frac{1}{2}}$ $(b)\ I=-t+\frac{x}{(\frac{k_2}{2}x+\frac{k_1}{3}x^2+\dot{x})}$, $k_2^2=4\lambda_1$
(ii)	$k_3=0,k_4=\frac{k_1}{4}(k_2\pm\omega)$, $(k_1,\ k_2,\ \lambda_1:\text{arbitrary})$	$\frac{1}{2}(k_2\mp\omega)+k_1x$,	$e^{\frac{(k_2\pm\omega)}{2}t}$	$I=\left(\dot{x}+\frac{k_2\mp\omega}{2}x+\frac{k_1}{2}x^2\right)e^{(\frac{k_2\pm\omega}{2})t}$, $\omega=(k_2^2-4\lambda_1)^{\frac{1}{2}}$
(iii)	$k_1,\ k_3=0,\ \lambda_1=\frac{6k_2^2}{25}$ $(k_2,\ k_4:\text{arbitrary})$	$\frac{(\frac{2k_2\dot{x}}{5}+\frac{4k_2^2x}{25}+k_4x^2)}{(\dot{x}+\frac{2k_2}{5}x)}$	$(\dot{x}+\frac{2k_2}{5}x)e^{\frac{6}{5}k_2t}$	$I=e^{\frac{6}{5}k_2t}\left(\frac{\dot{x}^2}{2}+\frac{2k_2}{5}x\dot{x}+\frac{2k_2^2}{25}x^2+\frac{k_4}{3}x^3\right)$
(iva)	$k_3=\frac{(r-1)k_1^2}{2r^2},\ k_4=\frac{k_1k_2}{3}$, $\lambda_1=\frac{2k_2^2}{9},\ r\neq0$ $(k_1,\ k_2,\ r:\text{arbitrary})$	$\frac{k_2}{3}+k_1x+\frac{3k_3x^3}{(3\dot{x}+k_2x)}$	$\frac{(k_2x+3\dot{x})e^{(\frac{2(2-r)k_2}{3})t}}{(\frac{k_2}{3}x+rk_3x^2+\dot{x})^r}$	$I=\left(\frac{k_3}{2}x^4+(\dot{x}+\frac{k_2}{3}x)(\dot{x}+\frac{k_2}{3}x+\frac{k_1}{2}x^2)\right)$ $\times\left(\dot{x}+\frac{k_2}{3}x+rk_3x^2\right)^{-r}e^{\frac{2(2-r)}{3}k_2t}$, $r\neq2$ $I=\frac{2}{3}k_2t+\log(4k_2x+3k_1x^2+12\dot{x})$ $-\frac{4(k_2x+3\dot{x})}{(4k_2x+3k_1x^2+12\dot{x})}$, $r=2$
(ivb)	$k_1=0,\ k_4=0,$ $\lambda_1=\frac{2k_2^2}{9},r=0$ $(k_2,\ k_3:\text{arbitrary})$	$\frac{(\frac{k_2}{3}\dot{x}+\frac{k_2^2}{9}x+k_3x^3)}{(\dot{x}+\frac{k_2x}{3})}$	$e^{\frac{4}{3}k_2t}(\dot{x}+\frac{k_2x}{3})$	$I=e^{\frac{4}{3}k_2t}\left[\frac{\dot{x}^2}{2}+\frac{k_2}{3}x\dot{x}+\frac{k_2^2}{18}x^2+\frac{k_3}{4}x^4\right]$

Since we obtained only one integral in each case, (except case (i) where we have found second explicit time dependent integral, see Ref. 9), which are also time dependent ones, we need to integrate them further to obtain the second integration constant and prove the complete integrability of the respective systems, which is indeed a difficult task.

In this connection we have introduced a new method[1,9] which can be effectively used to transform the time dependent integral into a time independent one, for a *class of problems*, so that the latter can be integrated easily. We invoke this procedure here in order to integrate the time dependent first integrals and obtain the general solution for all the cases in Table I (except case (iv), see below). For the *case* (iv), we prove the Liouville integrability of it.

C. Method of transforming time dependent first integral to time independent one

Let us assume that there exists a first integral for the equation (10) of the form,

$$I = F_1(t, x, \dot{x}) + F_2(t, x). \tag{49}$$

Now let us split the function F_1 further in terms of two functions such that F_1 itself is a function of the product of the two functions, say, a perfect differentiable function $\frac{d}{dt}G_1(t, x)$ and another function $G_2(t, x, \dot{x})$, that is,

$$I = F_1\left(\frac{1}{G_2(t, x, \dot{x})}\frac{d}{dt}G_1(t, x)\right) + F_2\left(G_1(t, x)\right), \tag{50}$$

where F_1 is a function which involves the variables t, x and $\dot{x}$ whereas F_2 should involve only the variable t and x. We note that while rewriting Eq. (49) in the form (50), we demand that the function $F_2(t, x)$ in (49) automatically to be a function of $G_1(t, x)$. Now identifying the function G_1 as the new dependent variable and the integral of G_2 over time as the new independent variable, that is,

$$w = G_1(t, x), \quad z = \int_o^t G_2(t', x, \dot{x})dt', \tag{51}$$

one indeed obtains an explicit transformation to remove the time dependent part in the first integral. We note here that the integration on the right hand side of (51) leading to z can be performed provided the function G_2 is an exact derivative of t, that is, $G_2 = \frac{d}{dt}z(t, x) = \dot{x}z_x + z_t$, so that z turns out to be a function t and x alone. In terms of the new variables, Eq. (50) can be modified to the form

$$I = F_1\left(\frac{dw}{dz}\right) + F_2(w). \tag{52}$$

In other words,

$$F_1\left(\frac{dw}{dz}\right) = I - F_2(w). \tag{53}$$

Now rewriting Eq. (52) one obtains a separable equation

$$\frac{dw}{dz} = f(w), \tag{54}$$

which can lead to the solution after an integration. Now rewriting the solution in terms of the original variables one obtains a general solution for the given equation.

In the following using the above idea we integrate the first integrals given in Table I and deduce the second integration constant and general solution.

D. Application

Case (ia): $k_3 = \frac{k_1^2}{9},\ k_4 = \frac{k_1 k_2}{3},\ k_1,\ k_2$ and λ_1 : arbitrary:

The parametric restrictions given above fix the equation of motion (10) in the form

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1 k_2}{3}x^2 + \lambda_1 x = 0, \tag{55}$$

Let us rewrite the first integral associated for this case (vide case (i) in Table I) in the form

$$I_1 = -\frac{k_1 e^{\frac{k_2 \mp \omega}{2}t} x^2}{(3\dot{x} - \frac{(-k_2 \pm \omega)}{2}3x + k_1 x^2)}\left[\frac{d}{dt}\left((\frac{-3}{k_1 x} + \frac{-k_2 \pm \omega}{2\lambda_1})e^{\frac{-k_2 \mp \omega}{2}t}\right)\right], \tag{56}$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$. Comparing this with the equation (50), and using (51), we obtain

$$w = (\frac{-3}{k_1 x} + \frac{-k_2 \pm \omega}{2\lambda_1})e^{\frac{-k_2 \mp \alpha}{2}t}, \quad z = (\frac{-3}{k_1 x} + \frac{-k_2 \mp \omega}{2\lambda_1})e^{\frac{-k_2 \pm \omega}{2}t}. \tag{57}$$

Substituting (57) into Eq. (55), the latter becomes the free particle equation, namely, $\frac{d^2 w}{dz^2} = 0$, whose general solution is $w = I_1 z + I_2$, where I_1 and I_2 are integration constants. Rewriting w and z in terms of x and t one gets

$$x(t) = \left(\frac{6\lambda_1(1 - I_1 e^{\omega t})}{k_1\omega(1 + I_1 e^{\omega t}) - (k_2 \pm \omega)I_2 e^{\frac{k_2 \pm \omega}{2}t} - k_1 k_2(1 - I_1 e^{\omega t})}\right), \tag{58}$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$.

Interestingly one can consider several sub-cases. In the following we discuss some important ones which are being widely discussed in the current literature. In particular, the difference in dynamics arises mainly depending on the sign of the parameter α $(=\sqrt{k_2^2-4\lambda_1})$. We consider the cases (i) $k_2^2 < 4\lambda_1$ (ii) $k_2^2 > 4\lambda_1$ and (iii) $k_2^2 = 4\lambda_1$ separately. The restriction $k_2^2 < 4\lambda_1$ reduces the solution (58) to the form[4],

$$x(t) = \frac{A\cos(\omega_0 t+\delta)}{\left(e^{\frac{k_2}{2}t} + \frac{2k_1 A}{3(k_2^2+4\omega_0^2)}(2\omega_0\sin(\omega_0 t+\delta) - k_2\cos(\omega_0 t+\delta))\right)}, \tag{59}$$

where $\omega_0 = \frac{\sqrt{4\lambda_1-k_2^2}}{2}$ and $\delta,\ A$ are arbitrary constants. A further restriction $k_2 = 0$ gives us the purely sinusoidally oscillating solution[19]

$$x(t) = \frac{A\sin(\omega_0 t+\delta)}{1-(\frac{k}{3\omega_0})A\cos(\omega_0 t+\delta)}, \quad 0 \le A < \frac{3\omega_0}{k}, \quad \omega_0 = \sqrt{\lambda_1}, \tag{60}$$

where A and δ are arbitrary constants. The associated equation of motion, namely $\ddot{x} + k_1 x\dot{x} + \frac{k_1^2}{9}x^3 + \lambda_1 x = 0$, admits very interesting nonlinear dynamics, see for example in Ref. 19.

On the other hand, in the limit $k_2^2 > 4\lambda_1$ the solution looks like a dissipative/front-like one[19]. A further restriction $\lambda_1 = 0$ takes us to the solution of the form[11]

$$x(t) = \left(\frac{3k_2(I_1 e^{k_2 t}-1)}{k_1 + k_2(3I_2 + k_1 I_1 t)e^{k_2 t}}\right). \tag{61}$$

Case (ib): $k_3 = \frac{k_1^2}{9},\ k_4 = \frac{k_1 k_2}{3},\ k_2^2 = 4\lambda_1,\ k_1$ and k_2 : arbitrary:

The third choice $k_2^2 = 4\lambda_1$ in (58) leads us to the solution

$$x(t) = \left(\frac{3(I_1+t)}{3I_2 e^{\frac{k_2}{2}t} - \frac{2k_1}{k_2^2}(2 + I_1 k_2 + k_2 t)}\right). \tag{62}$$

Further parametric restriction $k_2,\ \lambda_1 = 0$ provides us the general solution of the form

$$x(t) = \left(\frac{6(I_1+t)}{k_1(I_1+t)^2 + 6I_2}\right). \tag{63}$$

The underlying equation, that is, $\ddot{x} + k_1 x\dot{x} + \frac{k_1^2}{9}x^3 = 0$, is the $l = 1$ integrable case of Eq. (6) with the solution (7) (see for example in Refs. 18 and 19).

Case (ii): $\underline{k_3 = 0,\ k_4 = \frac{k_1}{4}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1}),\ k_1,\ k_2 \text{ and } \lambda_1 :\ \text{arbitrary:}}$

In this case we have the equation of the form

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1}{4}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1})x^2 + \lambda_1 x = 0. \tag{64}$$

The associated first integral reads (vide case (*ii*) in Table 1)

$$I = \left(\dot{x} + \frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}x + \frac{k_1}{2}x^2\right)e^{\frac{k_2 \pm \sqrt{k_2^2 - 4\lambda_1}}{2}t}. \tag{65}$$

Note that Eq. (65) can be rewritten as a Riccati equation of the form[21]

$$\dot{x} = Ie^{(\frac{-k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2})t} - \left(\frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}\right)x - \frac{k_1}{2}x^2. \tag{66}$$

The general solution of the Riccati equation is known to be free from movable critical points and satisfies the Painlevé property. In this sense Eq. (64) can be considered as integrable in the Painlevé criteria sense. However, in the general case, (66), it is not clear whether it can be explicitly integrated further. However, for the special case $\lambda_1 = \frac{2k_2^2}{9}$ it can be integrated as follows.

The restriction $\lambda_1 = \frac{2k_2^2}{9}$ fixes the equation of motion (64) and the first integral (65) in the forms

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1 k_2}{3}x^2 + \frac{2k_2^2}{9}x = 0, \tag{67}$$

and

$$I = \left(\dot{x} + \frac{k_2}{3}x + \frac{k_1}{2}x^2\right)e^{\frac{2k_2}{3}t}, \tag{68}$$

respectively. Now rewriting (68) in the form (50), we get

$$I = e^{\frac{k_2}{3}t}\left(\frac{d}{dt}(xe^{\frac{k_2}{3}t})\right) + \frac{k_1}{2}(xe^{\frac{k_2}{3}t})^2. \tag{69}$$

Identifying the dependent and independent variables from (69) and using the identities (51), we obtain the transformation

$$w = xe^{\frac{k_2}{3}t}, \quad z = -\frac{3}{k_2}e^{-\frac{k_2}{3}t}. \tag{70}$$

Using the transformation (70) the first integral (68) can be rewritten in the form

$$\hat{I} = w' + \frac{k_1}{2}w^2 \tag{71}$$

which in turn leads to the solution by an integration, that is,

$$w(z) = \sqrt{\frac{2I}{k_1}} \tanh\left[\sqrt{\frac{k_1 I}{2}}(z - z_0)\right], \tag{72}$$

where z_0 is arbitrary constant. Rewriting (72) in terms of old variables we get

$$x(t) = \sqrt{\frac{2I}{k_1}} e^{-(\frac{k_2}{3})t} \tanh\left[\frac{3}{k_2}(\sqrt{\frac{k_1 I}{2}})(e^{-\frac{k_2}{3}t_0} - e^{-\frac{k_2}{3}t})\right], \tag{73}$$

where t_0 is the second integration constant.

Case (iii): $k_1,\ k_3 = 0,\ \lambda_1 = \frac{6k_2^2}{25},\ k_2$ and $k_4:$ arbitrary:

The corresponding equation of motion is

$$\ddot{x} + k_2\dot{x} + k_4 x^2 + \frac{6k_2^2}{25}x = 0. \tag{74}$$

Rewriting the associated first integral I_1, given in Case (iii) in Table I, in the form (49), we get

$$I = \frac{1}{2}\left(\dot{x} + \frac{2k_2}{5}x\right)^2 e^{\frac{6}{5}k_2 t} + \frac{k_4}{3}x^3 e^{\frac{6}{5}k_2 t}. \tag{75}$$

Now splitting the first term in Eq. (75) further in the form (50), we obtain

$$I = e^{\frac{2k_2 t}{5}}\left(\frac{d}{dt}(\frac{1}{\sqrt{2}}x e^{\frac{2k_2 t}{5}})\right)^2 + \frac{k_4}{3}(x e^{\frac{2}{5}k_2 t})^3. \tag{76}$$

Identifying the dependent and independent variables from (76) and using the relations (51), we obtain the transformation

$$w = \frac{1}{\sqrt{2}}x e^{\frac{2k_2 t}{5}}, \quad z = -\frac{5}{k_2}e^{-\frac{k_2 t}{5}}. \tag{77}$$

Using this transformation, (77), the first integral (75) can be rewritten in the form

$$\hat{I} = w'^2 + \frac{\hat{k_4}}{3}w^3, \tag{78}$$

where $\hat{k_4} = 2\sqrt{2}k_4$, which inturn leads to

$$w'^2 = 4w^3 - g_3, \tag{79}$$

where $z = 2\sqrt{\frac{3}{\hat{k_4}}}\hat{z}$ and $g_3 = -\frac{12I_1}{\hat{k_4}}$. The solution of this differential equation can be represented in terms of Weierstrass function[12,13] $\varrho(\hat{z}; 0, g_3)$.

Case (iv): $k_3 = \frac{(r-1)}{2r^2}k_1^2,\ k_4 = \frac{k_1k_2}{3},\ \lambda_1 = \frac{2k_2^2}{9},\ k_1,\ k_2$ and r : arbitrary (but not zero):

The above parameters fix the equation of motion (10) in the form

$$\ddot{x} + (k_1x + k_2)\dot{x} + \frac{(r-1)k_1^2}{2r^2}x^3 + \frac{k_1k_2}{3}x^2 + \frac{2k_2^2}{9}x = 0, \quad r \neq 0. \tag{80}$$

The associated first integral reads (vide case (iva) in Table I)

$$I = \begin{cases} \left(\frac{(r-1)}{4r^2}k_1^2x^4 + (\dot{x} + \frac{k_2}{3}x)(\dot{x} + \frac{k_2}{3}x + \frac{k_1}{2}x^2)\right) \\ \qquad \times\left(\dot{x} + \frac{k_2}{3}x + \frac{(r-1)}{2r}k_1x^2\right)^{-r} e^{\frac{2(2-r)}{3}k_2t}, & r \neq 0, 2 \\ \frac{2}{3}k_2t + \log(4k_2x + 3k_1x^2 + 12\dot{x}) - \frac{4(k_2x+3\dot{x})}{(4k_2x+3k_1x^2+12\dot{x})}, & r = 2. \end{cases} \tag{81}$$

Rewriting Eq. (81) in the form (50), we get

$$I = \begin{cases} \left(\frac{(r-1)k_1^2}{4r^2}(xe^{\frac{k_2}{3}t})^4 + \frac{d}{dt}(xe^{\frac{k_2}{3}t})\left(\frac{d}{dt}(xe^{\frac{k_2}{3}t})e^{\frac{k_2}{3}t} + \frac{k_1}{2}(xe^{\frac{k_2}{3}t})^2\right)e^{\frac{k_2}{3}t}\right) \\ \qquad \times\left(\frac{d}{dt}(xe^{\frac{k_2}{3}t})e^{\frac{k_2}{3}t} + \frac{k_1(r-1)}{2r}(xe^{\frac{k_2}{3}t})^2\right)^{-r}, & r \neq 0, 2 \\ \frac{4\frac{d}{dt}(xe^{\frac{k_2}{3}t})e^{\frac{k_2}{3}t}}{k_1(xe^{\frac{k_2}{3}t})^2 + 4\frac{d}{dt}(xe^{\frac{k_2}{3}t})e^{\frac{k_2}{3}t}} - \log\left(k_1(xe^{\frac{k_2}{3}t})^2 + 4\frac{d}{dt}(xe^{\frac{k_2}{3}t})e^{\frac{k_2}{3}t}\right), & r = 2. \end{cases} \tag{82}$$

Identifying the dependent and independent variables from (82) and the relations (51), we obtain the transformation

$$w = xe^{\frac{k_2}{3}t}, \quad z = -\frac{3}{k_2}e^{-\frac{k_2}{3}t}. \tag{83}$$

In terms of the new variables, (83), the first integral I given above, (82), can be written as

$$I = \begin{cases} \left(w' + \frac{(r-1)}{2r}k_1w^2\right)^{-r}\left[\frac{(r-1)}{4r^2}k_1^2w^4 + w'(w' + \frac{k_1}{2}w^2)\right], & r \neq 0, 2 \\ \frac{4w'}{k_1w^2+4w'} - \log(k_1w^2 + 4w'), & r = 2. \end{cases} \tag{84}$$

On the other hand the transformation (83) modifies the equation (80) to the form

$$w'' + k_1ww' + \frac{(r-1)k_1^2}{2r^2}w^3 = 0, \quad r \neq 0 \text{ and } ' = \frac{d}{dz}. \tag{85}$$

Finally, for the case $r = 0$, we have an equation of the form (vide case (ivb) in Table I), $\ddot{x} + k_2\dot{x} + k_3x^3 + \frac{2}{9}k_2^2x = 0$, which is nothing but the force-free Duffing oscillator equation. Again using the transformation (83), the associated time dependent integral given in Table I can be rewritten as

$$I = \frac{w'^2}{2} + \frac{k_3}{4}w^4, \qquad r = 0. \tag{86}$$

Though it is difficult to integrate the above time independent first integrals, (84), as they are in complicated forms, one can easily check that Eq. (86) ($r = 0$) can be integrated in terms of Jacobian elliptic function[14] and the case $r = 1$ is already discussed as case (ii) in this section. For the other cases one can give a Hamiltonian formulation as in Sec. III A 4 and write the corresponding Hamiltonian as

$$H = \begin{cases} \left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - p\left(\frac{(r-1)}{2r}k_1w^2\right)\right], & r \neq 0, 1, 2, \\ \frac{k_1}{4}w^2p + \log\left(\frac{4k_1}{p}\right), & r = 2 \\ e^p + \frac{k_1}{2}w^2, & r = 1 \\ \frac{p^2}{2} + \frac{k_3}{4}w^4, & r = 0 \end{cases} \tag{87}$$

where

$$p = \begin{cases} \frac{1}{r-1}\left(\frac{(r-1)}{2r}k_1w^2 + w'\right)^{1-r}, & r \neq 0, 1 \\ \log(w'), & r = 1 \\ w', & r = 0. \end{cases} \tag{88}$$

Thus one is ensured of Liouville integrability of system (85) and so (80) for all values of r. Further, following the analysis in the above subsection III A 5, one can make a canonical transformation (vide Eqs. (42)-(44)) to standard nonlinear oscillator equations.

E. Summary of results for the $q = 1$ case:

To summarize the results obtained in this section, we have identified six integrable cases in Eq. (10) among which four of them were already known in the literature and the remaining two are new. In the following, we tabulate all of them for convenience.

1. *Integrable equations already known in the literature*

(1) $$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1 k_2}{3}x^2 + \lambda_1 x = 0, \qquad (55)$$

(2) $$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1 k_2}{3}x^2 + \frac{2k_2^2}{9}x = 0, \qquad (67)$$

(3) $$\ddot{x} + k_2\dot{x} + k_4 x^2 + \frac{6k_2^2}{25}x = 0, \qquad (74)$$

(4) $$\ddot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0. \qquad (33c)$$

We note that the dynamics and certain transformation properties of Eq. (55) have been studied in detail by three of the present authors in Refs. 9 and 11 recently. In particular, we have shown that this equation admits certain unusual nonlinear dynamics[19]. The dynamics of Eqs. (67),(74) and (33c) can be found in Ref. 12.

2. *New integrable equations*

(1) $$\ddot{x} + k_1 x\dot{x} + k_3 x^3 + \lambda_1 x = 0, \qquad (33b)$$

(2) $$\ddot{x} + (k_1 x + k_2)\dot{x} + k_3 x^3 + \frac{k_1 k_2}{3}x^2 + \frac{2k_2^2}{9}x = 0, \qquad (80)$$

where $r^2 k_3 = \frac{(r-1)k_1^2}{2}$ and k_1, k_2, λ_1 and r are arbitrary parameters. We note that (33b) includes the first equation of MEE hierarchy (6) as a sub-case. Importantly, we showed that (33b) is a Hamiltonian system (see Eq. (41)) and so it is Liouville integrable. Equation (80) can be transformed to the integrable Eq. (85). Explicit general solution of certain special cases, namely, $r = 3$ or $\frac{3}{2}$ and $r = -1$ or $\frac{1}{2}$ are reported in Ref. 20.

IV. GENERALIZED FORCE FREE DVP FORM OF EQUATIONS

Let us now consider the case $q = 2$ in Eq. (9) or equivalently (11), that is,

$$\ddot{x} = -((k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x) \equiv \phi(x, \dot{x}). \tag{11}$$

Interestingly Eq. (11) includes another class of physically important nonlinear oscillators. For example, choosing $k_3 = 0$ one can get force-free Duffing-van der Pol oscillator equation. With the choice $k_2,\ k_4,\ \lambda_1 = 0$, it coincides with the second equation in the MEE hierarchy equation. Equation (11) with the restriction $k_3 = \frac{k_1^2}{16},\ k_4 = \frac{k_1 k_2}{4}$ and $\lambda_1 = (\omega_0^2 + \frac{k_2^2}{4})$, has been investigated in a different perspective in Ref. 4. However, the general equation of the form (11) has never been considered for integrability test and so we perform the same here.

To identify integrals of motion and the general solution of Eq. (11) we again seek the PS procedure. As the calculations are similar to the $q = 1$ case of Eq. (9) which was carried out in the previous section, in the following, we give only the important steps.

A. The case $I_t = 0$

By considering the same arguments given in Sec. III A 1, the null form S can be fixed easily in the form

$$S = -\frac{((k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x)}{\dot{x}}. \tag{89}$$

The respective R equation becomes

$$\begin{aligned} R_t + \dot{x}R_x - ((k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x)R_{\dot{x}} \\ = ((k_1 x^2 + k_2) + \frac{((k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x)}{\dot{x}})R. \end{aligned} \tag{90}$$

To seek a particular form for R one may seek a suitable ansatz. We assume R to be of the form (28) and investigate the system (90) as before. Following a similar procedure we find that a nontrivial particular solution for (90) exists in the form

$$R = \frac{\dot{x}}{(\frac{(r-1)}{r}(\frac{k_1}{3}x^3 + k_2 x) + \dot{x})^r}, \tag{91}$$

where r, k_1 and k_2 are arbitrary parameters and the remaining parameters, k_3, k_4 and λ_1, are fixed by the relations

$$k_3 = \frac{(r-1)}{3r^2}k_1^2, \quad k_4 = \frac{4(r-1)}{3r^2}k_1 k_2, \quad \lambda_1 = \frac{(r-1)}{r^2}k_2^2. \tag{92}$$

Further, we confirmed the compatibility of the functions S and R with the extra constraint (24) also. We note that unlike the earlier case, $q = 1$, we do not get a nontrivial solution for the parametric restriction $k_2,\ k_4 = 0$. The above restrictions fix the Eq. (11) to the following specific forms:

$$(ia) \qquad \ddot{x} + (k_1x^2 + k_2)\dot{x} + \frac{(r-1)}{3r^2}k_1^2x^5 + \frac{4(r-1)k_1k_2}{3r^2}x^3 + \frac{(r-1)k_2^2}{r^2}x = 0,\ r \neq 0 \tag{93a}$$

$$(ib) \qquad \ddot{x} + k_3x^5 + k_4x^3 + \lambda_1 x = 0,\ r = 0 \tag{93b}$$

Now substituting (89) and (91) into (25) and evaluating the integrals we obtain the first integrals in the form

$$(ia) \qquad I_1 = \left(\dot{x} + \frac{(r-1)}{r}(\frac{k_1}{3}x^3 + k_2x)\right)^{-r} \times \left[\dot{x}(\dot{x} + \frac{k_1}{3}x^3 + k_2x) + \frac{(r-1)}{r^2}(\frac{k_1}{3}x^3 + k_2x)^2\right],\quad r \neq 0, 2, \tag{94a}$$

$$(ib) \qquad I_1 = \frac{6\dot{x}}{(6\dot{x} + 3k_2x + k_1x^3)} - \log(6\dot{x} + 3k_2x + k_1x^3),\quad r = 2, \tag{94b}$$

$$(ii) \qquad I_1 = \frac{\dot{x}^2}{2} + \frac{k_3}{6}x^6 + \frac{k_4}{4}x^4 + \frac{\lambda_1}{2}x^2,\quad r = 0. \tag{94c}$$

Further, as in the $q = 1$ case in Sec. III A 4, the integrals (94) can be recast into the Hamiltonian form

$$(ia) \qquad H = \left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{r}p(\frac{k_1}{3}x^3 + k_2x)\right],\quad r \neq 0, 1, 2, \tag{95a}$$

$$(ib) \qquad H = \frac{k_2}{2}xp + \frac{k_1}{6}x^3p + \log(\frac{6}{p}),\quad r = 2, \tag{95b}$$

$$(ic) \qquad H = e^p + \frac{k_1}{3}x^3 + k_2x,\quad r = 1, \tag{95c}$$

$$(ii) \qquad H = \frac{p^2}{2} + \frac{k_3}{6}x^6 + \frac{k_4}{4}x^4 + \frac{\lambda_1}{2}x^2,\quad r = 0. \tag{95d}$$

where the corresponding canonical momenta respectively are

$$(ia, b) \qquad p = \frac{1}{(r-1)}\left(\dot{x} + \frac{(r-1)}{r}(\frac{k_1}{3}x^3 + k_2x)\right)^{(1-r)},\quad r \neq 0, 1, \tag{96a}$$

$$(ic) \qquad p = \log\dot{x},\quad r = 1, \tag{96b}$$

$$(ii) \qquad p = \dot{x},\quad r = 0. \tag{96c}$$

Note that in the above the parameters $r,\ k_1,\ k_2,\ k_3$ and λ_1 are all arbitrary. We also note here that unlike the $q=1$ case discussed in Sec. III, so far we have been unable to find suitable canonical transformations for the above Hamiltonian systems so that the standard 'potential' equation results. The problem is being further investigated.

B. The case $I_t \neq 0$

Now let us study the case $I_t \neq 0$. In this case S has to be determined from Eq. (22), that is,

$$\begin{aligned} S_t + \dot{x}S_x - ((k_1x^2+k_2)\dot{x} + k_3x^5 + k_4x^3 + \lambda_1 x)S_{\dot{x}} \\ = (2k_1x\dot{x} + 5k_3x^4 + 3k_4x^2 + \lambda_1) - (k_1x^2+k_2)S + S^2. \end{aligned} \tag{97}$$

As we did in the $q=1$ case of Eq. (9) we proceed to solve Eq. (97) with the same form of ansatz (47). Doing so we find that Eq. (97) admits non-trivial forms of solutions for certain specific parametric restrictions. We report both the parametric values and their respective forms of S in Table II.

Now substituting the forms of S into Eq. (23) and solving the resultant equation we obtain the corresponding forms of R. Once S and R are determined then one has to verify the compatibility of this solution with the extra constraint (24). Then one can substitute the null forms and integrating factors into (25) and construct the associated integrals of motion. We report the integrating factors (R) and time-dependent integrals of motion (I) in Table II.

The remaining task is to derive the general solution and establish the complete integrability of Eq. (11) for each parametric restriction. We again adopt the procedure given in Sec. III C and transform the time dependent integrals into time independent ones and integrate the latter and deduce the general solution. As the procedure is exactly the same we provide only the results in the following.

Case (ia): $k_3=\frac{k_1^2}{16},\ k_4=\frac{k_1k_2}{4},\ k_1,\ k_2$ and λ_1 : arbitrary:

Substituting the parametric restrictions given above in Eq. (11), we get

$$\ddot{x} + (k_1x^2+k_2)\dot{x} + \frac{k_1^2}{16}x^5 + \frac{k_1k_2}{4}x^3 + \lambda_1 x = 0. \tag{98}$$

TABLE II: Parametric restrictions, null forms (S), integrating factors (R) and time dependent integrals of motion (I) of $\ddot{x}+(k_1x^2+k_2)\dot{x}+k_3x^5+k_4x^3+\lambda_1 x=0$ (identified with the assumed ansatz form of S and R)

Cases	Parametric restrictions	Null form (S)	Integrating factor (R)	Integrals of motion (I)
(i)	$k_3=\frac{k_1^2}{16},\ k_4=\frac{k_1k_2}{4}$ ($k_1,\ k_2,\ \lambda_1$: arbitrary)	$\dfrac{\frac{k_1}{2}x^3-\dot{x}}{x}$	$\dfrac{xe^{\mp\omega t}}{(\dot{x}-\frac{(k_2\pm\omega)}{2}x+\frac{k_1}{4}x^3)^2}$	$(a)\ I=e^{\mp\omega t}\left(\frac{4\dot{x}+2(k_2\pm\omega)x+k_1x^3}{4\dot{x}+2(k_2\mp\omega)x+k_1x^3}\right)$, $k_2,\lambda_1\neq 0,\ \omega=(k_2^2-4\lambda_1)^{\frac{1}{2}}$ $(b)\ I=-t+\frac{x}{(\frac{k_2}{2}x+\frac{k_1x^3}{4}+\dot{x})}$, $\quad k_2^2=4\lambda_1$
(ii)	$k_3=0, k_4=\frac{k_1}{6}(k_2\pm\omega)$, ($k_1,\ k_2,\ \lambda_1$: arbitrary)	$\frac{1}{2}(k_2\mp\omega)+k_1x^2$	$e^{\frac{(k_2\pm\omega)}{2}t}$	$I=\left(\dot{x}+\frac{k_2\mp\omega}{2}x+\frac{k_1}{3}x^3\right)e^{(\frac{k_2\pm\omega}{2})t}$, $\omega=(k_2^2-4\lambda_1)^{\frac{1}{2}}$
(iii)	$k_1,\ k_3=0,\ \lambda_1=\frac{2k_2^2}{9}$ ($k_2,\ k_4$: arbitrary)	$\left(\dfrac{\frac{k_2}{3}\dot{x}+\frac{k_2^2}{9}x+k_4x^3}{\dot{x}+\frac{k_2}{3}x}\right)$	$(\dot{x}+\frac{k_2}{3}x)e^{\frac{4}{3}k_2t}$	$I=e^{\frac{4}{3}k_2t}\left[\frac{\dot{x}^2}{2}+\frac{k_2}{3}x\dot{x}+\frac{k_2^2}{18}x^2+\frac{k_4}{4}x^4\right]$
(iva)	$k_3=\frac{(r-1)k_1^2}{3r^2},\ k_4=\frac{k_1k_2}{4}$, $\lambda_1=\frac{3k_2^2}{16},\ r\neq 0$ ($k_1,\ k_2,\ r$: arbitrary)	$\dfrac{k_2}{4}+k_1x^2+\dfrac{4k_3x^5}{(4\dot{x}+k_2x)}$	$\dfrac{(k_2x+4\dot{x})e^{\frac{3(2-r)}{4}k_2t}}{(\frac{k_2}{4}x+rk_3x^3+\dot{x})^r}$	$I=\left(\frac{k_3}{3}x^6+(\dot{x}+\frac{k_2}{4}x)(\dot{x}+\frac{k_2}{4}x+\frac{k_1}{3}x^3)\right)$ $\times\left(\dot{x}+\frac{k_2}{4}x+rk_3x^3\right)^{-r}e^{\frac{3(2-r)}{4}k_2t},\ \ r\neq 2$ $I=\frac{3}{4}k_2t+\log(6k_2x+4k_1x^3+24\dot{x})$ $-\dfrac{6(k_2x+4\dot{x})}{(6k_2x+4k_1x^3+24\dot{x})}$, $\quad r=2$
(ivb)	$k_1=0,\ k_4=0$, $\lambda_1=\frac{3k_2^2}{16}, r=0$ ($k_2,\ k_3$: arbitrary)	$\left(\dfrac{\frac{k_2}{4}\dot{x}+\frac{k_2^2}{16}x+k_3x^5}{\dot{x}+\frac{k_2}{4}x}\right)$	$e^{\frac{3k_2}{2}t}(\dot{x}+\frac{k_2}{4}x)$	$I=e^{\frac{3k_2}{2}t}\left(\frac{\dot{x}^2}{2}+\frac{k_2}{4}x\dot{x}+\frac{k_2^2}{32}x^2+\frac{k_3}{6}x^6\right)$

We observed that the first integral of this case (i) (see Table II), when rewritten, is nothing but the Bernoulli equation which can be integrated strightforwardly[21] and it leads to the general solution of the form

$$x(t) = \left(\frac{8k_2\lambda_1(e^{\omega t} - I_1)^2}{I_1^2k_1k_2(-k_2+\omega) - e^{2\omega t}k_1k_2(k_2+\omega) + 8I_2k_2\lambda_1e^{(k_2+\omega)t} + 8I_1k_1\lambda_1e^{\omega t}}\right)^{\frac{1}{2}}, \quad (99)$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$. A sub-case of the Eq. (98), namely, $k_2^2 < 4\lambda_1$ has been studied by Smith[4,22] who showed that the corresponding equation of motion admits a damped oscillatory form of solution, namely,

$$x(t) = \frac{Acos(\omega_0 t + \delta)}{\left(e^{k_2t} - \frac{k_1A}{4k_2} + \frac{k_1A}{4(k_2^2+4\omega_0^2)}\left(2\omega_0 sin2(\omega_0t+\delta) - k_2cos2(\omega_0t+\delta)\right)\right)^{\frac{1}{2}}}, \quad (100)$$

where $\omega_0 = \frac{1}{2}\sqrt{4\lambda_1 + k_2^2}$ and $\delta,\ A$ are arbitrary constants.

On the other hand for $k_2^2 > 4\lambda_1$, the solution (99) becomes dissipative type having a front-like structure. In particular, for $\lambda_1 = 0$ we get a solution of the form

$$x(t) = \left(\frac{2\sqrt{k_2}(I_1e^{k_2t} - 1)}{(-k_1 + 2k_1I_1e^{k_2t}(2 + k_2I_1te^{k_2t}) + 4k_2I_2e^{2k_2t})^{\frac{1}{2}}}\right). \quad (101)$$

Case (ib): $k_3 = \frac{k_1^2}{16},\ k_4 = \frac{k_1k_2}{4},\ k_2^2 = 4\lambda_1,\ k_1$ and k_2 : arbitrary:

In this case we get the general solution of the form from (101) as

$$x(t) = \left(\frac{2(I_1+t)^2}{2e^{k_2t}I_2 - \frac{k_1}{k_2^3}(2 + I_1^2k_2^2 + 2k_2t + k_2^2t^2 + 2I_1k_2(1+k_2t))}\right)^{\frac{1}{2}}. \quad (102)$$

One may note that a sub-case of this equation, namely, $k_2 = \lambda_1 = 0$ leads us to the second equation in the MEE hierarchy (6) and the corresponding solution follows from Eq. (102) as

$$x(t) = \sqrt{6}\left(\frac{(I_1+t)^2}{6I_2 + k_1t(3I_1^2 + 3I_1t + t^2)}\right)^{\frac{1}{2}}. \quad (103)$$

This form exactly coincides with the solution (7) for $l = 2$.

Case (ii): $k_3 = 0,\ k_4 = \frac{k_1}{6}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1}),\ k_1,\ k_2$ and λ_1 : arbitrary:

The repective equation of motion and the first integral are (see Table II)

$$\ddot{x}+(k_1x^2+k_2)\dot{x}+\frac{k_1}{6}(k_2\pm\sqrt{k_2^2-4\lambda_1})x^3+\lambda_1 x=0, \quad (104)$$

and

$$I=\left(\dot{x}+\frac{k_2\mp\sqrt{k_2^2-4\lambda_1}}{2}x+\frac{k_1}{3}x^3\right)e^{\frac{k_2\pm\sqrt{k_2^2-4\lambda_1}}{2}t}. \quad (105)$$

Eq. (105) can be rewritten as an Abel's equation in the form

$$\dot{x}=Ie^{(\frac{-k_2\mp\sqrt{k_2^2-4\lambda_1}}{2})t}-\left(\frac{k_2\mp\sqrt{k_2^2-4\lambda_1}}{2}\right)x-\frac{k_1}{3}x^3. \quad (106)$$

It is not clear whether Eq. (106) can be explicitly integrated in general. However, for the special case $\lambda_1=\frac{3}{16}k_2^2$ it can be integrated as follows.

The restriction $\lambda_1=\frac{2k_2^2}{9}$ fixes the equation of motion (104) and the first integral (105) in the forms

$$\ddot{x}+(k_1x^2+k_2)\dot{x}+\frac{k_1k_2}{4}x^3+\frac{3k_2^2}{16}x=0, \quad (107)$$

and

$$I=\left(\dot{x}+\frac{k_2}{4}x+\frac{k_1}{3}x^3\right)e^{\frac{3k_2}{4}t}, \quad (108)$$

respectively.

Now following our procedure given in Sec. 3.3 one arrives at the general solution[1] as

$$z+z_0=-\frac{a}{3I}\left[\frac{1}{2}\log\left(\frac{(w-a)^2}{w^2+aw+a^2}\right)+\sqrt{3}\arctan\left(\frac{-w\sqrt{3}}{2a+w}\right)\right], \quad (109)$$

with $w=xe^{\frac{k_2}{4}t}$, $z=-\frac{2}{k_2}e^{-\frac{k_2}{2}t}$ and $a=\sqrt[3]{\frac{3I}{k_1}}$ and z_0 is the second integration constant. Rewriting w and z in terms of old variables one can get the explicit solution.

Case (iii): $k_1,\ k_3=0,\ \lambda_1=\frac{2k_2^2}{9},\ k_2$ and k_4 : arbitrary:

The parametric restrictions given above fix the equation of motion (11) to the force-free Duffing oscillator, namely, $\ddot{x}+k_2\dot{x}+k_4x^3+\frac{2k_2^2}{9}x=0$. We have already discussed the general solution of this equation in Sec. III (vide case (*iv*)).

Case (iv): $k_3=\frac{(r-1)k_1^2}{3r^3},\ k_4=\frac{k_1k_2}{4},\ \lambda_1=\frac{3k_2^2}{16},\ k_1,\ k_2$ and $r:$ arbitrary:

The equation of motion turns out to be

$$\ddot{x}+(k_1x^2+k_2)\dot{x}+\frac{(r-1)k_1^2}{3r^2}x^5+\frac{k_1k_2}{4}x^3+\frac{3k_2^2}{16}x=0,\quad r\neq 0. \tag{110}$$

Rewriting the associated first integral I, given in Case (iv) in Table II, in the form (50), we get

$$I=\begin{cases}\left(\frac{(r-1)k_1^2}{9r^2}(xe^{\frac{k_2}{4}t})^6+\frac{d}{dt}(xe^{\frac{k_2}{4}t})\left(\frac{d}{dt}(xe^{\frac{k_2}{4}t})e^{\frac{k_2}{2}t}+\frac{k_1}{3}(xe^{\frac{k_2}{4}t})^3\right)e^{\frac{k_2}{2}t}\right) & \\ \qquad\times\left(\frac{d}{dt}(xe^{\frac{k_2}{4}t})e^{\frac{k_2}{2}t}+\frac{k_1(r-1)}{3r}(xe^{\frac{k_2}{4}t})^3\right)^{-r}, & r\neq 0,2,\\ \frac{6\frac{d}{dt}(xe^{\frac{k_2}{4}t})e^{\frac{k_2}{2}t}}{k_1(xe^{\frac{k_2}{4}t})^3+6\frac{d}{dt}(xe^{\frac{k_2}{4}t})e^{\frac{k_2}{2}t}}-\log(k_1(xe^{\frac{k_2}{4}t})^3+6\frac{d}{dt}(xe^{\frac{k_2}{4}t})e^{\frac{k_2}{2}t}), & r=2\\ \frac{1}{2}\left(\frac{d}{dt}(xe^{\frac{k_2}{4}t})\right)^2e^{k_2t}+\frac{k_3}{6}(xe^{\frac{k_2}{4}t})^6, & r=0\end{cases} \tag{111}$$

and identifying the dependent and independent variables from (111) and the relations (51), we obtain the transformation

$$w=xe^{\frac{k_2}{4}t},\quad z=-\frac{2}{k_2}e^{-\frac{k_2}{2}t}. \tag{112}$$

In terms of the new variables (112) the first integral I given above, (111) can be written as

$$I=\begin{cases}\left(w'+\frac{(r-1)}{3r}k_1w^3\right)^{-r}\left[w'(w'+\frac{k_1}{3}w^3)+\frac{(r-1)}{9r^2}k_1^2w^6\right], & r\neq 0,2\\ \frac{6w'}{k_1w^3+6w'}-\log(k_1w^3+6w'), & r=2,\\ \frac{w'^2}{2}+\frac{k_3}{6}w^6, & r=0.\end{cases} \tag{113}$$

On the other hand substituting the transformation (112) into the equation of motion (110) we get

$$w''+k_1w^2w'+\frac{(r-1)k_1^2}{3r^2}w^5=0,\quad r\neq 0,\qquad '=\frac{d}{dz}. \tag{114}$$

In the case $r = 0$, we have an equation of the form (vide case (ivb) in Table II)

$$\ddot{x} + k_2\dot{x} + k_3x^5 + \frac{3k_2^2}{16}x = 0. \tag{115}$$

We note that the Eq. (114) is the $l = 2$ case of Eq. (6). As we mentioned in the introduction the general solution of this equation can be obtained only for certain specific choices, namely, $\frac{(r-1)k_1^2}{3r^2} = \frac{1}{16}$. This in turn gives $r = 4k_1$ or $\frac{4}{3}k_1$. The respective solutions for these values of r of Eq. (114) can be fixed from Eq. (7) with $l = 2$. The other cases do not seem to be amenable to explicit integration. However, all of them can be recast in the Hamiltonian form as we see below.

As the first integrals (113) are now 'time' independent ones, one can give a Hamiltonian formalism for all the integrals (113) by following the ideas given in Sec. III A 4. Doing so we obtain

$$H = \begin{cases} \left[\frac{\left(\frac{(r-1)p}{}\right)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{3r}k_1w^3p\right], & r \neq 0, 1, 2, \\ \frac{k_1}{6}w^3p + \log(\frac{6}{p}), & r = 2 \\ e^p + \frac{k_1}{3}w^3, & r = 1 \\ \frac{p^2}{2} + \frac{k_3}{6}w^6, & r = 0 \end{cases} \tag{116}$$

where

$$p = \begin{cases} \frac{1}{(r-1)}\left(w' + \frac{(r-1)}{3r}k_1w^3\right)^{(1-r)}, & r \neq 0, 1 \\ \log w', & r = 1 \\ \frac{p^2}{2} + \frac{k_3}{6}w^6, & r = 0. \end{cases} \tag{117}$$

In this sense these cases may be considered as Liouville integrable systems. Finally, for $r = 0$ case in Eq. (113) can be integrated in terms of Jacobian elliptic function (see for example in Ref. 23). Again, here, we have not been able to identify canonical transformations which can lead to the identification of suitable 'potential' equations.

C. Summary of results in $q = 2$ case:

To summarize the results obtained for the $q = 2$ case, we have identified six integrable cases in Eq. (11) among which three of them were already known in the literature and the remaining three are new. In the following, we tabulate both of them.

1. *Integrable equations already known in the literature*

(1) $$\ddot{x} + (k_1x^2 + k_2)\dot{x} + \frac{k_1k_2}{4}x^3 + \frac{3k_2^2}{16}x = 0, \qquad (107)$$

(2) $$\ddot{x} + k_2\dot{x} + k_3x^3 + \frac{2k_2^2}{9}x = 0, \qquad (118)$$

(3) $$\ddot{x} + k_3x^5 + k_4x^3 + \lambda_1 x = 0. \qquad (93b)$$

We mention that Eq. (107) is nothing but the force-free DVP whose integrability is established in Ref. 1 and Eq. (118) is nothing but the force-free Duffing oscillator[12,14].

2. *New integrable equations*

(1) $$\ddot{x} + (k_1x^2 + k_2)\dot{x} + k_3x^5 + \frac{4(r-1)k_1k_2}{3r^2}x^3 + \frac{(r-1)k_2^2}{r^2}x = 0,\ r \neq 0 \qquad (93a)$$

(2) $$\ddot{x} + (k_1x^2 + k_2)\dot{x} + \frac{k_1^2}{16}x^5 + \frac{k_1k_2}{4}x^3 + \lambda_1 x = 0, \qquad (98)$$

(3) $$\ddot{x} + (k_1x^2 + k_2)\dot{x} + k_3x^5 + \frac{k_1k_2}{4}x^3 + \frac{3k_2^2}{16}x = 0, \qquad (110)$$

where $r^2k_3 = \frac{(r-1)k_1^2}{3}$ and k_1, k_2, λ_1 and r are arbitrary parameters. We proved that Eq. (93a) is Liouville integrable one. As far as equation (98) is concerned we derived the general solution for arbitrary values of $k_1,\ k_2$ and λ_1. Finally, for Eq. (110) though we identified only one time dependent integral, we have demonstrated that it can be transformed into time independent Hamiltonian and thereby ensuring its Liouville integrability.

V. EXTENDED PRELLE-SINGER METHOD TO GENERALIZED EQ. (9)

One can investigate the integrability properties of Eq. (9) by considering the cases $q = 3, 4, 5, \ldots,$ one by one and classify the integrable equations. Since the procedure and the mathematical techniques in exploring the integrating factors (R), null forms (S), first integrals (I) and general solution are the same in each case we do not consider each case in detail. We straightaway move to the case $q = arbitrary$, that is, $q \in R$ and not necessarily an integer, and present the outcome of our investigations.

As we did earlier, we consider the cases $I_t = 0$ and $I_t \neq 0$ separately for the choice $q = arbitrary$ also. First let us consider the case $I_t = 0$.

A. The case $I_t = 0$

By considering the same arguments given in Sec. 3.1.1 the null form S and the integrating factor R can be fixed easily in the form

$$S = -\frac{((k_1x^q + k_2)\dot{x} + k_3x^{2q+1} + k_4x^{1+q} + \lambda_1 x)}{\dot{x}},$$
$$R = \frac{\dot{x}}{(\frac{(r-1)}{r}(\frac{k_1}{(q+1)}x^{q+1} + k_2x) + \dot{x})^r}, \qquad (119)$$

where k_1 and k_2 are arbitrary and the remaining parameters, k_3, k_4 and λ_1, are related to the parameters k_1 and k_2 through the relations

$$k_3 = \frac{(r-1)}{r^2}(q+1)\hat{k}_1^2, \quad k_4 = \frac{(r-1)}{r^2}(q+2)\hat{k}_1k_2, \quad \lambda_1 = \frac{(r-1)}{r^2}k_2^2, \qquad (120)$$

where $\hat{k}_1 = \frac{k_1}{(q+1)}$. The above restrictions fix Eq. (9) to the following specific forms:

$$(ia) \qquad \ddot{x} + ((q+1)\hat{k}_1x^q + k_2)\dot{x} + \frac{(r-1)}{r^2}[(q+1)\hat{k}_1^2x^{2q+1} + (q+2)\hat{k}_1k_2x^{q+1} + k_2^2x] = 0, \; r \neq 0 \qquad (15)$$

$$(ib) \qquad \ddot{x} + k_3x^{2q+1} + k_4x^{q+1} + \lambda_1 x = 0, \; r = 0. \qquad (121)$$

Now substituting (119) into (25) and evaluating the integrals we obtain the first integrals of the form

$$(ia) \quad I_1 = \left(\dot{x} + \frac{(r-1)}{r}(\hat{k}_1 x^{q+1} + k_2 x)\right)^{-r} \times \left[\dot{x}(\dot{x} + \hat{k}_1 x^{q+1} + k_2 x) + \frac{(r-1)}{r^2}(\hat{k}_1 x^{q+1} + k_2 x)^2\right], \quad r \neq 0, 2, \tag{122a}$$

$$(ib) \quad I_1 = \frac{\dot{x}}{(\dot{x} + \frac{k_2}{2}x + \frac{\hat{k}_1}{2}x^{q+1})} - \log(\dot{x} + \frac{k_2}{2}x + \frac{\hat{k}_1}{2}x^{q+1}), \quad r = 2, \tag{122b}$$

$$(ii) \quad I_1 = \frac{\dot{x}^2}{2} + \frac{k_3}{2(q+1)}x^{2(q+1)} + \frac{k_4}{(q+2)}x^{q+2} + \frac{\lambda_1}{2}x^2, \quad r = 0. \tag{122c}$$

Further, using the above forms of the first integrals, one can show that the equation of motion (9), with the parametric restrictions (120), can also be derived from the Hamiltonians

$$(ia) \qquad H = \left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{r}p(\hat{k}_1 x^{q+1} + k_2 x)\right], \quad r \neq 0, 1, 2, \tag{123a}$$

$$(ib) \qquad H = \frac{k_2}{2}xp + \frac{\hat{k}_1}{2}x^{q+1}p + \log(\frac{2(q+1)}{p}), \quad r = 2 \tag{123b}$$

$$(ic) \qquad H = e^p + \hat{k}_1 x^{q+1} + k_2 x, \quad r = 1, \tag{123c}$$

$$(ii) \qquad H = \frac{p^2}{2} + \frac{k_3}{2(q+1)}x^{2(q+1)} + \frac{k_4}{(q+1)}x^{q+1} + \frac{\lambda_1}{2}x^2, \quad r = 0, \tag{123d}$$

where the corresponding canonical momenta respectively are

$$(ia, b) \qquad p = \frac{1}{(r-1)}\left(\dot{x} + \frac{(r-1)}{r}(\hat{k}_1 x^{q+1} + k_2 x)\right)^{(1-r)}, \quad r \neq 0, 1, \tag{124a}$$

$$(ic) \qquad p = \log \dot{x}, \quad r = 1, \tag{124b}$$

$$(ii) \qquad p = \dot{x}, \quad r = 0. \tag{124c}$$

With the above Hamiltonian formulation, for the parametric set (120), the integrability of the associated equation of motion is assured for these parametric cases through Liouville theorem.

B. The case $I_t \neq 0$

We use the same ansatz and ideas which we followed for the $q = 1$ and $q = 2$ cases to determine the forms of S and R. As the procedure is exactly the same as in the earlier cases we present the parametric restrictions and the respective form of expressions of the integrating factors, null forms and integrals of motions in Table III without further discussion.

Since we derived only one integral, which is also a time dependent one for each parametric restriction, we need to integrate each one of them further and obtain the second integration constant in order to prove the complete integrability of each of the cases reported in Table III. In the following we deduce the second integral and general solution by utilizing the proceduce given in Sec. III C.

Case (ia): $k_3 = \frac{k_1^2}{(q+2)^2},\ k_4 = \frac{k_1 k_2}{(q+2)},\ k_1,\ k_2$ and λ_1 : arbitrary:

We have an equation of the form

$$\ddot{x} + ((q+2)\hat{k}_1 x^q + k_2)\dot{x} + \hat{k}_1^2 x^{2q+1} + \hat{k}_1 k_2 x^{q+1} + \lambda_1 x = 0, \tag{13}$$

where $k_1 = (q+2)\hat{k}_1$. The corresponding first integral given in Table 3 is nothing but the Bernoulli equation which can be solved using the standard method[21]. The general solution turns out to be

$$x(t) = \Big(e^{\omega t} - I_1\Big)\left(e^{\frac{q}{2}(k_2+\omega)t}\left(I_2 + \hat{k}_1 q \int \left(\frac{e^{\omega t} - I_1}{e^{\frac{1}{2}(k_2+\omega)t}}\right)^q dt\right)\right)^{\frac{-1}{q}}, \tag{125}$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$. We note here that a sub-case of the above, namely, $k_2^2 < 4\lambda_1$, has been studied by Smith[4] who had shown that the corresponding system admits the general solution of the form

$$x(t) = Acos(\omega_0 t + \delta)e^{-\frac{k_2}{2}t}\left(1 + q\hat{k}_1 A \int e^{\frac{-qk_2}{2}t} cos^q(\omega_0 t + \delta) dt\right)^{-\frac{1}{q}}, \tag{126}$$

where $\omega_0 = \frac{1}{2}\sqrt{4\lambda_1 + k_2^2}$ and $\delta,\ A$ are arbitrary constants. For $k_2^2 > 4\lambda_1$, the solution become a dissipative type/front-like structure. In particular, for $\lambda_1 = 0$ the general solution takes the form

$$x(t) = \Big(e^{k_2 t} I_1 - 1\Big)\left[e^{qk_2 t}\left(I_2 + \hat{k}_1 q \int \left(I_1 - e^{-k_2 t}\right)^q dt\right)\right]^{-\frac{1}{q}}. \tag{127}$$

TABLE III: Parametric restrictions, null forms (S), integrating factors (R) and time dependent integrals of motion (I) of

$\ddot{x} + (k_1x^q + k_2)\dot{x} + k_3x^{2q+1} + k_4x^{q+1} + \lambda_1 x = 0$ (identified with the assumed ansatz form of S and R)

Cases	Parametric restrictions	Null form (S)	Integrating factor (R)	Integrals of motion (I)
(i)	$k_3 = \frac{k_1^2}{(q+2)^2},\ k_4 = \frac{k_1k_2}{(q+2)}$ $(k_1,\ k_2,\ \lambda_1 :$ arbitrary$)$	$\frac{(\frac{qk_1}{(q+2)}x^{q+1} - \dot{x})}{x}$	$\frac{xe^{\mp\omega t}}{(\dot{x} - \frac{(k_2\pm\omega)}{2}x + \frac{k_1}{(q+2)}x^{q+1})^2}$	$(a)\ I = e^{\mp\omega t}\left(\frac{\dot{x}-\frac{(-k_2\mp\omega)}{2}x+\frac{k_1}{q+2}x^{q+1}}{\dot{x}-\frac{(-k_2\pm\omega)}{2}x+\frac{k_1}{q+2}x^{q+1}}\right),$ $k_2, \lambda_1 \neq 0,\ \omega = (k_2^2 - 4\lambda_1)^{\frac{1}{2}}$ $(b)\ I = -t + \frac{x}{(\frac{k_2}{2}x+\frac{k_1x^{q+1}}{q+2}+\dot{x})},\qquad k_2^2 = 4\lambda_1$
(ii)	$k_4 = \frac{k_1(k_2\pm\omega)}{2(q+1)},\ k_3 = 0,$ $(k_1,\ k_2,\ \lambda_1 :$ arbitrary$)$	$\frac{1}{2}(k_2 \mp \omega) + k_1x^q,$	$e^{\frac{(k_2\pm\omega)}{2}t}$	$I = \left(\dot{x} + \frac{k_2\mp\omega}{2}x + \frac{k_1}{(q+1)}x^{q+1}\right)e^{(\frac{k_2\pm\omega}{2})t},$ $\omega = (k_2^2 - 4\lambda_1)^{\frac{1}{2}}$
(iii)	$k_1, k_3 = 0, \lambda_1 = \frac{2(q+2)k_2^2}{(q+4)^2}$ $(k_2,\ k_4 :$ arbitrary$)$	$\frac{\frac{2k_2\dot{x}}{(q+4)} + \frac{4k_2^2x}{(q+4)^2} + k_4x^{q+1}}{(\dot{x} + \frac{2k_2x}{(q+4)})}$	$(\dot{x} + \frac{2k_2x}{(q+4)})e^{\frac{2(q+2)}{(q+4)}k_2t}$	$I = e^{\frac{2(q+2)}{(q+4)}k_2t}\left[\frac{\dot{x}^2}{2} + \frac{2k_2x\dot{x}}{(q+4)} + \frac{2k_2^2x^2}{(q+4)^2} + \frac{k_4x^{q+2}}{(q+2)}\right]$
(iv)a	$k_3 = \frac{(r-1)k_1^2}{(q+1)r^2},$ $k_4 = \frac{k_1k_2}{(q+2)},$ $\lambda_1 = \frac{(q+1)k_2^2}{(q+2)^2},\ r \neq 0$ $(k_1,\ k_2,\ r :$ arbitrary$)$	$\frac{k_2}{(q+2)} + k_1x^q + \frac{k_3x^{2q+1}}{(\dot{x}+\frac{k_2}{(q+2)}x)}$	$\frac{(k_2x + (q+2)\dot{x})e^{\frac{(q+1)(2-r)}{(q+2)}k_2t}}{(\frac{k_2}{(q+2)}x + rk_3x^{q+1} + \dot{x})^r}$	$I = \left(\frac{k_3x^{2(q+1)}}{(q+1)} + (\dot{x} + \frac{k_2x}{q+2})(\dot{x} + \frac{k_2x}{q+2} + \frac{k_1x^{q+1}}{q+1})\right)$ $\times\left(\frac{k_2}{(q+2)}x + rk_3x^{q+1} + \dot{x}\right)^{-r} e^{\frac{(q+1)(2-r)}{(q+2)}k_2t}, r \neq 2$ $I = \frac{q+1}{q+2}k_2t + \log(k_1x^{q+1} + 2(q+1)(\dot{x} + \frac{k_2}{q+2}x))$ $-(\frac{2(q+1)(\dot{x} + \frac{k_2}{q+2}x)}{k_1x^{q+1} + 2(q+1)(\dot{x} + \frac{k_2}{q+2}x)}),\qquad r = 2$
(iv)b	$k_1 = 0,\ k_4 = 0,$ $\lambda_1 = \frac{(q+1)k_2^2}{(q+2)^2}, r = 0$ $(k_2,\ k_3 :$ arbitrary$)$	$\frac{k_2}{(q+2)} + \frac{k_3x^{2q+1}}{(\dot{x} + \frac{k_2}{(q+2)}x)}$	$e^{\frac{(2q+2)k_2}{(q+2)}t}(\dot{x} + \frac{k_2}{(q+2)}x)$	$I = \left(\frac{\dot{x}^2}{2} + \frac{k_2x\dot{x}}{(q+2)} + \frac{k_2^2x^2}{2(q+2)^2} + \frac{k_3x^{2q+2}}{(2q+2)}\right)e^{\frac{(2q+2)k_2}{(q+2)}t}$

Case (ib): $k_3 = \frac{k_1^2}{16},\ k_4 = \frac{k_1 k_2}{4},\ k_2 = 4\lambda_1,\ k_1$ and k_2 : arbitrary:

A general solution for this case can be fixed from (127) as

$$x(t) = (I_1 + t)e^{-\frac{k_2}{2}t}\left(I_2 + q\hat{k}_1 \int e^{-\frac{qk_2}{2}t}(I_1+t)^q dt\right)^{-\frac{1}{q}}. \tag{128}$$

On the other hand the general solution for the parametric choice $k_2,\ \lambda_1 = 0$ turns out to be

$$x(t) = \left(\frac{(q+1)(I_1+t)^q}{\hat{k}_1 q(I_1+t)^{q+1} + (q+1)I_2}\right)^{\frac{1}{q}}, \tag{129}$$

which exactly coincides with the result (7) obtained by Feix et al.[3] for integer $q(=l)$ values.

Case (ii): $k_3 = 0,\ k_4 = \frac{k_1}{2(q+1)}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1}),\ k_1,\ k_2$ and λ_1 : arbitrary:

The associated equation of motion and the first integral are (see Table III)

$$\ddot{x} + ((q+1)\hat{k}_1 x^q + k_2)\dot{x} + \frac{\hat{k}_1}{2}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1})x^{q+1} + \lambda_1 x = 0, \tag{130}$$

and

$$I = \left(\dot{x} + \frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}x + \hat{k}_1 x^{q+1}\right)e^{(\frac{k_2 \pm \sqrt{k_2^2 - 4\lambda_1}}{2})t}, \tag{131}$$

where $k_1 = (q+1)\hat{k}_1$. Like in the earlier cases, that is, $q = 1$ and $q = 2$, we are able to integrate the first integral (131) explicitly only for a specific parametric restriction, namely, $\lambda_1 = (q+1)\hat{k}_2^2$, where $k_2 = (q+2)\hat{k}_2$. In this case the equation of motion (130) and the first integral, Eq. (131), can be recast in the form

$$\ddot{x} + (k_1 x^q + (q+2)\hat{k}_2)\dot{x} + k_1\hat{k}_2 x^{q+1} + (q+1)\hat{k}_2^2 x = 0, \tag{12}$$

and

$$I = \left(\dot{x} + \hat{k}_2 x + \hat{k}_1 x^{q+1}\right)e^{(q+1)\hat{k}_2 t}, \tag{132}$$

respectively. Now comparing (132) with (50), we get

$$I = e^{q\hat{k}_2 t}\left(\frac{d}{dt}(xe^{\hat{k}_2 t})\right) + \hat{k}_1(xe^{\hat{k}_2 t})^{(q+1)}. \tag{133}$$

Next identifying the dependent and independent variables from (133) using the relations (51), we obtain the transformation

$$w = xe^{\hat{k}_2 t}, \quad z = -\frac{1}{q\hat{k}_2}e^{-q\hat{k}_2 t}. \tag{134}$$

Using the transformation (134) the first integral (133) can be rewritten in the form

$$I = w' + \hat{k}_1 w^{(q+1)} \tag{135}$$

which in turn leads to the solution by an integration, that is,

$$z - z_0 = \int \frac{dw}{I - \hat{k}_1 w^{(q+1)}}, \tag{136}$$

where z_0 is an arbitrary constant. Solving Eq. (136) we get[24]

$$z - z_0 = \frac{1}{Ig^{\frac{1}{(q+1)}}} \begin{cases} -\frac{2}{q+1}\sum_{i=0}^{\frac{q-1}{2}} P_i \cos\frac{2i}{q+1}\pi + \frac{2}{q+1}\sum_{i=0}^{\frac{q-1}{2}} Q_i \sin\frac{2i}{q+1}\pi \\ +\frac{1}{q+1}ln\frac{(1+w)}{(1-w)}, \quad \text{q-a positive odd number,} \\ -\frac{2}{q+1}\sum_{i=0}^{\frac{q-2}{2}} R_i \cos\frac{2i+1}{q+1}\pi + \frac{2}{q+1}\sum_{i=0}^{\frac{q-2}{2}} T_i \sin\frac{2i+1}{q+1}\pi \\ +\frac{1}{q+1}ln(1+w), \quad \text{q-a positive even number,} \end{cases} \tag{137}$$

where $g = \frac{\hat{k}_1}{I}$ and

$$P_i = \frac{1}{2}ln\bigg(w^2 - 2w\cos\frac{2i}{q+1}\pi + 1\bigg), \quad Q_i = arctan\bigg[\frac{w - \cos\frac{2i}{q+1}\pi}{\sin\frac{2i}{q+1}\pi}\bigg],$$

$$R_i = \frac{1}{2}ln\bigg(w^2 + 2w\cos\frac{2i+1}{q+1}\pi + 1\bigg), \quad T_i = arctan\bigg[\frac{w + \cos\frac{2i+1}{q+1}\pi}{\sin\frac{2i+1}{q+1}\pi}\bigg].$$

Rewriting w and z in terms of old variables one can get the explicit solution.

Case (iii): $k_1, k_3 = 0, \ \lambda_1 = \frac{2(q+2)k_2^2}{(q+4)^2}, \ k_2$ and k_4 : arbitrary

The parametric choice given above fixes the equation of motion of the form

$$\ddot{x} + (q+4)\hat{k}_2\dot{x} + k_4 x^{(q+1)} + 2(q+2)\hat{k}_2^2 x = 0, \tag{14}$$

where $k_2 = (q+4)\hat{k}_2$. Rewriting the first integral I given in Case (iii) in Table III, in the form (49), we get

$$I = \frac{1}{2}\Big(\dot{x} + 2\hat{k}_2 x\Big)^2 e^{2(q+2)\hat{k}_2 t} + \frac{k_4 x^{(q+2)}}{(q+2)}e^{2(q+2)\hat{k}_2 t}. \tag{138}$$

Now splitting the first term in Eq. (138) further in the form of (50),

$$I = \left[e^{q\hat{k}_2 t}\frac{d}{dt}\left(\frac{x}{\sqrt{2}}e^{2\hat{k}_2 t}\right)\right]^2 + \frac{2^{(\frac{q+2}{2})}k_4}{(q+2)}\left(\frac{x}{\sqrt{2}}e^{2\hat{k}_2 t}\right)^{(q+2)} \tag{139}$$

and identifying the dependent and independent variables from (139) using the relations (51), we obtain the transformation

$$w = \frac{x}{\sqrt{2}}e^{2\hat{k}_2 t}, \quad z = -\frac{1}{q\hat{k}_2}e^{-q\hat{k}_2 t}. \tag{140}$$

Using the transformation (140) the first integral (138) can be brought to the form

$$I = w'^2 + \frac{2^{(\frac{q+2}{2})}k_4}{(q+2)}w^{(q+2)}. \tag{141}$$

Separating the dependent and independent variables and integrating the resultant equation we get

$$z - z_0 = \int \frac{dw}{\sqrt{I - \hat{k}_4 w^{(q+2)}}}, \tag{142}$$

where $\hat{k}_4 = \frac{2^{(\frac{q+2}{2})}}{(q+2)}k_4$ and z_0 is an arbitrary constant.

Case (iv): $k_3 = \frac{(r-1)k_1^2}{(q+1)r^2},\ k_4 = \frac{k_1 k_2}{(q+2)},\ \lambda_1 = \frac{(q+1)k_2^2}{(q+2)^2},\ k_1,\ k_2$ and r : arbitrary:

The equation of motion in this case turns out to be

$$\ddot{x} + ((q+1)\hat{k}_1 x^q + (q+2)\hat{k}_2)\dot{x} + (q+1)(\frac{(r-1)}{r^2}\hat{k}_1^2 x^{2q} + \hat{k}_1\hat{k}_2 x^q + \hat{k}_2^2)x = 0, \quad r \neq 0 \tag{16}$$

where $k_1 = (q+1)\hat{k}_1,\ k_2 = (q+2)\hat{k}_2$. Rewriting the associated first integral I, given in Case (iv) in Table III, in the form (50), we get

$$I = \begin{cases} \left(\frac{(r-1)\hat{k}_1^2}{r^2}(xe^{\hat{k}_2 t})^{2(q+1)} + \frac{d}{dt}(xe^{\hat{k}_2 t})\left(\frac{d}{dt}(xe^{\hat{k}_2 t})e^{q\hat{k}_2 t} + \hat{k}_1(xe^{\hat{k}_2 t})^{q+1}\right)e^{q\hat{k}_2 t}\right), \\ \quad \times\left(\frac{d}{dt}(xe^{\hat{k}_2 t})e^{q\hat{k}_2 t} + \frac{\hat{k}_1(r-1)}{r}(xe^{\hat{k}_2 t})^{q+1}\right)^{-r}, & r \neq 0, 2 \\ \frac{\frac{d}{dt}(xe^{\hat{k}_2 t})e^{q\hat{k}_2 t}}{\frac{\hat{k}_1}{2}(xe^{\hat{k}_2 t})^{q+1} + \frac{d}{dt}(xe^{\hat{k}_2 t})e^{q\hat{k}_2 t}} - \log(\frac{\hat{k}_1}{2}(xe^{\hat{k}_2 t})^{q+1} + \frac{d}{dt}(xe^{\hat{k}_2 t})e^{q\hat{k}_2 t}), & r = 2 \\ \frac{1}{2}\left(\frac{d}{dt}(xe^{\hat{k}_2 t})\right)^2 e^{2qk_2 t} + \frac{k_3}{2(q+1)}(xe^{\hat{k}_2 t})^{2(q+1)}, & r = 0. \end{cases} \tag{143}$$

Identifying the dependent and independent variables from (143) and the relations (51), we obtain the transformation

$$w = xe^{\hat{k}_2 t}, \quad z = -\frac{1}{q\hat{k}_2} e^{-q\hat{k}_2 t}. \tag{144}$$

Subsituting the transformation (144) into (16), one obtains

$$w'' + (q+1)\hat{k}_1 w^q w' + (q+1)\frac{(r-1)}{r^2}\hat{k}_1^2 w^{2q+1} = 0, \;\; r \neq 0, \quad ' = \frac{d}{dz}. \tag{145}$$

In terms of the new variables (144) change the time dependent first integral into time independent ones of the form

$$I = \begin{cases} \left(w' + \frac{(r-1)}{r}\hat{k}_1 w^{q+1}\right)^{-r}\left[w'(w' + \hat{k}_1 w^{q+1}) + \frac{(r-1)}{r^2}\hat{k}_1^2 w^{2(q+1)}\right], & r \neq 0, 2, \\ \frac{w'}{w' + \frac{\hat{k}_1}{2}x^{q+1}} - \log(w' + \frac{\hat{k}_1}{2}w^{q+1}), & r = 2, \\ \frac{w'^2}{2} + \frac{k_3}{2(q+1)}w^{2(q+1)}, & r = 0. \end{cases} \tag{146}$$

Once again one can deduce the Hamiltonians in the form

$$H = \begin{cases} \left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{r}\hat{k}_1 w^{q+1} p\right], & r \neq 0, 1, 2, \\ \frac{1}{2}\hat{k}_1 w^{q+1} p + \log(\frac{2(q+1)}{p}), & r = 2, \\ e^p + \hat{k}_1 w^{q+1}, & r = 1, \\ \frac{p^2}{2} + \frac{k_3}{2(q+1)}w^{2(q+1)}, r = 0, \end{cases} \tag{147}$$

with

$$p = \begin{cases} \frac{1}{(r-1)}\left(w' + \frac{(r-1)}{r}\hat{k}_1 w^{q+1}\right)^{(1-r)}, & r \neq 0, 1 \\ \log(w'), & r = 1 \\ w', & r = 0, \end{cases} \tag{148}$$

and thereby ensuring liouville integrability of Eq. (16).

C. Summary of results in $q =$ *arbitrary* case:

To conclude the integrability of Eq. (9), we have established the fact that the following equations, are integrable

$$(1) \qquad \ddot{x} + (k_1x^q + (q+2)k_2)\dot{x} + k_1k_2x^{q+1} + (q+1)k_2^2x = 0, \tag{12}$$

$$(2) \qquad \ddot{x} + ((q+2)k_1x^q + k_2)\dot{x} + k_1^2x^{2q+1} + k_1k_2x^{q+1} + \lambda_1 x = 0, \tag{13}$$

$$(3) \qquad \ddot{x} + (q+4)k_2\dot{x} + k_4x^{q+1} + 2(q+2)k_2^2x = 0, \tag{14}$$

$$(4) \qquad \ddot{x} + ((q+1)k_1x^q + k_2)\dot{x} + \frac{(r-1)}{r^2}((q+1)k_1^2x^{2q} + (q+2)k_1k_2x^q + k_2^2)x = 0, \; r \neq 0 \tag{15}$$

$$(5) \qquad \ddot{x} + ((q+1)k_1x^q + (q+2)k_2)\dot{x} + (q+1)(k_3x^{2q} + k_1k_2x^q + k_2^2)x = 0, \tag{16}$$

where $r^2k_3 = (r-1)k_1^2$ and $k_1,\ k_2,\ k_4,\ \lambda_1$ and r are arbitrary parameters (for simplicity we have removed hats in k_i's, $i = 1, 2$, in Eqs. (12)-(16)) . The significance and newness of the equations (12)-(16) are already pointed out in Sec. I B.

VI. DISCUSSION AND CONCLUSIONS

In this paper, we have investigated the integrability properties of Eq. (9) and shown that it admits a large class of integrable nonlinear systems. In fact, many classical integrable nonlinear oscillators can be derived as sub-cases of our results. One of the important outcomes of our investigation is that the entire class of Eq. (6) can be derived from a conservative Hamiltonian (vide Eq. (123)) eventhough the system deceptively looks like a dissipative equation.

From our detailed analysis we have shown that Eq. (9) admits both conservative Hamiltonian systems and dissipative systems, depending on the choice of parameters. As far as the former is concerned we have deduced the explicit forms of the Hamiltonians for the respective equations. In fact, for the case, $q = 1$, we have constructed suitable canonical transformations and transformed the equations into conservative nonlinear oscillator equations. However, the canonical transformations for the conservative Hamiltonian systems for the cases $q = 2, \ldots,$ arbitrary, if at all they exist, still remain to be obtained. Exploring the classical dynamics underlying these conservative Hamiltonian systems is also of considerable interest for further study. As far as dissipative systems are concerned we have not only shown that Eq. (9) contains the well known force-free Helmholtz, Duffing and Duffing-van der Pol oscillators but also have several integrable generalizations which is another important outcome of our investigations. The study of chaotic dynamics of these nonlinear oscillators under further perturbations is one of the current topics[22] in the contemporary literature in nonlinear dynamics. In principle one can extend such analysis to the above generalized equations as well.

In this paper, we have also not touched the question of linearizability of the integrable cases of Eq. (9). In our earlier work, we have shown that the Eq. (55) is linearizable to the free particle equation, $\frac{d^2w}{dz^2} = 0$. Of course one can show that this is the only linearizable equation in (9) through invertible point transformation[9,11,18]. However, linearizablity of other integrable cases through more general transformations still remains to be explored.

In addition to the above, we have also carried out the Painlevé singularity structure analysis of Eq. (9) and compared the results obtained through both the methods. The details of this will be published elsewhere.

As we mentioned at the end of Sec. II, the crux of the PS procedure lies in finding the explicit solutions satisfying all the three determining Eqs. (22)-(24). In this paper we have considered only certain specific ansatz forms to determine the null forms S, and integrating factors R. As a consequence only a specific class of integrable equations have been derived. It is not clear, whether these ansatz forms used in this paper exhaust all possible integrable cases of Eq. (9). One needs to consider more generalized ansatz forms, and if possible to solve Eqs. (22)-(24) for the most general forms of R and S, and try to identify all possible integrable cases underlying Eq. (9). This is being explored further.

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