Bifurcation and chaos in the double well Duffing- van der Pol oscillator: Numerical and analytical studies

A. Venkatesan and M. Lakshmanan
Centre for Nonlinear Dynamics,
Department of Physics,
Bharathidasan University,
Tiruchirappalli - 620 024, India

Abstract

The behaviour of a driven double well Duffing-van der Pol (DVP) oscillator for a specific parametric choice ($\mid \alpha \mid = \beta$) is studied. The existence of different attractors in the system parameters ($f - \omega$) domain is examined and a detailed account of various steady states for fixed damping is presented. Transition from quasiperiodic to periodic motion through chaotic oscillations is reported. The intervening chaotic regime is further shown to possess islands of phase-locked states and periodic windows (including period doubling regions), boundary crisis, all the three classes of intermittencies, and transient chaos. We also observe the existence of local-global bifurcation of intermittent catastrophe type and global bifurcation of blue-sky catastrophe type during transition from quasiperiodic to periodic solutions. Using a perturbative periodic solution, an investigation of the various forms of instablities allows one to predict Neimark instablity in the ($f - \omega$) plane and eventually results in the approximate predictive criteria for the chaotic region.

I. INTRODUCTION

The Duffing-van der Pol oscillator (DVP)

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \alpha x + \beta x^3 = f\cos\omega t, \mu > 0, \left(\frac{d}{dt} = .\right) \tag{1}$$

is a ubiquitous nonlinear differential equation which makes its presence in physical, engineering and even biological problems [1-7]. It is a generalization of the classic van der Pol oscillator equation. It can be considered in atleast three physically interesting situations, wherein the potential $V(x) = \frac{\alpha x^2}{2} + \frac{\beta x^4}{2}$ is a (i) single well ($\alpha > 0, \beta > 0$), (ii) double well ($\alpha < 0, \beta > 0$) or a (iii) double hump ($\alpha > 0, \beta < 0$). Each one of the above three cases has become a classic central model to describe inherently nonlinear phenomena, exhibiting a rich and baffling variety of regular and chaotic motions.

Chaotic motion in system (1) with a single well type restoring force was investigated by Ueda and Akamatsu [8] as a model of negative resistance oscillator and later on was studied by a number of other authors [9-12], who noted symmetry breaking of attractors and the onset of chaotic dynamics. Bountis etal.[11] have investigated the nonintegrability of a family of DVP oscillators by studying analyticity properties of the solution in the complex time plane and proved that infinitely sheeted structure (ISS) exists in this system. Rajasekar, Parthasarthy and Lakshmanan [13] pointed out that the DVP oscillator with double well potential exhibits Smale-horse shoe chaos when transverse intersections of the homoclinic orbits occur. Further Kao and Wang [14] had analog simulated the DVP oscillator with a double hump potential and discussed the various mode locking, multiple hysteresis, period doubling route to chaos, intermittent hopping and crises phenomena.

Recently Szemplinska-Stupnika and Rudowski [12] reported that a single well type DVP oscillator exhibits chaotic motion between two types of regular motion, namely periodic and quasi periodic oscillations, in the prinicipal resonance region for a specific values of the parameter $f(= 1.0)$ and a range of ω values (0.8 - 1.0). Also they have obtained a perturbative solution for the periodic oscillation and carried out stability analysis of such solution to predict Neimark bifurcation. However no such analysis exists for the important case of double well type DVP oscillator so far in the literature, which is atypical in the sense that even in the absence of forcing it shows the existence of multiple attractors [6,7].

Considering the DVP oscillator with a double well type restoring force in the form

$$\ddot{x} - \mu(1 - x^2)\dot{x} - \mid \alpha \mid x + \beta x^3 = f\cos\omega t, \beta > 0 \tag{2}$$

we notice that the three equilibrium points of the system (2) for $f = 0$ correspond to $- \mid \alpha \mid x + \beta x^3 = 0$, so that we have the stable fixed points $x_{1,2}^{(s)} = \pm\sqrt{\frac{|\alpha|}{\beta}}$ and the unstable fixed point which is hyperbolic at $x_o^{(u)} = 0$. Actually $x_{1,2}$ are elliptic points for $\mid \alpha \mid = \beta$ and become stable foci for $\mid \alpha \mid > \beta$ while they are unstable foci for $\mid \alpha \mid < \beta$. As a result, the system (2) exhibits a large orbit (LO) motion, which always encircles all the three equilibrium points for the case $\mid \alpha \mid = \beta$. As far as $\mid \alpha \mid > \beta$ is concerned, the system exhibits both small orbit (SO), that is oscillation around any one of the stable fixed points, and large orbit motion (LO), depending upon the values of the other control parameters and also initial conditions .

In this paper we undertake an investigation of the dynamics of the double-well DVP oscillator (2) and show that it is a rich dynamical system, possessing a vast number of regular and chaotic steady states. In particular, considering the special case $| \alpha |= \beta$ (the case $| \alpha |\neq \beta$ is even more richer and the results will be presented separately [16]), we bring out the existence of transition from quasiperiodic to periodic motion in the $(f-\omega)$ parameter space via chaotic motion. The novel features we observe are that in the chaotic sea there are many isles of periodic and phase-locked states, which exhibit period doubling phenomenon, intermittencies, crises etc., along with regions of transient chaos, corresponding to local bifurcations. Besides, there are also transitions from quasiperiodicity to period T orbits corresponding to global bifurcation of blue-sky and local-global bifurcation of intermittent catastrophes. We also present a perturbative approach to the study of bifurcations which occur near the principal resonance. The analysis allows us to derive the algebraic equations for the instability boundaries.

The plan of the paper is as follows. In sec. II we present the numerical results for different steady states, bifurcation routes and chaos for system (2). In sec. III we develop a perturbative solution and obtain expressions for the stability regions and compare them with numerical results. Then, we compare the results with the dynamics of the double well Duffing oscillator in sec. IV. Finally, sec. V summarizes our results.

II. NUMERICAL RESULTS

Equation (2) is numerically integrated using the fourth order Runge-Kutta algorithm with adaptive step size with parameter values fixed at $| \alpha |$=0.5, β =0.5, and μ=0.1 inorder to study the large orbit behaviour mentioned in the introduction. The transitions are also characterized by tracing the time evolutions, phase portrait, Poincaré map, Fourier spectrum and Lyapunov exponents. For identifing different steady states, the dynamical transitions are traced out by two different scanning procedures: (1) varying ω at a fixed f (frequency scanning) and (2) varying f at a fixed ω (amplitude scanning). The resulting phase diagram in the $(f-\omega)$ parameter is shown in Fig.1. The diagram covers the transition thresholds in the region of principal and super harmonic resonances, $0.4 < \omega < 1.0$ and the forcing strength lying in the region $0.0 < f < 0.2$. The various features in the phase diagram are summarised and the dynamical transitions of the attractors are elucidated in the following.

A. Phenomena of steady states

One observes that equation (2) admits the free-running solution when the external force is absent $(f = 0.0)$.When it is present and for low f values and low ω values, the frequency of the system becomes incommensurate with the external frequency. Consequently, the system exhibits multifrequency quasiperiodic oscillations. When the value of the external frequency ω exceeds certain critical value for fixed f, a transition from quasiperiodic to periodic oscillations occur on increasing ω (see Fig.1) essentially due to supercritical Neimark bifurcation (see Sec. II B below and Sec III). This phenomenon continues until a critical f value $(f \sim 0.115)$.

On a further increase of the forcing parameter f, $f > 0.115$, the system exhibits chaotic motion between the two regular motions, that is quasiperiodic and periodic oscillations, within a range of the driving frequency ω. For example, at $f = 0.13$, chaotic motion occurs in the region $\omega \in (0.546, 0.553)$ (see Fig.2a).

As the forcing parameter f increases further, within a very narrow frequency region, chaos-periodic windows-chaos type transition is found to occur between the two regular oscillations. For example at f=0.14, period 5T solution occurs in the frequency range $\omega \in (0.545, 0.551)$ within the chaotic range $\omega \in (0.525, 0.553)$ (see Fig.2b). At f=0.17, period doubling phenomenon occurs in the window region of the frequency $\omega \in (0.503, 0.529)$. In the above range, we note that the period 2 orbit is born at $\omega = 0.529$;it undergoes a period 2×2^m doubling bifurcations, finally leading to the onset of chaos as ω decreases (see Fig.2c).

Higher values of the forcing strength f introduces the appearance of a new transition. When this increase is coupled with increasing frequency, the quasiperiodic motion suddenly changes into a phase locked attractor. As an example at $f = 0.19$, the transition from quasiperiodic oscillation to phase locked states of period 3 orbit occur at $\omega = 0.453$ and this locked state persists in the frequency range $\omega \in (0.453, 0.47)$, which is then followed by chaos, reverse period doubling, chaos and periodic solution (see Fig.2d).

The details of our numerical study are summarised in the $(f - \omega)$ phase diagrams, Fig.1 and Fig. 2. It depicts the system parameter region where quasiperiodic, large periodic and chaotic attractors exist. The curves denoted QP(1),QP(2) and P(1) are the boundaries of transition from quasiperiodic to the chaotic state, quasiperiodic to periodic state and chaotic to periodic state respectively. Also one observes in the entire transition regions where coexistence of multiple attractors occur. Further, beyond the curve P(1), there are some regions which exhibit phase locked periodic and transient chaotic states. However, in this paper, those states are not discussed in detail.

The various steady states as 1,2,3,...,15 denoted in Fig. 1 are then illustrated in Fig.3. Regular attractors are illustrated by their phase portraits and quasiperiodic and chaotic attractors by their Poincaré maps.

The first three points (1)-(3) in Fig.3 are examples of almost periodic (quasiperiodic) orbits for low values of f. Then the points (4)-(15) are essentially located at the principal and super harmonic resonance regions for large values of the forcing parameter, $f > 0.12$, at increasing driving frequency ω. We observe here the following: quasiperiodic orbit (point (4)), period 3T orbit (point (5)), chaotic orbits (points (6),(7)), period doubled orbits (points (8),(9)), chaotic orbits (points (10),(11)), period 5T orbit (point (12)) and period T orbits (points (13)-(15)).

B. Classification of Bifurcations

The complicated dynamical behaviours of the DVP oscillator (2) with $|\alpha| = \beta$ due to the presence of the double well restoring force has been confirmed by the phase diagram as discussed above. From the bifurcation theory point of view, these correspond to several types of bifurcations: secondary Hopf, intermittent and blue-sky catastrophes besides standard period doubling bifurcations which are discussed in the following section.

1. QP(1) region - Local bifurcations: Secondary Hopf (Neimark) bifurcation

In analogy with the Hopf bifurcation, a bifurcation is expected at a critical value as the limit cycle loses its stability, so that an attracting torus is born. This is the secondary Hopf bifurcation or a Neimark bifurcation [17]. Further, the bifurcated solution can be either stable and supercritical or unstable and subcritical. For the present DVP oscillator (2) with $\mid \alpha \mid = \beta$, there is a very large transition region QP(1) corresponding to this secondary Hopf bifurcation. As an example, let us examine the transient process near $f = 0.1$. Figures 4 show the Poincaré maps with values of ω decreasing and with the starting values of x and $\dot{x}$ indicated as in brackets. In Fig.4a, for $\omega = 0.59$, we see a node like convergence to a point, while in Fig.4b for $\omega = 0.58$ the convergence has a spiralling character and the rate of convergence is noticebly slower. In the last diagram of Fig.4c, at $\omega = 0.57$, we see that the system is moving outwards from near unstable fixed point towards the attracting invariant closed curve and the approach is termed as supercritical. This is the typical behaviour for the curve QP(1) in Fig.1 for $f < 0.10$.

The other nature of Neimark bifurcation, namely subcritical behaviour has also been observed in system (2). For example, at $f = 0.15$, chaotic region exists in the range $\omega \in (0.512, 0.551)$ (see Fig.5). It follows that this narrow strip of chaotic motion is related to the transition from quasiperiodic to periodic oscillation via chaotic motion. This transition does not occur in a smooth, continous way as it is the case when the Neimark bifurcation is a supercritical one but occurs through a chaotic region. This type of transition has a close resemblence with the Duffing oscillator [15,17,18] case where it was shown that a lower frequency band of chaotic region is related to the saddle-node bifurcation, which causes a sudden change from/to T-periodic orbit to/from chaotic attractor and transient motions separate the two different steady states. Thus by analogy the occurrence of chaotic motion in the region of driving frequency that separates quasiperiodic and periodic oscillations can be interpreted as a subcritical Neimark bifurcation. This type of bifurcation has been reported earlier in ref[12] for the single well DVP oscillator, but the unstable motion corresponds to randomly transitional motion, whereas in the present double-well case this corresponds to a fully chaotic attractor and periodic windows.

2. QP(2) region - Local-global bifurcation:Intermittent catastrophe

Figs.6 show the typical Poincaré maps with forcing values $f = 0.00172$ and 0.00173 at $\omega = 0.83$ in the QP(2) region, where a transition from quasiperiodic to periodic motion occurs. In order to lock the quasiperiodic motion to the period T motion, the Poincaré map points form a closed loop with points progressing more rapidly near the top of the attractor and more slowly where points are visibly dense. As the external force strength is increased, a pair of saddle and node develops such that quasiperiodic motion rapidly shrinks to the node, which is near the original attractor. This type of process is prototypical of intermittent catastrophe [17,19].

3. QP(2) region - Global bifurcations: Blue-sky catastrophe

As the frequency value is increased to $\omega = 0.998$, a different transition process is captured for $f = 0.05$. The typical Poincaré map is shown in Figs.7 with the values of ω at 0.998 and 1. and $f = 0.05$. From Figs.7, we find that a periodic attractor bifurcates to a quasiperiodic attractor which is located inside the other, and the two attractors are typically disjoint and separated in phase space by a finite distance. Furthermore, it is not generic for a quasiperiodic attractor to appear suddenly at the same control threshold where periodic motion vanishes. The quasiperiodic attractor will have existed previously, or it will not exist at all; in either case the bifurcation will consist of periodic attractor simply losing stability - it vanishes into the blue. Such phenomenon is termed as blue-sky catastrophe [17,20] in the literature and this event involves collisions with saddle type objects.

4. V-shaped region - Transitions to chaotic attractors

We now enumerate the various possible attractors present in the system in the V-shaped region in Fig. 1.

(i) Transient chaos and boundary crises: Chaotic behaviour is observed between boundary of curves QP(1) and P(1) in Fig.1. Near the boundary of each of the curves P(1), QP(1), the system behaves in a random way, with the trajectory moving in phase space as if it were on a strange attractor. However after a transient time, the motion settles into a regular attractor, that is near P(1) it settles into a periodic motion (for example see Fig.8) while near QP(1) into a quasiperiodic oscillation. Such a phenomenon is termed as transient chaos, which is a precursor to steady state chaos. Between these two transitional regions, periodic windows, phase locked states, chaotic attractor and period doubling phenomenon occur. In addition, the boundary crisis [21] of chaotic attractor appears as the value of the external frequency increases so that the dynamics corresponds to the curve P(1), where the boundary of the chaotic attractor touches the unstable periodic orbit.

In addition, one observes the interesting fact that in the V-shaped region there exists different parametric values for which intermittency of all the three classic types occur. This seems to be rare in such low dimensional systems.

(ii) Type I Intermittency: The parameter regions separating the periodic windows inside the V- shaped region correspond to various complicated dynamics including chaos. The precise stability boundary of each window has been found to correspond to a saddle-node instability. As an example, for f =0.17, if ω is increased across the saddle-node boundary, type I intermittency occurs[22-25]. One such intermittency signature is shown in Fig.9. The average laminar length ($< l >$) of this type of intermittency is found to comply with the law $< l > \sim \mu^{-\delta}$ with $\delta \sim 0.52 \pm 0.001$ where $\mu = \omega - \omega^c$ and $\omega^c$is the bifurcation threshold.

(iii) Type II Intermittency: In the earlier section, we showed that two possibilities exist in the QP(1) region when the periodic motion encounters a Hopf-bifurcation. Either quasiperiodic motion results if the bifurcation is supercritical or a complicated evolution appears if the bifurcation is subcritical. In the later case the transition to chaos is found to have intermittency signature in certain parametric regions. Such an intermittency signature is shown in Fig. 10. A close look into the signature reveals that there are distinct phases of the regular motion which are punctuated by other phases which are apparently chaotic. According to the classical Pomeau-Manneville categorization of different types of intermittencies based on local bifurcations, the present intermittency is of type II since the preceding bifurcation is a Hopf bifurcation [22-25]. The average length $< l >$ of the laminar phase of this intermittency is found to comply with the law $< l > \sim (\frac{1}{\mu})^{\delta}$ with δ=0.9321 where $\mu = \omega - \omega^c$ and ω^c is the bifurcation threshold.

(iv) Type III Intermittency: Next we discuss yet another type of route inwhich the periodic orbits in the periodic windows are seen to undergo intermittent transition to chaos. One such intermitteny motion is shown in Fig. 11, which is a Poincaré time series plot for f=0.14 and ω=0.53802. It is seen that the motion just before the onset of intermittency is of period 20 orbit which itself occurred due to the period doubling of period 10 orbit. The laminar phase of Fig. 11 corresponds to period 40 orbit along with chaotic bursts. Therefore this is identified to have arisen out of a subcritical half subharmonic instability, that is subcritical period doubling. Thus according to PM classification this is type III intermittency [22]. The average length $< l >$ of the laminar phase of this intermittency complies with the following scaling law predicted by Pomeau-Manneville: $< l > \sim (\frac{1}{\mu})^{\delta}$ with $\delta = 0.9912$ where $\mu = \omega - \omega^c$ and ω^c is the bifurcation threshold.

III. PERTURBATIVE ANALYSIS

From the numerical studies reported in the earlier sections, we observed that the T periodic orbit within a range of driving frequency ω for low values of f is close to a harmonic function of time. Then by obtaining a first order approximate period T $\left(\frac{=2\pi}{\omega}\right)$ solution and analysing the stability one can estimate the system parameters domain inwhich Neimark instability arises as was done in the case of single well DVP oscillator for fixed f and ω in ref [12]. Keeping this aim in mind we look for a periodic solution of (2) using a perturbative method (with both $\mid \alpha \mid$ and β fixed at 0.5). Applying the method of multiple scales [1,26] to equation (2), one can obtain the approximate solution about the stable fixed point $x_s = \sqrt{\frac{\alpha}{\beta}}$ in the form

$$x = -\frac{3}{4}a^2 + a\cos(\omega t + \phi) + \frac{a^2}{4}\cos 2(\omega t + \phi) - \frac{a^2}{3}\mu \sin 2(\omega t + \phi), \tag{3}$$

where

$$a = \frac{f}{\sqrt{(\Omega^2(a) - \omega^2)^2 + \left(\frac{3}{2}\mu\omega a^2\right)^2}}, \tag{4}$$

$$\tan\phi = -\frac{\frac{3}{2}\mu\omega a^2}{(\Omega^2(a) - \omega^2)}, \tag{5}$$

and $\Omega^2(a) = 1 - \frac{9}{8}a^2$ is the natural frequency of the autonomous conservative system (2) at $\mu = 0$ and $f = 0$.

A. Linear stability analysis

1. Soft-mode instability

In order to examine the stability of the solution (3), we may look at a specific form of instability which manifests itself by an exponential growth with time of the harmonic components in the solution (3). This can be done by adding small disturbances to the amplitude and phase of the solution (3) as

$$x = -\frac{3}{4}(a+\delta a)^2 + (a+\delta a)\cos(\omega t + \phi + \delta\phi) + \frac{(a+\delta a)^2}{4}\cos 2(\omega t + \phi + \delta\phi) - \frac{(a+\delta a)^2}{3}\mu \sin 2(\omega t + \phi + \delta\phi). \tag{6}$$

Working out the linearized equation for δa and $\delta\phi$, one ultimatively arrives at the following expression which corresponds to the first order instability limit as

$$\omega^4 + \left(\frac{27}{4}\mu^2 a^4 - 2a^2 - 2\right)\omega^2 + \left(2a^2 + \frac{53}{16}a^4 + 1\right) = 0. \tag{7}$$

The above analysis is valid only for a fluctuation having the same frequency as the approximate solution x(t) considered.

2. Hard-mode instability

Now we examine another type of instability in which the perturbation may have different harmonic components other than those in x(t). Following the spirit of the work of Szemplinska-Stupnicka and Rudowski [12], let us study the effect of a small disturbance to $x(t) = \bar{x}(t)$, where $\bar{x}(t)$ is the solution (3), in the form

$$x(t) = \bar{x}(t) + \delta x(t). \tag{8}$$

The linear variational equation for $\delta x(t)$ is then

$$\delta\ddot{x} + P_1(t)\delta\dot{x} + P_2(t)\delta x = 0, \tag{9}$$

where

$$P_1(t) = -\mu\left[1 - \left({a_0}^2 + \frac{a^2}{2} + \frac{a_2^2}{2} + \frac{a_3^2}{2}\right) + 2a_0 a\cos\theta + \left(\frac{a^2}{2}\right)\cos 2\theta + ...\right], \tag{10}$$

$$P_2(t) = -\mid\alpha\mid + 3\beta\left({a_0}^2 + \frac{a^2}{2} + 2a_0 a\cos\theta + \frac{a^2}{2}\cos 2\theta\right) + 2\mu\left[\left(-{a_0}^2 a - \frac{a^3}{4}\right)\omega\sin\theta + a_0 a^2\omega\sin 2\theta - \frac{a^3}{2}\omega\sin 3\theta\right] + ... \tag{11}$$

and $a_0 = -\frac{3}{4}a^2$; $a_2 = \frac{a^2}{4}$; $a_3 = \frac{a^3\mu}{3}$; $\theta = \omega t + \phi$, where dots correspond to higher harmonic components. Introducing now the transformation

$$\delta x = u \exp\left[\frac{-1}{2}\int_0^t P_1(t)dt\right], \qquad (12)$$

equation (9) can be converted into a Hill's equation

$$\ddot{u} + P(t)u = 0, \qquad (13)$$

where

$$P(t) = P_2 - \frac{P_1^2}{4} - \frac{\dot{P}_1}{2}. \qquad (14)$$

Using the form of P_1 and P_2 given in (10) and (11), the transformation (12) can be rewritten as

$$\delta x = u \exp\left[-\Delta t + \frac{\mu}{2}\int_0^t \left(2a_0 a\cos\theta + \frac{a^2}{2}\cos 2\theta\right)d\theta\right], \qquad (15)$$

$$\Delta = \frac{1}{2}\mu\left[\frac{a^2}{2} + a_0^2 + \frac{a_2^2}{2} + \frac{a_3^2}{2} - 1\right]. \qquad (16)$$

Applying the Floquet theorem, one can look for a particular solution of (13) in the form

$$u = \exp(\epsilon_1 t).\phi(t), \qquad (17)$$

where $\phi(t)$ is a periodic function of time and ϵ_1 is either real or imaginary. Thus the equation (15) becomes

$$\delta x(t) = \exp(\epsilon_1 - \Delta)t.\bar{\phi}(t), \qquad (18)$$

where

$$\bar{\phi}(t) = \phi(t)\exp\left[\frac{\mu}{2}\int_0^t \left(2a_0 a\cos\theta + \frac{a^2}{2}\cos 2\theta\right)d\theta\right]. \qquad (19)$$

The stability of solution (3) depends exclusively on the exponent coefficient $(\epsilon_1 - \Delta)$ in (18). Let now discuss the various possibilities to have stable solution.

Case(i): Considering the case $\epsilon_1 = \pm i\bar{\epsilon}_1$ so that $\bar{\epsilon}_1$ is real and positive then

$$\delta x(t) = \exp(\pm i\bar{\epsilon}_1 - \Delta)t.\bar{\phi}(t). \qquad (20)$$

It can be concluded that when ϵ_1 is imaginary, the solution (3) is stable if $\Delta > 0$ or $a > 0.91$ and unstable if $\Delta < 0$. This form of instability (termed as Neimark instability) leads to a buildup of new harmonic components whose frequencies are incommensurate with the frequency of the periodic solution (3).

Case(ii): Considering the case $\epsilon_1 = \pm\bar{\epsilon}_1$ so that $\bar{\epsilon}_1$ is real then

$$\delta x(t) = \exp(\pm\bar{\epsilon}_1 - \Delta)t.\bar{\phi}(t). \qquad (21)$$

When $\bar{\epsilon}_1$ is real, the solution (3) is stable if $\Delta > 0$ and $\Delta^2 > \bar{\epsilon}_1^2$ and this form of instability is approximately equal to the classic first order instability as given by Eq.(7).

Therefore the form of instability defined by the equation (20) and so the condition $\Delta > 0$ leads to the build up of new harmonic components with frequencies $\omega + \bar{\epsilon}_1$ and $\omega - \bar{\epsilon}_1$. However, these frequencies are in general incommensurate with the frequency ω of the periodic solution (3) whereas for $\Delta < 0$ the solution is unstable and so $\Delta = 0$ is the boundary of the instability. Thus, this instability can be interpreted as a Neimark instability, giving rise to a Neimark bifurcation.

Let us now look at the resonance curves, shown in Fig.12 and the two unstable regions defined by the condition (7) (first order instability) and the condition $\Delta < 0$ for Neimark instability. From the Fig.12, the Neimark instability is expected to occur at the frequency value where the resonance curve crosses the critical boundary value a ~ 0.91. To determine the Neimark stability limit in the $(f - \omega)$ parameter plane, we calculate the forcing parameter f by using the resonance equation (4). Fig.13 depicts the Neimark instability limit defined by the condition $\Delta < 0$ and the first order stability limit described by the condition (7). The numerical study results presented already in Fig.1 are shown for comparison. The theoretically predicted Neimark instability values are reasonably close to the numerical results.

IV. COMPARISON IN THE DYNAMICS WITH THE DOUBLE WELL DUFFING OSCILLATOR

Finally it is of importance to compare the dynamics of the double-well Duffing oscillator as given by Szemplinska-Stupnika and Rudowski [15],

$$\ddot{x} + \mu\dot{x} - \mid \alpha \mid x + \beta x^3 = f \cos \omega t, \tag{22}$$

with that of double well DVP (2) for the same parametric values $\mid \alpha \mid = \beta = 0.5$ and $\mu = 0.1$. The results are compared in Table I. In the lower frequency boundary region (see Fig.1 of ref[15]), the Duffing oscillator exhibits symmetry breaking bifurcations while in the DVP, Neimark bifurcation boundary(Fig.1 of the present paper) has been found. In the higher frequency boundary region, particularly in the prinicipal resonance region, the Duffing system exhibits period doubling bifurcations of small periodic orbits and cross well chaos. However in the DVP system, this region is always found to have highly regular orbits.

Further, the Duffing oscillator exhibits period doubling route to chaos of large period T orbit in the lower frequency region, that is $\omega < 0.4$, while in the present case, the system always exhibits almost-periodic oscillations in the region. In the present case, in every transition boundaries such as QP(1), P(1) and QP(2), there are some regions where coexistence of multiple attractors are found to occur. But in the Duffing oscillator case, the coexistence of multiple attractors are observed in the transition region from large period T orbit to cross well chaos. Blue-sky catasrophe, type II and III intermittencies and various phase locked states are found to occur in the present case. However, in the Duffing oscillator case such phenomena have not been found (atleast to our knowledge). Naturally the dynamics exhibited by the DVP equation (2) is also quite distinct compared to the forced van der Pol oscillator [27].

V. CONCLUSIONS

Numerical studies show that the double well Duffing-van der Pol oscillator (2) with the parameter choice $\mid \alpha \mid = \beta$ exhibits a rich variety of attractors of periodic, quasiperiodic and chaotic types. Four varieties of transitions from quasiperiodic to periodic motions occur: (1) QP - periodic orbits (2) QP - chaos - periodic, (3) QP - chaos - periodic windows - chaos - periodic and (4) QP - phase locked states - chaos - periodic orbits. Besides these, local stable or supercritical, unstable or subcritical Neimark bifurcations and mode locking, intermittent catastrophe and blue-sky catastrophe bifurcations are also shown to exist. Transient chaos, period doubling phenomena, boundary crisis and intermittencies of all the three classic types are shown to occur and these were demonstrated with suitable examples in the $(f - \omega)$ parameter space. In the literature, so far all the three intermittencies have been found to occur mostly in the higher dimensional or coupled systems [21-23]. However in the present case even in a single model of low-dimensional system, we are able to demonstrate all the three kinds of PM intermittencies. The various forms of instabilities of approximate periodic solution allows one to predict the Neimark bifurcation in the $(f - \omega)$ parameter domain. Although some discrepency between true and theoretical predictions occurs, the approximate analysis throws some light to distinguish between the regular and chaotic regions.

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FIGURES

FIG. 1. Regions of different steady states exhibited by the double well DVP oscillator (2) at $\mu = 0.1$.

FIG. 2. (i) Bifurcation diagram for maximum amplitude x vs external forcing frequency ω, (ii) Maximal Lypanov exponet λ_{max} vs external forcing frequency ω of system (2). (a) QP - chaos - periodic orbit transitions for $\omega \in$(0.4,0.65) at f=0.13. (b) QP - chaos - periodic windows - chaos transitions for $\omega \in$(0.4,0.65) at f=0.14. (c) QP - chaos - period doubling window - chaos transitions for $\omega \in$(0.4,0.65) at f=0.17. (d) QP - phase locked states - chaos - period doubling windows - chaos transitions for $\omega \in$(0.4,0.65) at f=0.19.

FIG. 3. Various types of steady states at $\mu = 0.1$: (1)f=0.001, ω =0.78; (2) f=0.001, ω=0.9; (3) f=0.12, ω=0.45; (4) f=0.15, ω=0.45; (5) f=0.19, ω=0.47; (6) ω=0.515; (7) ω=0.52; (8) ω=0.525; (9) ω=0.53; (10) f=0.15, ω=0.54; (11) f=0.14, ω=0.53; (12) f=0.135, ω=0.54; (13) f=0.14, ω=0.58; (14) f=0.15, ω=0.58; (15) f=0.18, ω=0.58.

FIG. 4. Trajectories near the Neimark bifurcation for system (2): f=0.1: (a) ω=0.59(1,1); (b) ω=0.59(1,1); (c) ω=0.57(1,1).

FIG. 5. (i) Bifurcation diagram for maximum amplitude x vs external forcing frequency ω, (ii) Maximal Lypanov exponet λ_{max} vs external forcing frequency ω of system (2). QP-chaos-periodic orbit transitions for $\omega \in$(0.4,0.65) at f=0.15.

FIG. 6. Poincaré map of the system (2) before and after mode locking for (a) f=0.00172, (b) f=0.00173, (c) f=0.00174 at ω=0.83.

FIG. 7. Poincaré map of blue-sky disappearence of Periodic attractor in system (2): (a) ω=0.998, (b) ω=1. at f=0.05

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FIG. 8. Trajectories to show transient chaos for $f = 0.125, \omega = 0.535$.

FIG. 9. Signature of type I intermittency: Time series plot for f=0.17, (a) ω=0.52601; (b) ω=0.526010001.

FIG. 10. Signature of type II intermittency: Time series plot for f=0.12, (a) ω=0.5532; (b) ω=0.5537.

FIG. 11. Signature of type III intermittency: Poincaré time series plot for f=0.14, ω=0.53802.

FIG. 12. Resonance curves and unstable regions of solution (3): I: branches unstable in the sense of first order instability; II : branches unstable in the sense of Neimark instability.

FIG. 13. Regions of different steady states : Numerical (solid line) and theoretical (dashed line) stability limits.

TABLES

I. Comparison of the orbits of Duffing and DVP oscillators

Parameters		Duffing Oscillator	DVP Oscillator
$\omega \in (0.4, 0.6)$	f small	small orbit	QP orbit
	f large	large orbit and chaos	large orbit and chaos
$\omega \in (0.6, 1.0)$	f small	small orbit	QP orbit
	f large	small orbit and chaos	large orbit