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On the complete integrability and linearization of nonlinear ordinary differential equations. III. Coupled first-order equations

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Continuing our study on the complete integrability of nonlinear ordinary differential equations (ODEs), in this paper we consider the integrability of a system of coupled first-order nonlinear ODEs of both autonomous and non-autonomous types. For this purpose, we modify the original Prelle–Singer (PS) procedure so as to apply it to both autonomous and non-autonomous systems of coupled first-order ODEs. We briefly explain the method of finding integrals of motion (time-independent as well as time-dependent integrals) for two and three coupled first-order ODEs by extending the PS method. From this we try to answer some of the open questions in the original PS method. We also identify integrable cases for the two-dimensional Lotka–Volterra system and three-dimensional Rössler system as well as other examples including non-autonomous systems in a straightforward way using this procedure. Finally, we develop a linearization procedure for coupled first-order ODEs.

Keywords: nonlinear differential equations; coupled first order; integrability; integrating factor; linearization

1. Introduction

In our previous two works (Chandrasekar *et al.* 2005, 2006), we have studied in some detail the extended modified Prelle–Singer (PS) procedure (Prelle & Singer 1983; Duarte *et al.* 2002), so as to apply it to a class of second- and third-order nonlinear ordinary differential equations (ODEs) and have solved several physically interesting nonlinear systems and identified a number of important linearization procedures. We now wish to extend the procedure to coupled ODEs. In the present paper, we discuss the modification and applicability of the extended PS method to a system of first-order ODEs of both autonomous and non-autonomous types. In subsequent papers, we will extend the procedure to coupled second- and higher order ODEs. We are motivated by certain open questions still prevailing in the original PS method for a system of autonomous coupled first-order ODEs. Before discussing them and describing how we answer them, we shall have an overview of the original PS method and its generalizations.

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We recall that in 1983 Prelle & Singer (1983) proposed an algorithmic procedure to find the integrating factor for the system of two autonomous first-order ODEs of the form $\mathrm{d}x/\mathrm{d}t = P(x, y)$, $\mathrm{d}y/\mathrm{d}t = Q(x, y)$, where P and Q are polynomials in x and y with coefficients in the field of complex numbers. Equivalently it can be recast in the form $y' = \mathrm{d}y/\mathrm{d}x = Q(x, y)/P(x, y)$. Once the integrating factor for the latter equation is determined, then it leads to a time-independent integral of motion for the above autonomous two coupled first-order ODEs (for the single first-order ODEs, the first integral is nothing but the general solution). The PS method guarantees that if the given system of two coupled first-order ODEs has a first integral in terms of elementary functions, then this first integral can be found. This method has been generalized to incorporate the integrals with non-elementary functions (Singer 1990, 1992; Duarte *et al.* 2002), and some first integrals of autonomous systems of ODEs of higher dimensions (of dimension 3) were also calculated. Man (1994) described a method for calculating first integrals of autonomous systems which are rational or quasi-rational, but said that 'The generalization of this procedure to higher dimensions to find elementary first integrals is still an open problem'. In addition, the question about whether the PS procedure can be extended to a non-autonomous system of first-order ODEs has not been addressed so far. Also to our knowledge, the problem of finding time-dependent integrals for a given system of coupled first-order ODEs through this procedure has not been taken up so far. Further, the problem of finding both time-dependent and time-independent integrals for a system of first-order ODEs greater than 2 in number has not been dealt with systematically. In addition, the problem of how to linearize a given system of coupled first-order ODEs systematically also remains to be tackled. In this paper, we address positively all these questions and come out with valuable answers to these problems and demonstrate the general results with suitable examples.

Firstly, in order to extend the PS method to a non-autonomous system of first-order ODEs with the rational form $\mathrm{d}x/\mathrm{d}t = P_1(t, x, y)/Q_1(t, x, y)$, $\mathrm{d}y/\mathrm{d}t = P_2(t, x, y)/Q_2(t, x, y)$, where $P_i(t, x, y)$ and $Q_i(t, x, y)$, $i = 1, 2$, are analytic functions of x and y with coefficients in the field of complex numbers, we develop a modified technique and derive determining equations for the integrating factors R and K. Secondly, in the case of a coupled system of two first-order ODEs, we observe that the integrals of motion I_1 and I_2 of the coupled ODEs are either both time dependent or I_1 may be time independent, while I_2 is time dependent (or vice versa). Based on this observation, we analyse the problem by splitting it into two different cases, namely (i) the solutions of the determining equations for R and K which lead to time-independent integral(s) and (ii) the solutions that lead to time-dependent integrals. We show that the determining equations in the time-independent integral case coincide with those derived by Prelle & Singer (1983) in their original paper. In the second case, we obtain the time-dependent integral, if it exists, and from this we tackle the second problem. Using this method we find integrable cases for the two-dimensional Lotka–Volterra (LV) equations (given as an example) with both time-dependent and time-independent integrals as well as certain non-autonomous systems.

For extending the method to a coupled system of more than two first-order ODEs, we first extend the above modified PS procedure to three coupled first-order ODEs and propose a systematic procedure to obtain the integrating factors R, K and M. In this case we divide our analysis into five categories. We illustrate this theory with two physically important examples, namely the Rössler system and the three-dimensional LV system and identify some new integrable cases. The method is extendable straightforwardly to a system of more than three coupled first-order ODEs. In addition to the above, we also present a method of finding a linearizing transformation for a system of first-order ODEs. We illustrate the theory with certain concrete examples.

The paper is organized as follows. In §2, we develop the modified PS method applicable for a non-autonomous system of two coupled first-order ODEs. In §3, we describe the method of solving the determining equations and how one can obtain compatible solutions from them. In §4 we illustrate the procedure with the LV equation as an example and identify many integrable cases in it. We also apply the method to non-autonomous two coupled equations. In §5, we extend the PS procedure to non-autonomous three coupled first-order ODEs and describe methods of solving the determining equations in §6. We emphasize the validity of the theory with two illustrative examples arising in different areas of physics in §7. Further, in §8 we discuss the direct applicability of the modified PS procedure to n (greater than 3) coupled first-order ODEs. In §9, we demonstrate the method of identifying linearizing transformations with examples. Finally, we present our conclusions in §10.

2. PS procedure for two coupled first-order ODEs

Let us consider a system of two coupled first-order ODEs of the form

$$\dot{x} = \frac{P_1(t,x,y)}{Q_1(t,x,y)}, \qquad \dot{y} = \frac{P_2(t,x,y)}{Q_2(t,x,y)}, \qquad \left(\cdot = \frac{\mathrm{d}}{\mathrm{d}t}\right), \tag{2.1}$$

where P_i and Q_i, $i=1, 2$, are analytic functions of x and y with coefficients in the field of complex numbers. Further, we assume that the ODE (2.1) admits a first integral $I(t, x, y)=C$ with C constant on the solutions, so that the total differential becomes

$$\mathrm{d}I = I_t\,\mathrm{d}t + I_x\,\mathrm{d}x + I_y\,\mathrm{d}y = 0, \tag{2.2}$$

where subscript denotes partial differentiation with respect to that variable. Let us rewrite (2.1) in the form

$$\frac{P_1}{Q_1}\mathrm{d}t - \mathrm{d}x = 0, \qquad \frac{P_2}{Q_2}\mathrm{d}t - \mathrm{d}y = 0. \tag{2.3}$$

Hence, on the solutions, the 1-forms (2.2) and (2.3) must be proportional. Multiplying the first equation in (2.3) by the integrating factor $R(t, x, y)$ and the second equation in (2.3) by a second integrating factor $K(t, x, y)$ (both of which are to be determined), we have on the solutions that

$$\mathrm{d}I = (R\phi_1 + K\phi_2)\mathrm{d}t - R\,\mathrm{d}x - K\,\mathrm{d}y = 0, \tag{2.4}$$

where $\phi_i \equiv P_i/Q_i$, $i=1, 2$. Comparing equations (2.4) and (2.2), we have, on the solutions, the relations

$$I_t = (R\phi_1 + K\phi_2), \qquad I_x = -R \quad \text{and} \quad I_y = -K. \tag{2.5}$$

The compatibility conditions, $I_{tx}=I_{xt}$, $I_{ty}=I_{yt}$, $I_{xy}=I_{yx}$, between the equations (2.5) provide us the conditions

$$R_t + \phi_1 R_x + \phi_2 R_y = -(R\phi_{1x} + K\phi_{2x}), \tag{2.6}$$

$$K_t + \phi_1 K_x + \phi_2 K_y = -(R\phi_{1y} + K\phi_{2y}), \tag{2.7}$$

$$R_y = K_x. \tag{2.8}$$

Integrating equations (2.5), we obtain the integral of motion

$$I = r_1 + r_2 - \int \left[K + \frac{\mathrm{d}}{\mathrm{d}y}\left(r_1 + r_2\right)\right]\mathrm{d}y, \tag{2.9}$$

where

$$r_1 = \int (R\phi_1 + K\phi_2)\mathrm{d}t \quad \text{and} \quad r_2 = -\int \left(R + \frac{\mathrm{d}}{\mathrm{d}x}(r_1)\right)\mathrm{d}x.$$

Solving the determining equations (2.6)–(2.8) consistently, we can obtain expressions for the functions R and K. Substituting them into (2.9) and evaluating the integrals, we can deduce the associated integral of motion. If two independent sets of solutions R and K for (2.6)–(2.8) are found, then they give rise to two independent integrals for the given system of first-order ODEs (2.1), which ensures the complete integrability of the system and obtaining explicit general solution of the system.

3. Method of solving determining equations

One may note that the determining equations (2.6)–(2.8) are over-determined and the crux of the problem lies in finding the explicit solutions satisfying all the three determining equations, since once a particular solution is known then the integral of motion can be readily constructed. To solve the equations (2.6) and (2.7), we introduce a transformation

$$R = SK, \tag{3.1}$$

where S is a function of t, x and y, so that the determining equations (2.6)–(2.8) now become

$$S_t + \phi_1 S_x + \phi_2 S_y = -\phi_{2x} + (\phi_{2y} - \phi_{1x})S + \phi_{1y}S^2, \tag{3.2}$$

$$K_t + \phi_1 K_x + \phi_2 K_y = -K(S\phi_{1y} + \phi_{2y}), \tag{3.3}$$

$$K_x = SK_y + KS_y. \tag{3.4}$$

One may observe that the equation for S, namely equation (3.2), is decoupled from that of K (equations (3.3) and (3.4)) and so the set (3.2)–(3.4) may be easier to analyse than solving the original ones (2.6)–(2.8) directly.

To begin with we observe that for the system of two coupled first-order ODEs (2.1), there can be two independent integrals I_1 and I_2 such that either both of them are time dependent or I_1 may be time independent, while I_2 is time dependent (or vice versa). So we consider these two cases separately with corresponding solutions S and K for equations (3.2)–(3.4). The determining equations in the time-independent integral case (see equation (3.8) below) coincide with the determining equation derived by Prelle & Singer (1983) in their original paper, whereas for the time-dependent integrals case, we develop an extended procedure to capture both the integrals, if they exist, and thereby make the PS procedure a more powerful tool in a self-contained way.

(*a*) *Time-independent integral*

In the case $I_t=0$, we denote $I_1=I$ and note that the function S can be easily fixed with the help of the first equation in (2.5), that is,

$$\frac{R}{K}=S=-\frac{\phi_2}{\phi_1}. \tag{3.5}$$

Since I is independent of t, it follows from equation (2.5) that S (and also R and K) is also independent of t. Indeed one can check that $S=-(\phi_2/\phi_1)$ is a solution of (3.2). Now substituting $S=-(\phi_2/\phi_1)$ into (3.3), we get the following equation for K, that is:

$$\phi_1 K_x+\phi_2 K_y=K\left(\frac{\phi_2}{\phi_1}\phi_{1y}-\phi_{2y}\right). \tag{3.6}$$

We now make a substitution

$$K=\frac{\phi_1}{f(x,\,y)}, \tag{3.7}$$

where $f(x, y)$ is an arbitrary non-zero function of x and y. Then (3.6) takes the simpler form

$$\phi_1 f_x+\phi_2 f_y=f(\phi_{1x}+\phi_{2y}). \tag{3.8}$$

We note that by redefining $f=1/\mathcal{R}$, equation (3.8) coincides with the determining equation derived by Prelle & Singer (1983) for the autonomous case of (2.1). We also mention that equation (3.8) is nothing but the one obtained by substituting the forms of S and K given in (3.5) and (3.7) into the constraint equation (3.4). So by solving equation (3.8), we can get the complete set $R(=SK)$ and K associated with equations (3.2)–(3.4).

Even though equation (3.8) is a quasi-linear PDE in two variables, the associated characteristic equation again leads to coupled differential equations of the form (2.1). Thus the routine methods of finding general solution of quasi-linear PDEs are not very useful here. Here, we find particular solutions for the determining equation (3.8) in a different way and obtain integrable cases for the given system. For this purpose, we assume a specific functional form for $f(x, y)$ with unknown functions and determine the latter consistently. A simple but effective choice is $f=(A(x)+B(x)y)^r$, where A and B are functions of their arguments, and r is a constant. Again the reason for choosing this form is as follows. Since K is in a rational form, while taking differentiation or integration the form of the denominator remains the same but the power of the denominator decreases or increases by a unit order from that of the initial one. So instead of considering f to be of the form $f=A(x)+B(x)y$, one may consider a more general form $f=(A(x)+B(x)y)^r$, where r is a constant to be determined. Depending on the problem in hand, one can also assume a more general form and proceed as in the present case.

Substituting now the form $f=(A(x)+B(x)y)^r$ into (3.8), we arrive at the following equation for the unknown functions A and B:

$$r\Big[\phi_1(A_x+B_xy)+\phi_2B\Big]=(\phi_{1x}+\phi_{2y})(A+By). \tag{3.9}$$

Inserting the given form of ϕ_i's, $i=1$, 2, into the above equation (3.9) and solving the resultant equation, one can fix the forms of A, B and r. Now plugging the resultant form of f into equation (3.7), one can get the integrating factor K, which in turn leads us to the other integrating factor R through the relation (3.5). Finally, substituting R and K into equation (2.9) and evaluating the integrals, one can deduce the time-independent integral for the given system. Since we are dealing with a system of two first-order ODEs, this time-independent integral itself guarantees the integrability of the given system. However, to explore the general solution, one may seek the time-dependent second integral. We describe the procedure in §3*b*.

(*b*) *Time-dependent integral*

Now we focus our attention on the case $I_t\neq 0$. In this case, the function S has to be determined from equation (3.2). Since it is too difficult to solve equation (3.2) for its general solution, we seek particular solutions for S, which is sufficient for our purpose. In particular, we seek a simple rational expression for S in the form

$$S=\frac{A_1(t,x)+B_1(t,x)y}{A_2(t,x)+B_2(t,x)y}, \tag{3.10}$$

where A_i's and B_i's, $i=1$, 2, are arbitrary functions of t and x, which are to be determined. Of course this can be further generalized, if the need arises. Substituting (3.10) into (3.2) and equating the coefficients of different powers of y to zero, we get a set of determining equations for the functions A_i's and B_i's, $i=1$, 2. Solving these determining equations, we obtain explicit expressions for the functions A_i's and B_i's, $i=1$, 2, which in turn fixes S through the relation (3.10).

Now substituting the forms of S into equation (3.3) and solving the resultant equation, one can obtain the corresponding forms of K. To solve the determining equation for K, we again seek the same form of ansatz (3.7) but with explicit t dependence on the coefficient functions, that is, $K=S_{\mathrm{d}}/((A(t,x)+B(t,x)y)^r)$, where S_{d} is the denominator of S. Once S and K are determined, then one has to verify the compatibility of this set (S,K) with the extra constraint equation (3.4). Now substituting R's $(=SK)$ and K's into equation (2.9) and evaluating the integrals, one can construct the associated integrals of motion.

We note here that for a given equation (2.1) one may also get two time-dependent integrals or one time-dependent and one time-independent integrals (discussed earlier), which in turn automatically guarantees the complete integrability of the given system and provides us with an explicit solution by algebraic manipulation. On the other hand, under certain circumstances, one may get only one time-dependent integral and one can transform this time-dependent integral into a time-independent one and thereby establish the integrability.

4. Two coupled ODEs: application

(*a*) *Example: two-dimensional LV system*

Our motivation is to identify integrable cases and deduce both time-dependent and time-independent integrals for a given system of two coupled first-order ODEs through the extended PS procedure in a self-contained way.

To demonstrate this, we consider the celebrated two-dimensional LV system

$$\dot{x} = x(a_1 + b_{11}x + b_{12}y) = \phi_1, \qquad \dot{y} = y(a_2 + b_{21}x + b_{22}y) = \phi_2, \tag{4.1}$$

where the a_i's and b_{ij}'s, i, j=1, 2, are six real parameters. This system was originally introduced by Lotka (1920) and Volterra (1931) to model two species competition. However, in recent years this model has appeared widely in applied mathematics and in a large variety of physics topics such as laser physics, plasma physics, convective instabilities, neural networks, etc. (Minorsky 1962; Brenig 1988; Murray 1989). The integrability properties of the system (4.1) alone have been analysed by many authors; see for example Cairo & Llibre (2000) and references therein.

In the following, we identify integrable cases in (4.1), through our procedure.

(i) *Time-independent integral* (I_t=0)

In this case the function S can be fixed easily in the form (vide equation (3.5))

$$S = -\frac{\phi_2}{\phi_1} = -\frac{y(a_2 + b_{21}x + b_{22}y)}{x(a_1 + b_{11}x + b_{12}y)}. \tag{4.2}$$

To explore the integrating factor K, we need to fix the form f first; see equation (3.7). For this purpose we substitute the ϕ_i's, i=1, 2, into (3.9) so that one gets the following equation for the unknown functions A and B which constitute the function f:

$$\begin{aligned} r\Big[x(a_1 + b_{11}x + b_{12}y)(A_x + B_x y) + y(a_2 + b_{21}x + b_{22}y)B\Big] \\ = (a_1 + a_2 + (b_{21} + 2b_{11})x + (b_{12} + 2b_{22})y)(A + By). \end{aligned} \tag{4.3}$$

Equating the coefficients of various powers of y^i, i=0, 1, 2, and solving the resultant differential equations for A and B, we arrive at the following two general expressions that involve the system parameters:

$$\left.\begin{aligned} b_{21}(b_{12} - b_{22})(b_{12}b_{21} - b_{11}(b_{12} - 2b_{22}))(a_1(b_{11} - b_{21})b_{22} + a_2 b_{11}(b_{22} - b_{12})) = 0, \\ a_1 b_{22}(b_{11} - b_{21})^2(a_2 b_{12} - a_1 b_{22}) + a_2^2 b_{11}(b_{12} - b_{22})(b_{12}b_{21} - b_{11}b_{22}) = 0 \end{aligned}\right\} \tag{4.4}$$

and

$$r = \frac{(b_{11}b_{12} + b_{12}b_{21} - 2b_{11}b_{22})}{b_{11}(b_{12} - b_{22})} \quad \text{or} \quad r = \frac{(b_{11}b_{12} + b_{12}b_{21} - 2b_{11}b_{22})}{(b_{12}b_{21} - b_{11}b_{22})}. \tag{4.5}$$

Any consistent solution that comes out from the above expression (4.4) gives us an integrating factor, which in turn leads us to an integral of motion. In this sense, (4.4) forms an integrability condition of some generality, which in fact encompasses all known integrable cases with time-independent integrals

(Cairo & Llibre 2000; Llibre & Valls 2007). For example, while analysing the above equation, we find that one can straightforwardly recover several known integrable cases such as (i) $b_{11}=b_{22}=0$, $r=1$, (ii) $b_{22}=b_{12}$, $b_{11}=b_{21}$, $r=1$, (iii) $a_1=a_2$, $b_{12}=3b_{22}$, $b_{11}=-b_{21}$, $r=-1$, and (iv) $a_1=a_2=0$, r as given in equation (4.5), and so on straightforwardly from equations (4.4) and (4.5) and construct the associated integral of motion, which in turn coincides with the existing results. However, as we are interested in constructing an integral of motion with more general parametric choice, we do not fix any relation between parameters (other than the general relation) and proceed further.

The respective forms of A and B are (for $b_{22}b_{12}(b_{22}-b_{12})\neq 0$),

$$A=((r-1)g(x)+rb_{22}(a_1+b_{11}x))x^{((2-r)b_{22}+b_{12})/rb_{12}},\quad B=rb_{22}b_{12}x^{((2-r)b_{22}+b_{12})/rb_{12}}, \tag{4.6}$$

so that

$$f=x^{((2-r)b_{22}+b_{12})/b_{12}}\Big((r-1)g(x)+rb_{22}(a_1+b_{11}x+b_{12}y)\Big)^r, \tag{4.7}$$

where

$$g(x)=\left(a_2b_{12}-2a_1b_{22}+\frac{b_{22}}{(b_{22}-b_{12})}(b_{12}b_{21}+b_{11}(b_{12}-2b_{22}))x\right). \tag{4.8}$$

Making use of the explicit forms of S and f, vide equations (4.2) and (4.7), respectively, with the parametric restrictions (4.4), the integrating factors K and R can be fixed as $R=-(\phi_2/f)$ and $K=\phi_1/f$. Substituting the forms R and K into (2.9) and evaluating the integrals, we arrive at the following time-independent integrals for (4.1) for the parametric cases (4.4) for different values of r, namely

$$\begin{aligned} I=&\frac{x}{f}\Bigg[\frac{(r-1)}{r^2}g(x)^2+\Bigg[b_{22}(a_1+b_{11}x+b_{12}y)\Big((a_2b_{12}-a_1b_{22}\\ &+\frac{b_{22}}{(b_{12}-b_{22})}(b_{11}b_{22}-b_{12}b_{21})x+b_{12}b_{22}y)\Big)\Bigg]\Bigg],\quad r\neq 0,2, \end{aligned} \tag{4.9}$$

$$I=\frac{2b_{22}(a_1+b_{11}x+b_{12}y)}{b_{12}h(x,\ y)}-\log\Big[x^{-(b_{22}/b_{12})}h(x,\ y)\Big],\quad r=2, \tag{4.10}$$

$$I=x^{-(2b_{22}/b_{12})}y(2a_1+2b_{11}x+b_{12}y),\quad r=0, \tag{4.11}$$

where $h(x,\ y)=a_2+(b_{11}+b_{21})x+2b_{22}y$ and $g(x)$ is given in (4.8).

For $b_{22}b_{12}(b_{22}-b_{12})=0$, one obtains known integrable cases, following the same procedure as above. For example, let us consider the first case, which we cited above as the known case $b_{22}=b_{11}=0$. For this case, one can find a trivial solution for equation (4.3) as $A=0$ and $B=x$ with $r=1$ so that f becomes $f=xy$. The respective integrating factors R and K read as $R=-((a_2+b_{21}x)/x)$ and $K=((a_1+b_{12}y)/y)$, so that the associated integral of motion takes the form $I=b_{21}x-b_{12}y+a_2\log x-a_1\log y$. This integral is well known and has been popular in the literature for a long time (e.g. Minorsky 1962; Prelle & Singer 1983; Murray 1989). Similarly for the case $b_{22}=b_{12}$ and $b_{11}=b_{21}$, we obtain the integral of motion of the form $I=x^{a_2}y^{-a_1}(a_1a_2+a_1b_{22}y+a_2b_{11}x)^{(a_1-a_2)}$. In the case $b_{12}=0$, equation (4.1) becomes uncoupled and the general solution for this case can easily be obtained.

(ii) *Time-dependent integrals* ($I_t \neq 0$)

Now let us concentrate on the case $I_t \neq 0$. In this case S has to be determined from equation (3.2), that is,

$$S_t + x(a_1 + b_{11}x + b_{12}y)S_x + y(a_2 + b_{21}x + b_{22}y)S_y = -b_{21}y + ((a_2 + b_{21}x + 2b_{22}y) - (a_1 + 2b_{11}x + b_{12}y))S + b_{12}xS^2. \quad (4.12)$$

As we have mentioned earlier, to obtain a particular solution for the above equation (4.12), we seek a simple ansatz for S of the form (3.10). Substituting (3.10) into (4.12) and solving the resulting equation, we obtain non-trivial forms of S for the following specific parametric restrictions (we omitted the uncoupled case $b_{12}b_{21}=0$):

$$\text{(i)} \quad a_1 = a_2, \qquad b_{21} - b_{11}\left(2 - \frac{b_{12}}{b_{22}}\right) = 0, \quad (4.13)$$

$$\text{(ii)} \quad b_{21} = b_{11}, \qquad b_{12} = b_{22}, \quad (4.14)$$

$$\text{(iii)} \quad b_{21} = \frac{b_{22}b_{11}}{b_{12}}, \quad (4.15)$$

$$\text{(iv)} \quad a_1 = a_2, b_{21} - b_{11}\left(2 - \frac{b_{12}}{b_{22}}\right) \neq 0, \quad (4.16)$$

and the respective S forms are

$$\left.\begin{aligned} &\text{(i)} \quad S = \frac{b_{11}}{b_{22}}, \qquad \text{(ii)} \quad S = -\frac{y}{x}, \\ &\text{(iii)} \quad S = -\frac{b_{22}y}{b_{12}x}, \qquad \text{(iv)} \quad S = -\frac{y(b_{21}x + b_{22}y)}{x(b_{11}x + b_{12}y)}. \end{aligned}\right\} \quad (4.17)$$

Now substituting the above forms of S into equation (3.3) and solving the resultant equation, we obtain the corresponding forms of K. By making use of the ansatz mentioned in §3*b*, we obtain following expressions for K:

$$\left.\begin{aligned} &\text{(i)} \quad K = -\frac{a_2 b_{22} e^{-a_2 t}}{(a_2 + b_{11}x + b_{22}y)^2}, \quad \text{(ii)} \quad K = \frac{x}{y^2} e^{(a_2 - a_1)t}, \\ &\text{(iii)} \quad K = -x^{-(b_{22}/b_{12})} \exp\left(\left(\frac{a_1 b_{22}}{b_{12}} - a_2\right)t\right), \\ &\text{(iv)} \quad K = \frac{(b_{11}x + b_{12}y)\exp\left(\frac{a_2(b_{12}-b_{22})}{b_{12}}(r-2)t\right)x^{((r-2)b_{22})/b_{12}}}{\left(\frac{(r-1)}{r}\frac{(b_{12}b_{21}+b_{11}(b_{12}-2b_{22}))}{(b_{12}-b_{22})x} - b_{11}x - b_{12}y\right)^r}, \end{aligned}\right\} \quad (4.18)$$

with r being given in equation (4.5). It may be noted that the set (4.13)–(4.16) also includes the known time-dependent integrable cases.

Once $R(=SK)$ and K are determined, then one has to verify the compatibility of this solution with the extra constraint (2.8), which indeed gets satisfied in each one of the above four cases. Substituting the resultant integrating factors into (2.9) and evaluating the integrals, we obtain the associated time-dependent integrals of motion in the forms

$$\text{(i) (a)} \quad I = \frac{e^{-a_2 t}(b_{11}x + b_{22}y)}{(a_2 + b_{11}x + b_{22}y)}, \quad a_2 \neq 0, \tag{4.19}$$

$$\text{(b)} \quad I = \frac{1 + (b_{11}x + b_{22}y)t}{(b_{11}x + b_{22}y)}, \quad a_2 = 0, \tag{4.20}$$

$$\text{(ii)} \quad I = \frac{x}{y}\exp((a_2 - a_1)t), \tag{4.21}$$

$$\text{(iii)} \quad I = \exp\left(\left(\frac{a_1 b_{22}}{b_{12}} - a_2\right)t\right)x^{-(b_{22}/b_{12})}y, \tag{4.22}$$

$$\text{(iv) (a)} \ I = \left(\frac{(r-1)}{r}\frac{d_3 x}{d_1} - e_2\right)^{-r}\left[\frac{(r-1)}{r^2}d_3^2 x^2 + e_2 d_1 \times(b_{11}b_{22}x + b_{12}^2 y - b_{12}e_1)\right]x^{((r-2)b_{22})/b_{12}}\exp\left(\frac{a_2 d_1}{b_{12}}(r-2)t\right), \quad r \neq 0, 2, \tag{4.23}$$

$$\text{(b)} \quad I = \log\left[2x^{-(b_{22}/b_{12})}yd_1\right] - \frac{b_{21}x}{b_{12}y} - \frac{a_2 d_1}{b_{12}}t, \quad r = 2, \tag{4.24}$$

$$\text{(c)} \quad I = \exp\left(-\frac{2a_2 d_1}{b_{12}}t\right)x^{-(2b_{22}/b_{12})}\left(b_{11}^2 d_2 x^2 + b_{12}^2 d_1 y^2 + b_{11}b_{12}e_4 x\right), \quad r = 0, \tag{4.25}$$

where

$$\left.\begin{aligned} &d_1 = (b_{12} - b_{22}), \quad e_1 = (b_{21}x + b_{22}y), \quad d_2 = (b_{12} - 2b_{22}),\\ &d_3 = (b_{12}b_{21} + b_{11}d_2), \quad e_2 = (b_{11}x + b_{12}y),\\ &d_1 = (b_{12} - b_{22}), \quad e_1 = (b_{21}x + b_{22}y), \quad d_2 = (b_{12} - 2b_{22}),\\ &d_3 = (b_{12}b_{21} + b_{11}d_2), \quad e_2 = (b_{11}x + b_{12}y),\\ &e_3 = ((b_{11} - b_{21})x + 2d_1 y), \quad e_4 = (b_{21}x + 2d_1 y). \end{aligned}\right\} \tag{4.26}$$

Finally we note that our method not only gives us a rather general set of integrable parametric relations (which includes all known cases) but also provides two independent integrals from which one can deduce the general solution for some cases.

(iii) *General solutions/integrability*

Interestingly, we find that for certain parametric choices, we have two independent integrals (time independent as well as time dependent) and consequently one can express the general solution explicitly by using both of them. For example, let us consider the parametric choice given in (4.13) and the associated time-dependent integrals given in (4.19) and (4.20). For this parametric choice, that is, $a_1 = a_2$, $b_{21}b_{22} - b_{11}(2b_{22} - b_{12}) = 0$, we can also find the following time-independent integral from (4.9):

$$I_2 = \frac{y}{x}(a_1 + b_{11}x + b_{22}y)^{(b_{12}/b_{22})-1}. \tag{4.27}$$

Using the integrals I and I_2, the general solution for the two-dimensional LV system, (4.1), for the parametric choice (4.13), can be written as

$$\left.\begin{aligned} x(t) &= \frac{a_1\hat{a}_1 \mathrm{e}^{a_1 t} I}{b_{11}\hat{a}_1 g_1 + b_{22} I_2 (g_1)^{b_{12}/b_{22}}}, \\ y(t) &= \frac{a_1 \mathrm{e}^{a_1 t} I I_2}{(b_{11}\hat{a}_1 (g_1)^{2-(b_{12}/b_{22})} + b_{22} I_2 g_1)}, \quad a_1 \neq 0 \end{aligned}\right\} \tag{4.28}$$

and

$$x(t) = \frac{1}{I_2(g_2)^{b_{12}/b_{22}} - b_{11}g_2}, \qquad y(t) = \frac{I_2(g_2)^{((b_{12}/b_{22})-1)}}{b_{22}(b_{11}g_2 - I_2(g_2)^{b_{12}/b_{22}})}, \quad a_1 = 0, \tag{4.29}$$

where $\hat{a}_1 = a_1^{(b_{12}/b_{22})-1}$, $g_1 = (1 - I\mathrm{e}^{a_1 t})$ and $g_2 = (t + I)$, respectively. Further, for the parametric choice given in (4.14), that is, $b_{21} = b_{11}$, $b_{12} = b_{22}$, we obtain the general solution of the form

$$x(t) = \frac{a_1 a_2 \mathrm{e}^{a_1 t}}{(II_1^{-a_2})^{1/(a_1 - a_2)} - g_3(t)}, \qquad y(t) = \frac{a_1 a_2 \mathrm{e}^{a_2 t}}{(II_1^{-a_2})^{1/(a_1-a_2)} - g_3(t)}, \tag{4.30}$$

where $g_3(t) = (a_1 b_{22}\mathrm{e}^{a_2 t} + a_2 b_{11} I_1 \mathrm{e}^{a_1 t})$. Depending on the signs and magnitudes of the system parameters a_1, a_2, b_{11}, b_{12} and b_{22}, the above solutions describe normalized interacting populations that asymptotically decay or grow or saturate.

Similarly for all the other integrable cases identified in this section, one can derive the general solutions that are physically and mathematically relevant, often after some manipulations. The details will be presented elsewhere.

(*b*) *Application to non-autonomous system of first-order ODEs*

As we have mentioned in §1, one of our motivations is to show that the procedure developed in §3 is applicable to non-autonomous systems as well.

(i) *Example 1: complex Riccati equation*

To demonstrate this point in brief, let us consider the following first-order non-autonomous equations:

$$\left.\begin{aligned} \dot{x} &= \alpha_1(t)x - \alpha_2(t)y + \beta_1(t)(x^2 - y^2) - 2\beta_2(t)xy = \phi_1(t, x, y), \\ \dot{y} &= \alpha_1(t)y + \alpha_2(t)x + \beta_2(t)(x^2 - y^2) + 2\beta_1(t)xy = \phi_2(t, x, y), \end{aligned}\right\} \tag{4.31}$$

where $\alpha_i(t)$ and $\beta_i(t)$, $i = 1, 2$, are arbitrary functions of t. Equation (4.31) describes the dynamics of two interacting species with time-modulated parameters. This coupled equation is essentially the real form of the complex Riccati–Bernoulli equation $\dot{z} = \alpha(t)z + \beta(t)z^2$, where $z = x + \mathrm{i}y$, $\alpha(t) = \alpha_1(t) + \mathrm{i}\alpha_2(t)$

and $\beta(t)=\beta_1(t)+\mathrm{i}\beta_2(t)$. Substituting the form of ϕ_i's, $i=1$, 2, into the determining equation (3.2) and solving the latter, we obtain the following forms for S (generalizing the ansatz (3.10)):

$$\left.\begin{aligned} S_1 &= \frac{(y^2-x^2)\cos(\omega(t))-2xy\sin(\omega(t))}{(x^2-y^2)\sin(\omega(t))-2xy\cos(\omega(t))}, \\ S_2 &= \frac{(y^2-x^2)\sin(\omega(t))+2xy\cos(\omega(t))}{(y^2-x^2)\cos(\omega(t))-2xy\sin(\omega(t))}, \end{aligned}\right\} \tag{4.32}$$

where $\omega(t)=\int\alpha_2(t)\mathrm{d}t$. Now inserting the above forms of S_1 and S_2 into equation (3.3) and solving the resultant equations, we obtain the following integrating factors:

$$\left.\begin{aligned} K_1 &= \frac{\exp\left(\int\alpha_1(t)\mathrm{d}t\right)}{(x^2+y^2)^2}((y^2-x^2)\sin(\omega(t))+2xy\cos(\omega(t))), \\ K_2 &= \frac{\exp\left(\int\alpha_1(t)\mathrm{d}t\right)}{(x^2+y^2)^2}((x^2-y^2)\sin(\omega(t))+2xy\cos(\omega(t))). \end{aligned}\right\} \tag{4.33}$$

Substituting the complete sets $(R_i(=S_iK_i),K_i)$, $i=1$, 2, into equation (2.9) and evaluating the integrals, we obtain

$$\left.\begin{aligned} I_1 &= \frac{\exp\left(\int\alpha_1(t)\mathrm{d}t\right)}{(x^2+y^2)}(x\cos(\omega(t))+y\sin(\omega(t)))+\gamma_1(t), \\ I_2 &= \frac{\exp\left(\int\alpha_1(t)\mathrm{d}t\right)}{(x^2+y^2)}(x\sin(\omega(t))-y\cos(\omega(t)))+\gamma_2(t), \end{aligned}\right\} \tag{4.34}$$

where

$$\left.\begin{aligned} \gamma_1(t) &= \int(\beta_1(t)\cos(\omega(t))-\beta_2(t)\sin(\omega(t)))\exp\left(\int\alpha_1(t)\mathrm{d}t\right)\mathrm{d}t, \\ \gamma_2(t) &= \int(\beta_2(t)\cos(\omega(t))+\beta_1(t)\sin(\omega(t)))\exp\left(\int\alpha_1(t)\mathrm{d}t\right)\mathrm{d}t. \end{aligned}\right\} \tag{4.35}$$

From the integrals I_1 and I_2, the general solution for the equation (4.31) can be fixed easily in the form

$$\left.\begin{aligned} x(t) &= \frac{\exp\left(\int\alpha_1(t)\mathrm{d}t\right)}{\gamma_3(t)}\Big((I_1-\gamma_1(t))\cos(\omega(t))+(I_2-\gamma_2(t))\sin(\omega(t))\Big), \\ y(t) &= \frac{\exp\left(\int\alpha_1(t)\mathrm{d}t\right)}{\gamma_3(t)}\Big((I_1-\gamma_1(t))\sin(\omega(t))-(I_2-\gamma_2(t))\cos(\omega(t))\Big), \end{aligned}\right\} \tag{4.36}$$

where $\gamma_3(t)=(I_1-\gamma(t))^2+(I_2-\gamma(t))^2$. Again depending on the nature of the system parameters, the above solutions represent oscillatory or decaying or growing populations.

(ii) *Example 2: a predator–prey equation*

To demonstrate the theory for non-autonomous systems further, we consider another example that is a non-autonomous predator–prey equation,

$$\dot{x} = \alpha_1 x + \gamma_1 \mathrm{e}^{-\beta_1 t} xy = \phi_1, \qquad \dot{y} = \alpha_2 y + \gamma_2 \mathrm{e}^{-\beta_2 t} xy = \phi_2, \tag{4.37}$$

where α_i, β_i and γ_i, $i=1, 2$, are arbitrary parameters. Substituting the form of ϕ_i's, $i=1, 2$, into the determining equation (3.2) and solving the latter, we obtain the following form for S for the parametric choice $\beta_1=\beta_2=\beta$:

$$S_1 = -\frac{(\alpha_2-\beta)y + \gamma_2 \mathrm{e}^{-\beta t} xy}{(\alpha_1-\beta)x + \gamma_1 \mathrm{e}^{-\beta t} xy}. \tag{4.38}$$

Now inserting the above form of S_1 into equation (3.3) and solving the resultant equation, we obtain

$$K_1 = \frac{(\alpha_1-\beta) + \gamma_1 \mathrm{e}^{-\beta t} y}{y}. \tag{4.39}$$

Substituting the forms $R_1(=S_1K_1)$ and K_1 into equation (2.9) and evaluating the integrals, we obtain

$$I_1 = (\gamma_2 x - \gamma_1 y)\mathrm{e}^{-\beta t} + (\alpha_2-\beta)\log x - (\alpha_1-\beta)\log y - (\alpha_2-\alpha_1)\beta t. \tag{4.40}$$

Unfortunately we could not find a second integral in this case. However, for the further parametric restriction, namely $\alpha_1=\alpha_2=\alpha$, we obtain S_2 of the form

$$S_2 = -\frac{\gamma_2}{\gamma_1}. \tag{4.41}$$

Substituting S_2 into equation (3.3) and solving the resultant equation, we obtain

$$K_2 = \gamma_1 \mathrm{e}^{-\alpha t}. \tag{4.42}$$

Plugging the forms $R_2(=SK)$ and K_2 into equation (2.9), we obtain I_2 as

$$I_2 = (\gamma_2 x - \gamma_1 y)\mathrm{e}^{-\alpha t}. \tag{4.43}$$

From the integrals I_1 and I_2, the general solution for the equation (4.37) with $\alpha_1=\alpha_2=\alpha$ and $\beta_1=\beta_2=\beta$ can be fixed easily in the form

$$\left.\begin{aligned} x(t) &= \frac{I_2 \mathrm{e}^{\alpha t}}{\left(\gamma_2 - \gamma_1 \exp\left(\frac{(I_1 - I_2 \exp((\alpha-\beta)t))}{\beta-\alpha}\right)\right)}, \\ y(t) &= \frac{I_2\, \mathrm{e}^{\alpha t}}{\left(\gamma_2 \exp\left(\frac{(I_1 - I_2 \exp((\alpha-\beta)t))}{\alpha-\beta}\right) - \gamma_1\right)}. \end{aligned}\right\} \tag{4.44}$$

It is obvious that depending upon the signs and magnitudes of the parameters α and β, the general solution either decays or grows or saturates in the asymptotic limit.

5. PS procedure for three coupled first-order ODEs

Next, we focus our attention on a system of three coupled first-order ODEs of the form

$$\dot{x} = \frac{P_1(t, x, y, z)}{Q_1(t, x, y, z)}, \qquad \dot{y} = \frac{P_2(t, x, y, z)}{Q_2(t, x, y, z)}, \qquad \dot{z} = \frac{P_3(t, x, y, z)}{Q_3(t, x, y, z)}, \tag{5.1}$$

where P_i's and Q_i's, $i=1, 2, 3$, are analytic functions in x, y and z with coefficients in the field of complex numbers. Further, we assume that the ODE (5.1) admits a first integral $I(t, x, y, z)=C$, with C constant on the solutions so that the total differential becomes

$$\mathrm{d}I = I_t\,\mathrm{d}t + I_x\,\mathrm{d}x + I_y\,\mathrm{d}y + I_z\,\mathrm{d}z = 0. \tag{5.2}$$

Now let us rewrite the equation (5.1) in the form

$$\frac{P_1}{Q_1}\mathrm{d}t - \mathrm{d}x = 0, \qquad \frac{P_2}{Q_2}\mathrm{d}t - \mathrm{d}y = 0, \qquad \frac{P_3}{Q_3}\mathrm{d}t - \mathrm{d}z = 0. \tag{5.3}$$

Hence, on the solutions, the 1-forms (5.2) and (5.3) must be proportional. Multiplying the first, second and third equations in (5.3) by the functions $R(t, x, y, z)$, $K(t, x, y, z)$ and $M(t, x, y, z)$, respectively, which act as the integrating factors of the corresponding equations, we have on the solutions that

$$\mathrm{d}I = (R\phi_1 + K\phi_2 + M\phi_3)\mathrm{d}t - R\,\mathrm{d}x - K\,\mathrm{d}y - M\,\mathrm{d}z = 0, \tag{5.4}$$

where $\phi_i \equiv P_i/Q_i$, $i=1, 2, 3$. Comparing equations (5.4) and (5.2), we have, on the solutions, the relations

$$I_t = (R\phi_1 + K\phi_2 + M\phi_3), \qquad I_x = -R, \qquad I_y = -K, \qquad I_z = -M. \tag{5.5}$$

The compatibility conditions between the equations (5.5) provide us the following determining equations for the integrating factors R, K and M:

$$R_t + \phi_1 R_x + \phi_2 R_y + \phi_3 R_z = -(R\phi_{1x} + K\phi_{2x} + M\phi_{3x}), \tag{5.6}$$

$$K_t + \phi_1 K_x + \phi_2 K_y + \phi_3 K_z = -(R\phi_{1y} + K\phi_{2y} + M\phi_{3y}), \tag{5.7}$$

$$M_t + \phi_1 M_x + \phi_2 M_y + \phi_3 M_z = -(R\phi_{1z} + K\phi_{2z} + M\phi_{3z}), \tag{5.8}$$

$$R_y = K_x, \qquad R_z = M_x, \qquad K_z = M_y. \tag{5.9}$$

On the other hand integrating equations (5.5), we obtain the integral of motion,

$$I = r_1 + r_2 + r_3 - \int \left[M + \frac{\mathrm{d}}{\mathrm{d}z}(r_1 + r_2 + r_3) \right] \mathrm{d}z, \tag{5.10}$$

where

$$r_1 = \int \Big(R\phi_1 + K\phi_2 + M\phi_3 \Big) \mathrm{d}t, \quad r_2 = -\int \left(R + \frac{\mathrm{d}}{\mathrm{d}x}(r_1) \right) \mathrm{d}x,$$
$$r_3 = -\int \left(K + \frac{\mathrm{d}}{\mathrm{d}y}(r_1 + r_2) \right) \mathrm{d}y.$$

Naturally, for the complete integrability of equation (5.1), we require three independent integrals and so three independent sets of integrating factors (R_i, K_i, M_i), $i=1, 2, 3$.

6. Method of solving determining equations

The determining equations (5.6)–(5.9) are more complicated than the two-dimensional case discussed in §3 and so to simplify the determining equations, we introduce the transformations

$$R = SM \qquad \text{and} \qquad K = UM, \tag{6.1}$$

where S and U are functions of t, x, y and z, so that the equations (5.6)–(5.9) become

$$D[S] = S(S\phi_{1z} + U\phi_{2z} + \phi_{3z}) - (S\phi_{1x} + U\phi_{2x} + \phi_{3x}), \tag{6.2}$$

$$D[U] = U(S\phi_{1z} + U\phi_{2z} + \phi_{3z}) - (S\phi_{1y} + U\phi_{2y} + \phi_{3y}), \tag{6.3}$$

$$D[M] = -M(S\phi_{1z} + U\phi_{2z} + \phi_{3z}), \tag{6.4}$$

$$M_x = SM_z + MS_z, \quad M_y = UM_z + MU_z, \tag{6.5}$$

$$U_x - S_y = SU_z - US_z, \quad D = \frac{\partial}{\partial t} + \phi_1\frac{\partial}{\partial x} + \phi_2\frac{\partial}{\partial y} + \phi_3\frac{\partial}{\partial z}. \tag{6.6}$$

One may note that two of the determining equations are still in coupled form and the transformations are natural extensions of the two coupled case. Of course, one may also consider alternate possibilities, that is, either $R=\hat{S}K$ and $M=\hat{U}K$ or $K=\tilde{S}R$ and $M=\tilde{U}R$. However, such possibilities again lead to the same results.

In the two coupled case (2.1), we divided our analysis into two categories, vide §3*a* (time-independent integrals) and §3*b* (time-dependent integrals). However, in the present case, we divide our analysis into five categories, that is, (i) $I_x=0$ and I_t, I_y, $I_z\neq 0$, (ii) $I_y=0$ and I_t, I_x, $I_z\neq 0$, (iii) $I_z=0$ and I_t, I_x, $I_y\neq 0$, (iv) $I_t=0$ and I_x, I_y, $I_z\neq 0$ and (v) I_t, I_x, I_y, $I_z\neq 0$. We intend to proceed in this way because we observed the absence of a dynamical variable in some integrals in certain specific dynamical systems of the type (5.1). We try to identify these cases first. In fact, proceeding in this way, we are able to formulate a condition on the system variables and if the given system satisfies this condition, one can conclude that the given dynamical system has the integral without that respective variable. Since we have four variables, t, x, y and z, we consider each one of the cases separately and treat none of the variables being absent as the fifth independent case.

Since we are dealing with a system of coupled three first-order ODEs, the complete integrability is guaranteed by the presence of two time-independent integrals (whereupon the system can be reduced to a single quadrature) or three time-dependent ones (in which case, the solution can be obtained in an algebraic way; Bountis *et al.* 1984). In the following, we will search for such integrals.

(a) Case 1: $I_x=0$ and I_t, I_y, $I_z\neq0$

In the case $I_x=0$, we have $R=0$ (vide equation (5.5)), which in turn implies that (i) either $S=0$ and $M\neq0$ or (ii) $S\neq0$ and $M=0$ as can be seen from (6.1). In the former case, $S=0$, $M\neq0$, one can easily fix the form of U, from equations (6.2) and (6.6) as

$$U=-\frac{\phi_{3x}}{\phi_{2x}}, \qquad U_x=0. \tag{6.7}$$

On the other hand the choice $M=0$ and $S\neq0$ when $R=0$ leads to the case where one of the dynamical variables becomes uncoupled (see equations (5.6)–(5.9)), which effectively results in a system of two coupled first-order ODEs, which we have already discussed. So this choice is not considered further. Inserting the above form (6.7) into (6.3), we arrive at the condition

$$\begin{aligned}\phi_{3x}(\phi_{2xt}+\phi_2\phi_{2xy}+\phi_3\phi_{2xz}-\phi_{2z}\phi_{3x}-\phi_{2x}\phi_{2y})\\ -\phi_{2x}(\phi_{3xt}+\phi_2\phi_{3xy}+\phi_3\phi_{3xz}-\phi_{2x}\phi_{3y}-\phi_{3x}\phi_{3z})=0.\end{aligned} \tag{6.8}$$

The condition (6.8) gives us the integrable cases for which the system possesses the integrals of motion with $I_x=0$. Now substituting (6.7) into (6.4), we obtain the following determining equation for M:

$$D[M]=M\left(\frac{\phi_{3x}}{\phi_{2x}}\phi_{2z}-\phi_{3z}\right). \tag{6.9}$$

Again to solve equation (6.9) one has to make a suitable ansatz for M. Choosing appropriate ansatz for M and solving the equation (6.9), one can get an explicit form for M. Once M is known, the integrating factors can be fixed from the relations $K=UM$ and $R=SM=0$. Now plugging the forms of R, K and M into equation (5.10) and evaluating the integrals, one can construct the integrals of motion for the given system.

(b) Case 2: $I_y=0$ and I_t, I_x, $I_z\neq0$

The determining equations and conditions can be fixed in a similar manner for this case, $I_y=0$ and I_t, I_x, $I_z\neq0$, with the replacement $(S, U, M, \phi_1, \phi_2, \phi_3, x, y, z)\to(U, S, M, \phi_2, \phi_1, \phi_3, y, x, z)$ in the above analysis.

(c) Case 3: $I_z=0$ and I_t, I_x, $I_z\neq0$

In the present case with the form of the integrating factors $R=SM$ and $K=UM$, $I_z=0$ implies $M=0$ and so $R=0$ and $K=0$ as well, leading to an integral of motion, which turns out to be constant. Therefore in this case, we consider the other possibility $R=\hat{S}K$ and $M=\hat{U}K$ and proceed as above. The final results are obtained with the replacement $(S, U, M, \phi_1, \phi_2, \phi_3, x, y, z)\to(\hat{U}, \hat{S}, K, \phi_3, \phi_1, \phi_2, z, x, y)$ in case 1.

(d) Case 4: $I_t=0$ and I_x, I_y, $I_z\neq0$

Next, in the time-independent case $I_t=0$, the first equation in (5.5) gives

$$S=\frac{R}{M}=-\frac{(\phi_3+\phi_2 U)}{\phi_1}. \tag{6.10}$$

Substituting this form of S into (6.3) and (6.4), we get the following form of determining equations for U and M:

$$D[U] = \frac{\phi_3 + \phi_2 U}{\phi_1}(U\phi_{1y} - \phi_{1z}) + U(U\phi_{2z} + \phi_{3z} - \phi_{2y}) - \phi_{3y}, \tag{6.11}$$

$$D[M] = M\left(\frac{\phi_3 + \phi_2 U}{\phi_1}\phi_{1z} - \phi_{2z}U - \phi_{3z}\right). \tag{6.12}$$

To solve the equations (6.11) and (6.12) we adopt the following methodology.

To start with, in order to solve (6.11), we consider U in the form

$$U = \frac{A_1(x, y) + B_1(x, y)z}{A_2(x, y) + B_2(x, y)z}, \tag{6.13}$$

where A_i's and B_i's, $i=1, 2$, are arbitrary functions of x and y. Substituting (6.13) into (6.11) and equating the coefficients of different powers of z to zero, we get a set of determining equations for the functions A_i's and B_i's, $i=1, 2$. Solving these determining equations, we obtain explicit expressions of the functions A_i's and B_i's, $i=1, 2$, and consequently the associated function U.

Now substituting the forms of U into equation (6.12) and solving the resultant equation, we obtain the corresponding form of M. To solve the determining equation for M, we again seek the ansatz of the form $M = U_{\rm d}/((A(x, y) + B(x, y)z)^r)$ where $U_{\rm d}$ is the denominator of U. Once U and M are fixed, then one has to verify the compatibility of this set (S, U, M) with the constraint equations (6.5) and (6.6). Now substituting $R(=SK)$, $K(=UM)$ and M's into equation (5.10), one can construct the associated integrals. Finally, one can proceed with a more generalized ansatz than (6.13), if the need arises.

(*e*) *Case 5: I_t, I_x, I_y, $I_z \neq 0$*

Solving the determining equations (6.2)–(6.4) is naturally more tedious with none of the variables (t, x, y, z) absent in I, when compared with the earlier cases. To start with, one may use the following simple ansatz to solve the determining equations (5.6)–(5.8):

$$\left.\begin{aligned} R = A_1(t, x, y) + B_1(t, x, y)z, \quad K = A_2(t, x, y) + B_2(t, x, y)z,\\ M = A_3(t, x, y) + B_3(t, x, y)z, \end{aligned}\right\} \tag{6.14}$$

where A_i's and B_i's, $i=1, 2, 3$, are arbitrary functions of t, x and y. Depending on the nature of the equation (5.1), one may work with more general forms such as a rational one.

7. Three coupled ODEs: applications

(*a*) *Example: Rössler system*

Let us consider the Rössler (1976) system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y - z = \phi_1, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x + \alpha_1 y = \phi_2, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \alpha_2 + xz + \alpha_3 z = \phi_3, \tag{7.1}$$

where α_i's, $i=1, 2, 3$, are arbitrary parameters. Several works have been devoted to the study of the dynamics of this equation. Very recently, Llibre & Zhang (2002) and Zhang (2004) have studied equation (7.1) using the so-called Darboux method and obtained conditions for integrability. We now apply our above method to system (7.1) and explore new integrals, if they exist.

(i) *$I_x=0$ and I_t, I_y, $I_z\neq 0$*

Substituting (7.1) into (6.8), we get

$$z\alpha_1 + \alpha_2 = 0. \tag{7.2}$$

From equation (7.2) we conclude that $\alpha_1=\alpha_2=0$, so that from (6.7) we get $U=-z$. The determining equation for M turns out to be

$$M_t + xM_y + z(x+\alpha_3)M_z = -M(x+\alpha_3), \tag{7.3}$$

in which we have taken $M_x=0$ (since $I_x=0$). A simple solution for (7.3) is $M=1/z$. Making use of the explicit forms of U and M and with the parametric restriction $\alpha_1=\alpha_2=0$, we conclude that $R=0$, $K=-1$, $M=1/z$. Now substituting the functions R, K and M into equation (5.10) and evaluating the integrals, we obtain the following integral of motion:

$$I = y + \alpha_3 t - \log(z). \tag{7.4}$$

The integral (7.4) with $\alpha_3=0$ has already been given by Llibre & Zhang (2002) and Zhang (2004). The integral (7.4) with $\alpha_3\neq 0$ is *new to the literature*, at least to our knowledge.

(ii) *$I_y=0$, I_t, I_x, $I_z\neq 0$ and $I_z=0$, I_t, I_x, $I_y\neq 0$*

Proceeding appropriately we could not find any integrable case in the Rössler system belonging to these categories.

(iii) *$I_t=0$ and I_x, I_y, $I_z\neq 0$*

In this case the function S can be fixed in the form (vide equation (6.10))

$$S = \frac{\alpha_2 + (x+\alpha_3)z + (x+\alpha_1 y)U}{y+z}. \tag{7.5}$$

Substituting (7.1) into (6.11), we get

$$D[U] = \frac{\alpha_2 + (x+\alpha_3)z + (x+\alpha_1 y)U}{y+z}(U-1) + (x+\alpha_3-\alpha_1)U. \tag{7.6}$$

Substituting (6.13) into (7.6) and solving the resultant equations, we obtain non-trivial forms of U for the specific parametric restriction $\alpha_1=\alpha_2=\alpha_3=0$ and then making use of the U forms into (7.5) we obtain $(S_1, U_1)=(x, y)$, $(S_2, U_2)=(0, -z)$. Now substituting the forms of U into equation (6.12) and solving the resultant equation, we obtain $M_1=-1$ and $M_2=-\mathrm{e}^{-y}$. Now inserting the integrating factors R_i's$(=S_iM_i)$, K_i's$(=U_iM_i)$ and M_i's, $i=1, 2$, into (5.10) and evaluating the integrals, we arrive at the expressions

$$I_1 = x^2 + y^2 + 2z, \qquad I_2 = z\mathrm{e}^{-y}. \tag{7.7}$$

These two integrals have already been known (Llibre & Zhang 2002; Zhang 2004).

(iv) I_t, I_x, I_y, $I_z \neq 0$

In this case, substituting ϕ_i's, $i=1, 2, 3$, into (5.6)–(5.8) and solving the resultant system of equations with the ansatz (6.14), we obtain the integrating factors for the parametric choice $\alpha_1=\alpha_3=0$, that is, $R=-x$, $K=-y$, $M=-1$ and the corresponding integral of motion takes the form

$$I = (x^2 + y^2 + 2z - 2\alpha_2 t). \tag{7.8}$$

The integral (7.8) has also been reported by Zhang (2004).

We conclude this section by mentioning that our studies reveal that the system (7.1) possesses a time-dependent integral for the parametric choice $\alpha_3 \neq 0$.

(*b*) *Example 2: three-dimensional LV system*

Let us consider a three-dimensional LV model for competition between three populations whose dynamical evolution is determined by the following equations (Cairo 2000):

$$\left.\begin{aligned} \dot{x} = x(\alpha_1 + a_1 x + b_1 y + c_1 z), \quad \dot{y} = y(\alpha_2 + a_2 x + b_2 y + c_2 z), \\ \dot{z} = z(\alpha_3 + a_3 x + b_3 y + c_3 z), \end{aligned}\right\} \tag{7.9}$$

where α_i, a_i, b_i and c_i, $i=1, 2, 3$, are arbitrary parameters. Needless to say, the three-dimensional LV system is one of the challenging problems and a testing ground for several analytical methods. In the following, we identify the integrals of motion for certain specific parametric choices in (7.9) using the procedure given above. For the purpose of demonstration, in the following we present our results only for a couple of cases and a detailed analysis will be presented separately.

(i) *Case 1*

For the parametric choice $a_i=a$, $b_i=b$, and $c_i=c$, $i=1, 2, 3$, we find the following three complete sets of integrating factors (R_i, K_i, M_i), $i=1, 2, 3$:

$$\left.\begin{aligned} R_1 &= 0, \quad K_1 = \frac{\exp((\alpha_3-\alpha_2)t)z}{y^2}, \quad M_1 = -\frac{\exp((\alpha_3-\alpha_2)t)}{y}, \\ R_2 &= \frac{\exp((\alpha_3-\alpha_1)t)z}{x^2}, \quad K_2 = 0, \quad M_2 = -\frac{\exp((\alpha_3-\alpha_1)t)}{x}, \\ R_3 &= -\frac{e^{\alpha_1 t} h(y,z)}{x^2}, \quad K_3 = \alpha_3 \frac{e^{\alpha_1 t}}{ax}, \quad M_3 = \alpha_2 \frac{e^{\alpha_1 t}}{ax}, \end{aligned}\right\} \tag{7.10}$$

where $h(y, z)=(\alpha_2\alpha_3+b\alpha_3 y+c\alpha_2 z)$. Now substituting the functions R, K and M into equation (5.10), one can obtain the following integrals of motion:

$$\left.\begin{aligned} I_1 &= \frac{\exp((\alpha_2-\alpha_3)t)z}{y}, \quad I_2 = \frac{\exp((\alpha_1-\alpha_3)t)z}{x}, \\ I_3 &= \frac{e^{\alpha_1 t}(\alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3 a x + \alpha_1\alpha_3 b y + \alpha_1\alpha_2 c z)}{x}. \end{aligned}\right\} \tag{7.11}$$

From the integrals I_1, I_2 and I_3, we can deduce the general solution for the equation (7.9) for the parametric choice $a_i=a$, $b_i=b$, and $c_i=c$, $i=1, 2, 3$, of the form

$$\left.\begin{aligned} x(t) &= \frac{\alpha_1\alpha_2\alpha_3 I_1 \mathrm{e}^{\alpha_1 t}}{I_3 I_1 - (a\alpha_2\alpha_3 I_1 \mathrm{e}^{\alpha_1 t} + b\alpha_1\alpha_3 I_2 \mathrm{e}^{\alpha_2 t} + c\alpha_1\alpha_2 I_1 I_2 \mathrm{e}^{\alpha_3 t})}, \\ y(t) &= \frac{\alpha_1\alpha_2\alpha_3 I_2 \mathrm{e}^{\alpha_2 t}}{I_3 I_1 - (a\alpha_2\alpha_3 I_1 \mathrm{e}^{\alpha_1 t} + b\alpha_1\alpha_3 I_2 \mathrm{e}^{\alpha_2 t} + c\alpha_1\alpha_2 I_1 I_2 \mathrm{e}^{\alpha_3 t})}, \\ z(t) &= \frac{\alpha_1\alpha_2\alpha_3 I_1 I_2 \mathrm{e}^{\alpha_3 t}}{I_3 I_1 - (a\alpha_2\alpha_3 I_1 \mathrm{e}^{\alpha_1 t} + b\alpha_1\alpha_3 I_2 \mathrm{e}^{\alpha_2 t} + c\alpha_1\alpha_2 I_1 I_2 \mathrm{e}^{\alpha_3 t})}. \end{aligned}\right\} \tag{7.12}$$

For the above choice of parameter, the system (7.9) is a completely integrable one.

(ii) *Case 2*

For the parametric choice $\alpha_i=\alpha$, $i=1, 2, 3$, $a_3/3=-a_2/2=a_1$, $c_1=-c_3$, $b_1=-b_3$ and $c_2=b_2=0$, we obtain two complete sets of integrating factors of the form

$$\left.\begin{aligned} &R_1 = \mathrm{e}^{-4\alpha t} y^2 z, \quad K_1 = 2\mathrm{e}^{-4\alpha t} xzy, \quad M_1 = \mathrm{e}^{-4\alpha t} xy^2, \\ &R_2 = a_1\alpha y\left(6a_1 x - 3c_1 z + 2b_1\left(y - \frac{c_1}{\alpha} z\right)\right)\mathrm{e}^{-3\alpha t}, \\ &K_2 = \alpha\left(3a_1^2 x^2 + b_1^2 y^2 + a_1 x(4b_1 y - c_1 z\left(3 + \frac{4}{\alpha} b_1 y\right)\right)\mathrm{e}^{-3\alpha t}, \\ &M_2 = -a_1 c_1 xy(3\alpha + 2b_1 y)\mathrm{e}^{-3\alpha t}. \end{aligned}\right\} \tag{7.13}$$

Now substituting the integrating factors R, K and M into equation (5.10), one can obtain the corresponding integrals of motion, namely

$$\begin{aligned} I_1 &= \mathrm{e}^{-4\alpha t} xy^2 z, \\ I_2 &= \mathrm{e}^{-3\alpha t}\left(\alpha a_1 xy\left(3c_1 z - 3a_1 x - 2b_1 y + \frac{2}{\alpha} b_1 c_1 yz\right) - \frac{1}{3} b_1^2 y^3\right). \end{aligned} \tag{7.14}$$

We could not find the third integrating factor (R_3, K_3, M_3) within our ansatz, (6.14), and further detailed exploration is needed to conclude whether it exists or not, and to find the third integral if it exists.

8. PS procedure for n (greater than 3) coupled first-order ODEs

The above procedure to find integrating factors and integrals can be extended in principle to a system of n coupled first-order ODEs ($n>3$). In this case, we get n determining equations for the n integrating factors along with $n(n-1)/2$ constraint equations, which one can solve algorithmically by following the above procedure. Further, the same procedure can also be applied in principle to any higher order as well as coupled higher order equations. This is because, any higher order equation can always be rewritten equivalently as a system of first-order ODEs. For example, the second-order equation $\ddot{x}=f(\dot{x}, x, t)$ can be written in the first-order form as $\dot{x}=y$, $\dot{y}=f(y, x, t)$. Then the determining equations for this first-order form, namely equations (3.2)–(3.4), can be related to the determining equations (2.6)–(2.8) given by us earlier (Chandrasekar *et al.* 2005) for the second-order form $\ddot{x}=f(\dot{x}, x, t)$. However, one has to note that each one of the procedures has its own merits and demerits, for example in constructing non-standard Hamiltonian structures. Similarly, a straightforward first-order form for the third-order equation $\dddot{x}=f(\ddot{x}, \dot{x}, x, t)$ is $\dot{x}=y$, $\dot{y}=z$, $\dot{z}=f(x, y, z, t)$. In this case as well the determining equations for the first-order form can be related to that of the third-order ODEs as given by us earlier (Chandrasekar *et al.* 2006). Again analysing third-order equations as such has its own practical advantages. This analogy can also be extended to coupled higher order equations as well in principle, though in the actual analysis one may have to make a judicious choice of which one of the methods is advantageous for investigating the integrability aspects.

9. Linearization

In this section, we describe a procedure to deduce the linearizing transformations from the known integrals and illustrate the theory with an example.

Let us assume that equation (2.1) admits the following integral:

$$I = F(x, y, t). \tag{9.1}$$

Now let us split the function F_1 in the form

$$I = F_1\left(\frac{1}{G_2(t, x, y)}\frac{\mathrm{d}}{\mathrm{d}t}G_1(t, x, y)\right). \tag{9.2}$$

Now we identify the function G_1 as the new dependent variable and the integral of G_2 over time as the new independent variable, that is,

$$w = G_1(t, x, y), \qquad \tau = \int_0^t G_2(t', x, y)\mathrm{d}t'. \tag{9.3}$$

We note here that the integration on the right-hand side of (9.3) leading to τ can be performed provided the function G_2 is an exact derivative of t, that is, $G_2=(\mathrm{d}/\mathrm{d}t)\tau(t, x, y)=\dot{x}\tau_x+\dot{y}\tau_y+\tau_t$. In terms of the new variables, equation (9.2) can be modified to the form

$$I = F_1\left(\frac{\mathrm{d}w}{\mathrm{d}\tau}\right). \tag{9.4}$$

Inverting the relation (9.4) suitably, one can obtain a linear equation,

$$\frac{\mathrm{d}w}{\mathrm{d}\tau}=\hat{I}, \tag{9.5}$$

where $\hat{I}$ is a constant. Or equivalently

$$\frac{\mathrm{d}w}{\mathrm{d}\tau}=u=\hat{I}, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau}=0. \tag{9.6}$$

Equation (9.6) is the corresponding linear equation of (2.1). From equations (9.3) and (9.6), we have the following linearizing transformation for (2.1):

$$w = \hat{G}_1(t, x, y), \quad u = F(x, y, t), \quad \tau = \int_0^t \hat{G}_2(t', x, y)\mathrm{d}t'. \tag{9.7}$$

In this case, the new variables w, u and τ help us to transform the given system of coupled first-order nonlinear ODEs into a linear system of coupled first-order ODEs, which in turn leads to the solution by trivial integration. The above procedure can also be extended to more than two coupled first-order ODEs straightforwardly and we do not present the details here.

(a) *Example 1*

To illustrate the underlying ideas, let us consider the two-dimensional LV system (4.1) with the specific parametric choice, $b_{11}=b_{21}$ and $b_{12}=b_{22}$, namely

$$\dot{x} = x(a_1 + b_{11}x + b_{22}y), \qquad \dot{y} = y(a_2 + b_{11}x + b_{22}y). \tag{9.8}$$

Let us consider the following first integral for equation (9.8), namely

$$I = \frac{y}{x}\exp((a_1 - a_2)t). \tag{9.9}$$

Now rewriting equation (9.9) using (9.8) in the form (9.2), we get

$$I = -\frac{1}{b_{22}}\mathrm{e}^{-a_2 t}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{1}{x}+\frac{b_{11}}{a_1}\right)\mathrm{e}^{a_1 t}\right]. \tag{9.10}$$

Then

$$w = \left(\frac{1}{x}+\frac{b_{11}}{a_1}\right)\mathrm{e}^{a_1 t}, \qquad \tau = -\frac{b_{22}}{a_2}\mathrm{e}^{a_2 t}. \tag{9.11}$$

From equations (9.11), (9.9) and (9.6), we have the following linearizing transformation for (9.8):

$$w = \left(\frac{1}{x}+\frac{b_{11}}{a_1}\right)\mathrm{e}^{a_1 t}, \qquad u = \frac{y}{x}\exp((a_1 - a_2)t), \qquad \tau = -\frac{b_{22}}{a_2}\mathrm{e}^{a_2 t}. \tag{9.12}$$

In this case the new variables w, u and τ help us to transform the given system of coupled first-order nonlinear ODE, (9.8), into a linear system of coupled first-order ODEs of the form (9.6). The general solution can then be straightforwardly deduced.

(b) *Example 2*

Similarly, for the specific parametric choice $a_1=a_2=0$, $b_{12}=-b_{22}$, and $b_{21}=3b_{11}$, equation (4.1), that is,

$$\dot{x} = x(b_{11}x + b_{12}y), \qquad \dot{y} = y(3b_{11}x - b_{12}y), \tag{9.13}$$

can be transformed to linear equation of the form (9.6) by the following linearizing transformations (for the integral see equation (4.20)):

$$w = \frac{1}{2}t^2 - \frac{(b_{12}y + b_{11}x)}{2b_{11}x(b_{12}y - b_{11}x)^2}, \qquad u = t + \frac{1}{b_{11}x - b_{12}y}, \qquad \tau = t. \tag{9.14}$$

10. Conclusion

In this paper, we have modified the PS procedure such that it is applicable to both autonomous as well as non-autonomous systems of coupled first-order ODEs. We have also developed systematic procedures for finding both time-independent and time-dependent integrals for them. From this analysis, we have answered the following open questions. (i) How can the PS method be extended to a non-autonomous system of coupled first-order ODEs? (ii) How to find the second or the time-dependent integrals for the given coupled first-order ODEs? (iii) How can this procedure be generalized to higher dimensions in order to find first integrals? We have also shown that the determining equations for the time-independent integral in the two coupled equations of our method coincide with the determining equations derived by Prelle and Singer in their original paper. We have illustrated this procedure with physically interesting examples, namely the two-dimensional LV system, Rössler system and three-dimensional LV system and identified several integrable cases. Further, we have developed a linearization procedure for coupled first-order ODEs. Finally, we note that the procedures that we have developed in this paper, namely both the extended PS procedure and linearization, can also be extended to any number of coupled first-order ODEs.

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