Depletion of Fractional Fermion Number of a Soliton at Finite Chemical Potential and Temperature

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We find an expression for the fractional fermion number of a soliton at finite fermion chemical potential μ and finite temperature T in the $(1+1)$-dimensional chiral field theory. We discuss the physical origin of the depletion of the soliton charge by finite μ and the relevance to lower-dimensional charge-density–wave systems and to topological solitons in higher-dimensional field theories.

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The fractional fermion number[1] has recently been studied by several authors[2–4] at finite temperatures. Midorikawa[2] and Niemi and Semenoff[3] studied this for a $(1+1)$-dimensional quantum field theory,[5] while Barišić and Batistić[4] studied this for a model of quasi-one-dimensional conductors. In Refs. 2 and 3 the background soliton field was assumed to be classical and temperature independent. In Ref. 4 the finite-temperature self-consistent modification of the lattice distortion was taken into account. In this Letter we address ourselves to the question of the change of the fractional fermion number in the presence of finite density of particles or antiparticles (finite chemical potential μ) at finite temperature T in the chiral model in $(1+1)$ dimensions.[6,7] This chiral model is known to be a continuum model for some conjugated polymer systems.[7,8] We get an exact expression for the screened fractional charge from first principles in a simple way without the use of trace identities[3] or Golstone-Wilczek perturbation expansion.[2,6] For zero μ and finite T, our result agrees with the result of Niemi and Semenoff[3] and disagrees with the result of Midorikawa.[2] We find that the fermionic charge on the soliton is depleted for nonzero μ, going to zero for $\mu \to \pm\infty$. We also point out the interesting possibility that while the finite-T and zero-μ result for the charge Q depends only on the asymptotic properties of the soliton profile, the finite-μ result could depend on some local details of the soliton profile as well. We discuss the physics of μ and T dependence of Q and the relevance of the present result to depletion of fractional charges in real charge-density–wave systems as well as topological solitons in higher-dimensional quantum field theory at the end of this Letter.

We now consider the $(1+1)$-dimensional quantum-field theory model studied by Goldstone and Wilczek[6] and Jackiw and Semenoff.[7] The Hamiltonian of the model is

$$H=\int dx\left[\Psi^\dagger\left(\sigma^2\frac{1}{i}\frac{d}{dx}+\sigma^1\phi+\sigma^3\epsilon\right)\Psi\right], \tag{1}$$

where $\Psi^\dagger, \Psi$ are the two-component Dirac field operators, ϕ is the real scalar soliton field, and ϵ is the charge-conjugation symmetry-breaking parameter and σ's are the Pauli spin matrices. In the first quantized form, the Hamiltonian is the one-dimensional Dirac operator in the external fields ϕ and ϵ:

$$\hat{H}(\phi,\epsilon)=\sigma^2\frac{1}{i}\frac{d}{dx}+\sigma^1\phi+\sigma^3\epsilon. \tag{2}$$

This Hamiltonian has the following positive- and negative-energy continuum solutions and a bound-state solution (assuming a soliton profile which has only one bound state)[7]:

$$\psi_{k\alpha}=\begin{pmatrix}[(\alpha E+\epsilon)/2\alpha E]^{1/2}u_k\\ +[2\alpha E(\alpha E+\epsilon)]^{-1/2}(\partial_x+\phi)u_k\end{pmatrix}, \tag{3}$$

$$\psi_s=N_0\begin{pmatrix}\exp[-\int^x dx'\phi(x')]\\ 0\end{pmatrix}, \tag{4}$$

where N_0 is a normalization factor, $\alpha=\pm1$ distinguishes the positive- and negative-energy solutions, and $u_k(x)$ is the normalized eigenfunction of the Schrödinger-like equation

$$(-\partial_x^2+\phi^2-\partial_x\phi)u_k=(E^2-\epsilon^2)u_k,$$
$$E=(k^2+\phi_0^2+\epsilon^2)^{1/2},\quad \phi(\pm\infty)=\pm\phi_0. \tag{5}$$

The fermion density associated with an occupied state ψ_i is given by

$$\rho_i(x) = \psi_i^\dagger(x)\psi_i(x). \tag{6}$$

The soliton charge Q is defined as the total change in fermion number "around" the soliton when the soliton vacuum is created adiabatically from the topologically trivial vacuum.[6] Thus in the ground state the soliton charge is defined as

$$Q = \int_{-\infty}^{\infty} dx \sum_{i \in \mathrm{occ}} [\rho_i^s(x) - \rho_i^0(x)], \tag{7}$$

where $\rho_i^s(x)$ and $\rho_i^0(x)$ are the fermion number density at a point x in the presence and absence of the soliton, due to an occupied state i. The sum is over the occupied states of the Dirac spectrum. Notice that in this definition of the soliton charge Q, the soliton charge also includes the unit fermion number coming from the occupied soliton bound state when $\epsilon < 0$. For example, for $T=0$ and $\mu=0$, the sum is over the entire negative-energy states. This case has been analyzed by Jackiw and Semenoff.[7]

The generalization of Eq. (7) to finite μ and T is straightforward since we have a noninteracting sea of fermions:

$$Q(\mu,T) = \int_{-\infty}^{\infty} dx \sum_i [\rho_i^s(x) - \rho_i^0(x)]\, n(\epsilon_i - \mu), \tag{8}$$

where

$$n(\epsilon-\mu) \equiv \{\exp[\beta(\epsilon-\mu)]+1\}^{-1},$$

is the Fermi distribution function. Substitution for the ρ's in Eq. (8) yields

$$Q = \int_{-\infty}^{\infty} dx \sum_{\alpha=\pm 1} \int_{-\infty}^{\infty} \frac{dk}{2\pi} (|u_k^s|^2 - |u_k^0|^2)\, n(\alpha E - \mu) + \sum_{\alpha=\pm 1} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[\frac{(\partial_x |u_k^s|^2 + 2|u_k|^2\phi)_{x=-\infty}^{x=+\infty}}{4\alpha E(\alpha E+\epsilon)} \right] n(\alpha E-\mu) + n(\epsilon-\mu). \tag{9}$$

With use of the method described in Ref. 6, the square bracket in the second term of the above expression can be simplified further using

$$(\partial_x |u_k^s|^2 + 2|u_k|^2\phi)_{x=-\infty}^{x=+\infty} = 2. \tag{10}$$

For $T=0$ and $\mu=0$, the first term in Eq. (9) is easily evaluated using the completeness properties of the u's.[7] But, for finite T and μ we do not find any general argument to evaluate the above first term for a general soliton profile. So we choose a standard soliton profile

$$\phi(x) = \phi_0 \tanh(\phi_0 x),$$

for which the eigenfunctions $u_k^s(x)$ are known exactly[6] to be

$$u_k(x) = -\exp(ikx)\left[\frac{\tanh\phi_0 x - (ik/\phi_0)}{1+(ik/\phi_0)} \right].$$

Substitution for the u's in Eq. (9) yields

$$Q(\mu,T) = -2\phi_0 \sum_\alpha \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{n(\alpha E-\mu)}{(k^2+\phi_0^2)} + 2\phi_0 \sum_\alpha \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{n(\alpha E-\mu)}{2\alpha E(\alpha E+\epsilon)} + n(\epsilon-\mu). \tag{11}$$

This is the desired expression for the soliton charge for finite μ and T.

In particular, the integrals can be evaluated exactly for zero temperature and finite μ to get

$$Q(\mu,0) = -\mathrm{sgn}(\mu) Q_0(\epsilon) - \theta(\mu) G(k_\mathrm{F},\epsilon) + \theta(-\mu) G(k_\mathrm{F},-\epsilon) \tag{12}$$

for $|\mu| > m$, where

$$Q_0(\epsilon) = \frac{-1}{\pi}\tan^{-1}\left(\frac{\phi_0}{\epsilon}\right), \quad G(k_\mathrm{F},\epsilon) = \frac{w}{\pi}\tan^{-1}\left(\frac{\phi_0 \tan[\frac{1}{2}\tan^{-1}(k_\mathrm{F}/m)]}{m+\epsilon} \right),$$

$$k_\mathrm{F} = (\mu^2 - m^2)^{1/2}, \quad m \equiv (\phi_0^2 + \epsilon^2)^{1/2}.$$

Figure 1 schematically describes the behavior of the soliton fermionic charge as a function of μ for zero temperature.

It is easily seen that for $\mu = 0$ Eq. (11) reduces to the expression derived by Niemi and Semenoff.[2] They obtained their result using a trace identity and without the use of any specific soliton profile. This suggests that for zero μ we also should be able to get an expression for Q by our method without assuming any specific soliton profile. We argue below that this dependence of Q *only* on the asymptotic properties of the soliton profile may not be true for finite μ and in this case Q depends on the local properties of the soliton profile such as the width of the soliton.

Now we will discuss the physical meaning of our result. Once we know the relation of the soliton-bound-state wave function to the continuum wave functions in the absence of the soliton, the dependence of Q becomes clear. For the charge-conjugation symmetric case ($\epsilon = 0$), the self-conjugate soliton bound state is made up of the continuum states of the solitonless background field with half of its norm from the positive-energy states and the other half from the negative-energy states. It is this depletion from the occupied negative-energy continuum which shows up as a fermion number $-\frac{1}{2}$ for the soliton[1] (when the soliton-bound state is unoccupied). Also, the soliton state, being of finite size, is mostly made up of states close to the bottom and top of the positive- and negative-energy continuum, respectively. The energy width of this region is proportional to the energy ϵ_0 required to localize a Dirac particle in a linear dimension of the order of the soliton size. When these states are emptied or filled by introducing finite chemical potential at $T = 0$, the net accumulation of fermionic charge on the soliton changes. For example, when some of the negative-energy states are emptied at the top of the Dirac sea by introducing a negative μ, the charge depletion decreases because some of the negative-energy states which have contributed to the formation of soliton bound states are no longer occupied. *Thus the soliton charge almost goes to zero when the chemical potential becomes greater in magnitude than the energy* ϵ_0. Thus in the limit $\mu \to -\infty$ the soliton charge is zero. By the same argument the net soliton charge decreases as μ is made positive and tends to zero as $\mu \to \infty$. For the above case of $\epsilon = 0$, the soliton state is occupied for any positive μ and therefore it follows that Q is an odd function of μ and that $|Q|$ is a symmetric function of μ (Fig. 1).

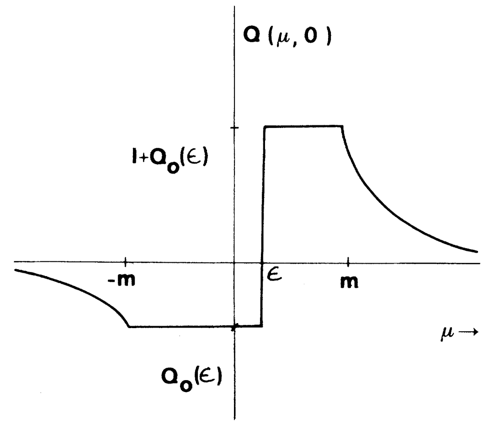

FIG. 1. Schematic μ dependence of Q at $T = 0$.

Similar arguments go through for the charge-conjugation symmetry-broken case ($\epsilon \neq 0$). Now $|Q|$ is an asymmetric function of μ; the reason being that for $\epsilon > 0$ (for example), the soliton bound state has more of its norm from the positive-energy states than the negative-energy states. The effect of finite temperature on the soliton charge has a similar interpretation.[5] When some of the occupied states are emptied or empty states are occupied by thermal excitations, the local charge depletion gets affected.

Now we will discuss the relevance of our finite-μ result to the lower-dimensional charge-density-wave systems. According to the simple arguments of Peierls,[9] in a one-dimensional metal, the gap will open exactly at $2k_F$ (where k_F is the Fermi wave vector) so that there are no more free carriers at $T = 0$. However, the lock-in commensurability energy may minimize the total energy by opening a gap at a wave vector Q_0 (commensurate with the underlying periodicity) close to $2k_F$ rather than at $2k_F$. In this case we will have free carriers like electrons or holes depending on whether $2k_F$ is greater or less than Q_0.[10] In some cases these carriers may not be free for energetic reasons—they may be accommodated as occupied or unoccupied bound states of the solitons which are spontaneously created.[11] When this does not happen our analysis is relevant and it amounts to having a chemical potential not coiniciding with the center of the Peierls gap. Thus we expect distinct finite-density screening of soliton charges in a small range of densities at very low temperatures.[12] Here we should remember that, unlike the case of the quantum-field-theory models, in the charge-density–wave system large changes of the chemical potential will change the soliton charge from one rational fraction to another rational fraction. That is, the soliton charge will change continuously from its rational value when we shift from the center of the Peierls gap until the point at which the charge-density wave length jumps to a new commensurate value. At this point the charge of the soliton has a new rational value (corresponding to the new periodicity[13]) and the chemical potential will sit at the center of the new Peierls gap. Another place to look for the screening of the fermionic charge of soliton by finite μ is the spin-Peierls instability system[14] where the chemical potential can be changed by changing the magnetic field.

We expect the physics of the depletion of the fermionic charge of soliton to be the same in higher-dimensional quantum-field-theory models (e.g., skyrmions) as well. In particular, beyond a particular density of fermions or antifermions, we expect the solitons fermionic charge to be depleted almost to zero value.

After this paper was submitted for publication we received a preprint by Niemi[15] which addresses topological solitons at $\mu > 0$. Some of his results are similar to ours, though, the context is not exactly the same. We differ, however, with the interpretation of sharp changes in fermion number as phase transitions.

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[12]In practice the effect is not so naive. This is because of the screening of the soliton by the electrons due to the Coulomb interaction, which in turn, increases the local electron density around the soliton, further changing the soliton charge. A self-consistent calculation which respects, also, the Friedel sum rules will be presented later.

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