Exact soliton solutions, shape changing collisions and partially coherent solitons in coupled nonlinear Schrödinger equations

T. Kanna and M. Lakshmanan*

Centre for Nonlinear Dynamics, Department of Physics, Bharathidasan University, Tiruchirapalli 620 024, India

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We present the exact bright one-soliton and two-soliton solutions of the integrable three coupled nonlinear Schrödinger equations (3-CNLS) by using the Hirota method, and then obtain them for the general N-coupled nonlinear Schrödinger equations (N-CNLS). It is pointed out that the underlying solitons undergo inelastic (shape changing) collisions due to intensity redistribution among the modes. We also analyse the various possibilities and conditions for such collisions to occur. Further, we report the significant fact that the various partial coherent solitons (PCS) discussed in the literature are special cases of the higher order bright soliton solutions of the N-CNLS equations.

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In recent years the concept of soliton has been receiving considerable attention in optical communications since soliton is capable of propagating over long distances without change of shape and with negligible attenuation [1-3]. It has been found that soliton propagation through optical fiber arrays is governed by a set of equations related to the CNLS equations[1,2],

$$iq_{jz} + q_{jtt} + 2\mu \sum_{p=1}^{N} |q_p|^2 q_j = 0, \quad j = 1, 2, ...N, \qquad (1)$$

where q_j is the envelope in the jth core, z and t represent the normalized distance along the fiber and the retarded time, respectively. Here 2μ gives the strength of the nonlinearity. Eq. (1) reduces to the standard envelope soliton possessing integrable nonlinear Schrödinger equation for $N = 1$. For $N = 2$, the above Eq. (1) governs the integrable Manakov system [4] and recently for this case the exact two-soliton solution has been obtained and novel shape changing inelastic collision property has been brought out [5]. However, the results are scarce for $N \geq 3$, even though the underlying systems are of considerable physical interest. For example, in addition to optical communication, in the context of biophysics the case $N = 3$ can be used to study the launching and propagation of solitons along the three spines of an alpha-helix in protein[6]. Similarly the CNLS Eq. (1) and its generalizations for $N\geq 3$ are of physical relevance in the theory of soliton wavelength division multiplexing[7], multi-channel bit-parallel-wavelength optical fiber networks[8] and so on. In particular, for arbitrary N, Eq. (1) governs the propagation of N-self trapped mutually incoherent wavepackets in Kerr-like photorefractive media[9] in which q_j is the jth component of the beam, z and t represents the normalized coordinates along the direction of propagation and the transverse coordinate, respectively, and $\sum_{p=1}^{N} |q_p|^2$ represents the change in the refractive index profile created by all incoherent components of the light beam [9] when the medium response is slow.The parameter $\mu = k_0^2 n_2/2$, where n_2 is the nonlinear Kerr coefficient and k_0 is the free space wave vector.

In this letter, we report the exact bright one and two soliton solutions, first for the $N = 3$ case and then for the arbitrary N case, where the procedure can be extended in principle to higher order soliton solutions, using the Hirota bilinearization method. In particular, we point out that the shape changing inelastic collision property persists for the $N \geq 3$ cases also as in the $N = 2$ (Manakov) case reported recently [5], giving rise to many possibilities of energy exchange. Furthermore, we point out that in the context of spatial solitons the partially coherent stationary solitons(PCS) reported in the recent literature[9-10] are special cases of the above general soliton solutions which undergo shape changing collisions.

The bright one-soliton and two-soliton solutions of the 3-CNLS system,

$$iq_{jz} + q_{jtt} + 2\mu(|q_1|^2 + |q_2|^2 + |q_3|^2)q_j = 0, \; j = 1, 2, 3, \quad (2)$$

can be obtained from its equivalent bilinear form resulting from the transformation $q_j = g^{(j)}/f$,

$$(iD_z + D_t^2)g^{(j)}.f = 0, \;\; D_t^2 f.f = 2\mu \sum_{n=1}^{3} g^{(n)} g^{(n)*}, \qquad (3)$$

where $*$ denotes the complex conjugate, D_z and D_t are Hirota's bilinear operators [11], and $g^{(j)}$'s are complex functions, while $f(z,t)$ is a real function.The resulting set of Eqs. (3) can be solved recursively by making the power series expansion $g^{(j)} = \chi g_1^{(j)} + \chi^3 g_3^{(j)} + \ldots$, $f = 1 + \chi^2 f_2 + \chi^4 f_4 + \ldots$, $j = 1, 2, 3$, where χ is a formal expansion parameter. In order to get the one-soliton solution, the power series expansions are terminated as $g^{(j)} = \chi g_1^{(j)}$, and $f = 1 + \chi^2 f_2$. After deducing $g^{(j)}$ and f as $g^{(j)} = \alpha_1^{(j)} e^{\eta_1}$, $j = 1, 2, 3$ and $f = 1 + e^{\eta_1 + \eta_1^* + R}$, where $e^R = \mu \sum_{j=1}^{3} |\alpha_1^{(j)}|^2/(k_1 + k_1^*)^2$, the bright one-soliton solution is obtained as

$$(q_1, q_2, q_3)^T = \frac{e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + R}} \left(\alpha_1^{(1)}, \alpha_1^{(2)}, \alpha_1^{(3)}\right)^T,$$

$$= \frac{k_{1R} e^{i\eta_{1I}}}{\cosh\left(\eta_{1R} + \frac{R}{2}\right)} \left(A_1, A_2, A_3\right)^T , \qquad (4)$$

where $\eta_1 = k_1(t + ik_1 z)$, $A_j = \alpha_1^{(j)}/\Delta$ and $\Delta = (\mu(\sum_{j=1}^{3} |\alpha_1^{(j)}|^2))^{1/2}$. Here $\alpha_1^{(j)}$, k_1, $j = 1, 2, 3$, are four arbitrary complex parameters. Further $k_{1R} A_j$ gives the amplitude of the jth mode and $2k_{1I}$ the soliton velocity.

The general bright two-soliton solution of Eq. (2) can be generated by terminating the series as $g^{(j)} = \chi g_1^{(j)} + \chi^3 g_3^{(j)}$ and $f = 1 + \chi^2 f_2 + \chi^4 f_4$ and solving the resultant linear partial differential equations. This solution contains eight arbitrary complex parameters, $\alpha_l^{(j)}$ and k_l, $l = 1, 2$ and $j = 1, 2, 3$. It is given by

$$q_j = (\alpha_1^{(j)} e^{\eta_1} + \alpha_2^{(j)} e^{\eta_2} + e^{\eta_1 + \eta_1^* + \eta_2 + \delta_{1j}} + e^{\eta_1 + \eta_2 + \eta_2^* + \delta_{2j}})/D, \; j = 1, 2, 3, \qquad (5)$$

where $D = 1 + e^{\eta_1 + \eta_1^* + R_1} + e^{\eta_1 + \eta_2^* + \delta_0} + e^{\eta_1^* + \eta_2 + \delta_0^*} + e^{\eta_2 + \eta_2^* + R_2} + e^{\eta_1 + \eta_1^* + \eta_2 + \eta_2^* + R_3}$, $\eta_i = k_i(t + ik_i z)$, $e^{R_1} = \kappa_{11}/(k_1 + k_1^*)$, $e^{R_2} = \kappa_{22}/(k_2 + k_2^*)$, $e^{\delta_0} = \kappa_{12}/(k_1 + k_2^*)$, $e^{\delta_{1j}} = ((k_1 - k_2)(\alpha_1^{(j)} \kappa_{21} - \alpha_2^{(j)} \kappa_{11}))/((k_1 + k_1^*)(k_1^* + k_2))$, $e^{\delta_{2j}} = ((k_2 - k_1)(\alpha_2^{(j)} \kappa_{12} - \alpha_1^{(j)} \kappa_{22}))/((k_2 + k_2^*)(k_1 + k_2^*))$, $e^{R_3} = (|k_1 - k_2|^2 (\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}))/((k_1 + k_1^*)(k_2 + k_2^*)|k_1 + k_2^*|^2)$ and $\kappa_{il} = \mu \sum_{n=1}^{3} \alpha_i^{(n)} \alpha_l^{(n)*}/(k_i + k_l^*)$, $i, \; l = 1, 2$, $j = 1, 2, 3$. Though one can proceed to obtain higher order soliton solutions in principle by making use of the general power series expansion, the details become complicated and we will present the results separately.

The nature of the interaction of the underlying solitons can be well understood by making an asymptotic analysis of the two-soliton solution[5]. Asymptotically, the two-soliton solution(5) can be written as a combination of two one-soliton solutions and their forms in the two different regimes $z \to -\infty$ and $z \to \infty$ are similar to those of the one-soliton solution given in Eq. (4) but differing in amplitude (intensity) and phase. The analysis reveals that there is an intensity exchange among the three components of each soliton during this two-soliton interaction, which can be quantified by defining a transition matrix T_j^l such that $A_j^{l+} = A_j^{l-} T_j^l$, $j = 1, 2, 3$ and $l = 1, 2$, where the superscripts $l\pm$ represent the solitons designated as S_1 and S_2 at $z \to \pm\infty$, and $k_{lR} A_j^{l\pm}$ denote the corresponding amplitudes.

Consequently, the three modes q_1, q_2 and q_3 of S_1 having magnitudes of amplitudes $|A_j^{1-}| k_{1R} = |\alpha_1^{(j)}| k_{1R}/\Delta_1$, where $\Delta_1 = (\mu(\sum_{j=1}^{3} |\alpha_1^{(j)}|^2))^{1/2}$, exchange intensity given by the square of the transition matrices, $|T_j^1|^2 = |1 - \lambda_2(\alpha_2^{(j)}/\alpha_1^{(j)})|^2/|1 - \lambda_1\lambda_2|$, $j = 1, 2, 3$, along with a phase shift $\Phi^1 = (R_3 - R_2 - R_1)/2$ during collision. Here $\lambda_1 = \kappa_{21}/\kappa_{11}$ and $\lambda_2 = \kappa_{12}/\kappa_{22}$. In a similar fashion due to collision the three modes q_1, q_2 and q_3 of S_2 also exchange an amount of intensity, $|T_j^2|^2 = |1 - \lambda_1\lambda_2|/|1 - \lambda_1(\alpha_1^{(j)}/\alpha_2^{(j)})|^2$, $j = 1, 2, 3$, respectively and change their amplitudes to $|A_j^{2+}| k_{2R} = |\alpha_2^{(j)}| k_{2R}/\Delta_2$ from $|A_j^{2-}| k_{2R}$, respectively. Here $\Delta_2 = (\mu(\sum_{j=1}^{3} |\alpha_2^{(j)}|^2))^{1/2}$. The associated phase shift for this soliton is $\Phi^2 = -(R_3 - R_2 - R_1)/2$. We also note there is a net change in the relative separation distance between the solitons due to collision by $\Delta x_{12} = (k_{1R} + k_{2R})|(R_3 - R_2 - R_1)|/2k_{1R}k_{2R}$.

Also, we note that for the special case $|T_j^l| = 1$, $l = 1, 2$, $j = 1, 2, 3$, which is possible only when $\alpha_1^{(1)}/\alpha_2^{(1)} = \alpha_1^{(2)}/\alpha_2^{(2)} = \alpha_1^{(3)}/\alpha_2^{(3)}$, the collision corresponds to the standard elastic collision. For all other cases, the quantity $|T_j^l| \neq 1$, which corresponds to a change in the values of the amplitudes of the individual modes leading to a redistribution of the intensities among them and corresponding to a change in the shape of the soliton. However, during the interaction the total intensity of the individual solitons S_1 and S_2 remains conserved, that is $|A_1^{l\mp}|^2 + |A_2^{l\mp}|^2 + |A_3^{l\mp}|^2 = 1/\mu$.

The above shape changing (inelastic) collision during the two-soliton interaction of the 3-CNLS can occur in two different ways. The first case is an enhancement of intensity in anyone of the modes of either one of the solitons (say S_1) and suppression in the remaining two modes of the corresponding soliton with commensurate changes in the other soliton S_2. The other possibility is an interaction which allows one of the modes of either one of the solitons (say S_1) to get suppressed while the other two modes of the corresponding soliton to get enhanced (with corresponding changes in S_2). In either of the cases, the intensity may be completely or partially suppressed(enhanced). Thus as a whole during the inelastic interaction among the two one solitons S_1 and S_2 of the 3-CNLS, the soliton S_1 (S_2) has the following six possible combinations to exchange the intensity among its modes: $(q_1, q_2, q_3) \to (q_1^a, q_2^b, q_3^c)_i, (a, b, c = S\,(\text{suppression}), E\,(\text{enhancement}))$ with $i = 1$, $a = E$, $b = S$, $c = S$; $i = 2$, $a = S$, $b = E$, $c = S$; $i = 3$, $a = S$, $b = S$, $c = E$; $i = 4$, $a = S$, $b = E$, $c = E$; $i = 5$, $a = E$, $b = S$, $c = E$ and $i = 6$, $a = E$, $b = E$, $c = S$.

Two of the above interactions involving a dramatic switching in the intensity are depicted in Fig. 1 for illustrative purpose for specific choice of soliton parameters. These may also be viewed as the two-soliton interaction in a waveguide supporting propagation of three nonlinear waves simultaneously. For other choices, in general, partial suppression(enhancement) of intensity among the components will occur depending on the values of the transition matrix elements T_j^l. Fig. 1a is plotted for the parameters $k_1 = 1 + i$, $k_2 = 2 - i$, $\alpha_2^{(1)} = \alpha_2^{(2)} = (39 + i80)/89$, $\alpha_1^{(1)} = \alpha_1^{(2)} = \alpha_1^{(3)} = \alpha_2^{(3)} = 1$ and $\mu = 1$. In this figure the intensities of the components q_1 and q_2 of S_1 (S_2) are almost completely suppressed (enhanced) and that of the third component is enhanced (suppressed). The second possibility of enhancement (suppression) of intensity in the q_1 and q_2 components of $S_1(S_2)$ and suppression(enhancement) of intensity in its

q_3 component are shown in Fig. 1b, in which the parameters are chosen as $k_1 = 1+i$, $k_2 = 2-i$, $\alpha_1^{(1)} = 0.02+0.1i$, $\alpha_1^{(2)} = 0.1i$, $\alpha_1^{(3)} = \alpha_2^{(1)} = \alpha_2^{(2)} = \alpha_2^{(3)} = 1$.

Now it is straight forward to extend the above analysis to obtain the one-soliton and two-soliton solutions of the arbitrary N-CNLS Eq. (1). After making the bilinear transformation $q_j = g^{(j)}/f$, $j = 1, 2, 3, \ldots, N$ in Eq. (1), one can get a set of bilinear equations of the form (3) but now with $j,\ n = 1, 2, 3, \ldots, N$. Then by expanding $g^{(j)}$s and f in power series up to Nterms and following the procedure mentioned above, the one-soliton and two-soliton solutions of Eq. (1) can be obtained.

(a) One-soliton solution:

$$(q_1, q_2, \ldots, q_N)^T = \frac{k_{1R} e^{i\eta_{1I}}}{\cosh\left(\eta_{1R} + \frac{R}{2}\right)} \left(A_1, A_2, \ldots, A_N\right)^T, \quad (6)$$

where $\eta_1 = k_1(t + ik_1 z)$, $A_j = \alpha_1^{(j)}/\Delta$, $\Delta = (\mu(\sum_{j=1}^{N} |\alpha_1^{(j)}|^2))^{1/2}$, $e^R = \Delta^2/(k_1 + k_1^*)^2$, $\alpha_1^{(j)}$ and k_1, j=1, 2 . . . , N, are $(N + 1)$ arbitrary complex parameters.

(b) Two-soliton solution: This solution will also be of the same form as Eq.(5) with the replacements, $j = 1, 2, \ldots, N$ and $\kappa_{il} = \mu \sum_{n=1}^{N} \alpha_i^{(n)} \alpha_l^{(n)*}/(k_i + k_l^*)$, where $i,\ l = 1, 2$. One can also verify that this two-soliton solution will depend on $2(N+1)$ complex parameters and the shape changing interaction can lead to intensity redistribution among the modes of the soliton of the N-CNLS system in $2^N - 2$ ways (by generalizing the $N = 3$ case). We believe that such studies will have important applications in logic gates and all optical computations[12].

From an application point of view, it has been observed recently that the CNLS equations can support a kind of stationary solutions known as partially coherent solitons (PCS). In particular, explicit form of such solutions have been given for $N = 2, 3$ and 4 cases of Eq. (1) in Ref.[9]. They have also been shown to have variable shapes. Now, the generalized Manakov equation(1) is integrable[13] and hence its N-soliton solution can be obtained in principle by extending our above analysis. So the natural question arises as to what is the relation between the PCS and the exact N-soliton solutions.

To answer the above question, let us look at the $N = 2, 3$ and 4, cases of Eq. (1) explicitly. One can check that very special cases corresponding to specific parametric restrictions in the two-soliton solution of the $N = 2$ case, the three-soliton solution of the $N = 3$ case and four-soliton solution of the $N = 4$ case give rise to the 2-soliton, 3-soliton and the 4-soliton PCSs, respectively, reported in [9]. In order to appreciate this we consider as an illustration the three-soliton solution of the $N = 3$ case of Eq. (1). Instead of writing down the full 3-soliton solution of the $N = 3$ case explicitly and choosing the special parametric values, we make the following simplified procedure. Starting from the bilinear Eqs. (3) and terminating the series for $g^{(j)}$ and f as $g^{(j)} = \chi g_1^{(j)} + \chi^3 g_3^{(j)} + \chi^5 g_5^{(j)}$ and $f = 1 + \chi^2 f_2 + \chi^4 f_4 + \chi^6 f_6$, one can identify $g_1^{(j)} = \alpha_1^{(j)} e^{\eta_1} + \alpha_2^{(j)} e^{\eta_2} + \alpha_3^{(j)} e^{\eta_3}$ where $\eta_n = k_n(t + ik_n z)$, $j,\ n = 1, 2, 3$ in which $\alpha_i^{(j)}$ and k_i are complex parameters. Finding $g_3^{(j)}$, $g_5^{(j)}$, $j = 1, 2, 3$, f_2, f_4 and f_6, the three soliton solution is obtained. Instead, as a special case, we look for a stationary solution with $k_{nI} = 0$, $\alpha_2^{(1)} = \alpha_3^{(1)} = \alpha_1^{(2)} = \alpha_3^{(2)} = \alpha_1^{(3)} = \alpha_2^{(3)} = 0$ and $\alpha_1^{(1)} = -\alpha_2^{(2)} = \alpha_3^{(3)} = 1$, in order to gain insight into the physics of the problem. Then, the resulting explicit expression for the three-soliton solution has been found after simple algebraic manipulation to be exactly the same as the stationary PCS for N=3 given in Eq.(17) of Ref.[9a]. One can also check that with the choice $k_{nI} = 0$, $\alpha_2^{(1)} = \alpha_1^{(2)} = 0$, $\alpha_1^{(1)} = -\alpha_2^{(2)} = 1$, in the two-soliton solution of the $N = 2$ case (Manakov equation) [5] of Eq. (1), the $N = 2$ soliton complex(PCS) is obtained. By a similar analysis we have verified that the $N = 4$ PCS also results as a special case of the 4-soliton solution of the 4-CNLS equation. Extending this idea it is clear that the PCS which is formed due to a nonlinear superposition of N-fundamental solitons[9] is a special case of N-soliton solution of the N-CNLS Eq. (1).

Further, it has been found that these PCS are of variable shape and also change their shape during collision with another PCS [9]. The reason for the shape change of the PCS can be deduced from the interaction properties of the solitons discussed above. The solitons are characterized by their amplitudes $A_j^l k_{lR}$ and their velocities $2k_{lI}$(so that the angle of incidence is $\theta_l = tan^{-1}(2k_{lI})$). During a pair-wise interaction of two fundamental solitons of N-CNLS equation there is an energy sharing between them resulting in a novel shape changing collision, depending on the transition matrix elements T_j^l, the phase shift Φ^l and change in the relative separation distance Δx_{ij} defined earlier . Since the PCS is a special case of the N-soliton solution, parametrized as above, it naturally possesses a variable shape. For example in Figs.1, the solitons S_1 and S_2 are travelling with velocities $2k_{1I} = 2$ and $2k_{2I} = -2$, respectively. For the chosen parameters, in Fig.1a, the amplitudes of the modes of the solitons S_1 and S_2 before interaction given respectively by $k_{1R}|A_j^{1-}| = (0.577, 0.577, 0.577)$ and $k_{2R}|A_j^{2-}| = (0.857, 0.857, 1.591)$ change to $k_{1R}|A_j^{1+}| = (0.183, 0.183, 0.966)$ and $k_{2R}|A_j^{2+}| = (1.155, 1.155, 1.155)$ after interaction, preserving the total intensity of each of the soliton. Similarly in Fig.1b, the amplitudes of the solitons S_1 and S_2 before interaction are $(0.101, 0.099, 0.990)$ and $(1.322, 1.335, 0.686)$ respectively, while after interaction they become $(0.466, 0.484, 0.741)$ and $(1.155, 1.155, 1.155)$.The phase shifts suffered by the solitons are $\Phi^1 = -\Phi^2 = -0.787$(Fig.1a),$-0.600$(Fig.1b). During the collision process the initial separation distance $x_{12}^- = -0.668$ changes to $x_{12}^+ = 0.513$ in Fig.1a and -0.036 to 0.865 in Fig.1b.

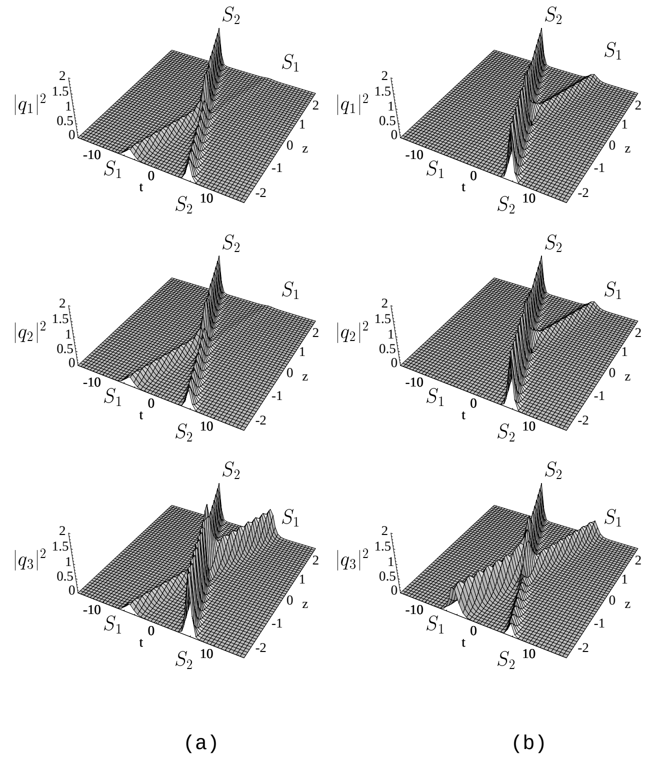

FIG. 1. Intensity profiles of the three modes of the two-soliton solution in a waveguide described by the CNLS Eq. (2) showing two different dramatic scenarios of the shape changing collision. The parameters are chosen as in the text.

In a similar way, the variable shape of the PCS during interaction with another PCS also arises from the fundamental bright soliton collision of the Manakov system. The collision of two PCS each comprising of m and n soliton complexes respectively, such that $m + n = N$, is equivalent to the interaction of N fundamental bright solitons (for suitable choice of parameters) represented by the special case of N-soliton solution of the N-CNLS system.Further details will be published elsewhere.

Our above analysis has considerable practical relevance in view of the various recently reported interesting experimental observations. Firstly, the Manakov spatial solitons have been observed in AlGaAs planar waveguides[14] and their collisions involving energy exchange of precisely the type discussed here has been experimentally demonstrated[15]. Also collisions between PCS's of shape changing type as treated here were observed in a photorefractive strontium barium niobate crystal using screening solitons[16]. Further partially incoherent solitons have been observed through excitation by partially coherent light[17] and with a light bulb[18]. Using different techniques, such as the coherent density function theory[19], to describe such incoherent solitons one can obtain the N-coupled NLS equations of the form(1) considered in this Letter. We believe that our exact analytical results will give further impetus in the experimental investigations of these solitons.

In conclusion, we have shown that N-CNLS Eq. (1) possesses fascinating type of soliton solutions undergoing shape changing (inelastic) collision property due to intensity redistribution among its modes. The many possibilities for such collisions to occur provides interesting avenues of research in developing logic gates and in all-optical digital computations[12]. We have also shown the interesting fact that the multisoliton complexes are special cases of the shape changing bright soliton solutions and pointed out that the variable shape of PCS is due to the shape changing collision of the fundamental solitons which is an inherent nature of the N-CNLS system.

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* Electronic address: lakshman25@satyam.net.in

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