Relation between Quantum and Semiclassical Description of Optical Coherence*

C. L. MEHTA

Department of Physics and Astronomy, University of Rochester, Rochester, New York

AND

E. C. G. SUDARSHAN†

Institute for Theoretical Physics, University of Bern, Bern, Switzerland

(Received 12 October 1964)

The problem of relating the semiclassical and quantum treatments of statistical states of an optical field is re-examined. The case where the rule of association between functions and operators is that of antinormal ordering is studied in detail. It is shown that the distribution function for each mode corresponding to this case is a continuous bounded function, and is also a boundary value of an entire analytic function of two variables. The nature of the distribution for the normal ordering rule of association and its relation to this entire function are discussed. It is shown that this distribution can be regarded as the limit of a sequence of tempered distributions in the following sense: One can find a sequence of density operators $\hat{\rho}_{(\nu)}$ which converges in the norm to the density operator $\hat{\rho}$ of any given field (consisting of a single mode), such that each member of the sequence can be expressed in the form $\hat{\rho}_{(\nu)}=\int\phi_{(\nu)}(z)|z\rangle\langle z|d^2z$, where $\phi_{(\nu)}$ is a tempered distribution.

I. INTRODUCTION

IN the quantum description of optical coherence, one specifies the statistical state of a radiation field by a density operator $\hat{\rho}$. In terms of this operator one can define[1] the coherence functions of arbitrary order $N=n+m$:

$$G_{j_1,j_2,\cdots j_n;k_1,k_2,\cdots k_m}{}^{(n,m)}(x_1,x_2,\cdots x_n;y_1,y_2,\cdots y_m)$$
$$=\mathrm{tr}\{\hat{\rho}\hat{E}_{j_1}{}^{(-)}(x_1)\cdots\hat{E}_{j_n}{}^{(-)}(x_n)\times\hat{E}_{k_1}{}^{(+)}(y_1)\cdots\hat{E}_{k_m}{}^{(+)}(y_m)\}\,,\quad(1.1)$$

where tr stands for trace and $\hat{E}_{j_l}{}^{(-)}(x_l)$ and $\hat{E}_{k_l}{}^{(+)}(y_l)$ are the typical Cartesian components of the negative and positive frequency parts of the electric field operator at the space-time points x_l and y_l, respectively. These operators may be expanded in a complete set of mode eigenfunctions

$$\hat{\mathbf{E}}^{(-)}(x)=\sum_\lambda \hat{a}_\lambda\mathbf{f}^{(\lambda)}(x);$$
$$\hat{\mathbf{E}}^{(+)}(x)=\{\hat{\mathbf{E}}^{(-)}(x)\}^\dagger=\sum_\lambda \hat{a}_\lambda{}^\dagger\mathbf{f}^{(\lambda)*}(x).\quad(1.2)$$

Here, the suffix λ specifies the mode which, in particular, may characterize the momentum and the polarization of the photon. $\hat{a}_\lambda$ and $\hat{a}_\lambda{}^\dagger$ are the annihilation and creation operators,[2] respectively, of the photon in mode λ, satisfying the commutation relations

$$[\hat{a}_\lambda,\hat{a}_{\lambda'}]=[\hat{a}_\lambda{}^\dagger,\hat{a}_{\lambda'}{}^\dagger]=0;\quad[\hat{a}_\lambda,\hat{a}_{\lambda'}{}^\dagger]=\delta_{\lambda,\lambda'}.\quad(1.3)$$

It will be found convenient to choose the eigenstates $|\{z\}\rangle$ of the annihilation operator $\hat{a}_\lambda$ as our basis

$$|\{z\}\rangle\equiv|z_1,z_2,\cdots z_\lambda,\cdots\rangle=\prod_\lambda|z_\lambda\rangle,\quad(1.4)$$

$$\hat{a}_\lambda|z_\lambda\rangle=z_\lambda|z_\lambda\rangle;\quad\langle z_\lambda|\hat{a}_\lambda{}^\dagger=z_\lambda{}^*\langle z_\lambda|;\quad(1.5)$$

$$\langle z_\lambda|z_\lambda\rangle=1.\quad(1.6)$$

Since $\hat{a}_\lambda$ is not Hermitian, its eigenvalue z_λ will, in general, be a complex number

$$z_\lambda=x_\lambda+iy_\lambda=r_\lambda e^{i\theta_\lambda};\quad(x_\lambda,y_\lambda,r_\lambda,\theta_\lambda\text{ real}).\quad(1.7)$$

The states $|z_\lambda\rangle$ can be expressed as a linear combination of the basis states of the Fock representation

$$|\{n\}\rangle\equiv|n_1,n_2,\cdots n_\lambda,\cdots\rangle=\prod_\lambda|n_\lambda\rangle;\quad n_\lambda=0,1,2,\cdots\quad(1.8)$$

$$\hat{a}_\lambda{}^\dagger\hat{a}_\lambda|n_\lambda\rangle=n_\lambda|n_\lambda\rangle,\quad\hat{a}_\lambda|n_\lambda\rangle=(n_\lambda)^{1/2}|n_\lambda-1\rangle,$$
$$\hat{a}_\lambda{}^\dagger|n_\lambda\rangle=(n_\lambda+1)^{1/2}|n_\lambda+1\rangle\quad(1.9)$$

in the form

$$|z_\lambda\rangle=\exp(-\tfrac{1}{2}|z_\lambda|^2)\sum_{n_\lambda=0}^{\infty}\frac{z_\lambda{}^{n_\lambda}}{(n_\lambda!)^{1/2}}|n_\lambda\rangle.\quad(1.10)$$

They also furnish the resolution of the identity

$$\frac{1}{\pi}\int|z_\lambda\rangle\langle z_\lambda|d^2z_\lambda=\hat{1},\quad(1.11)$$

where $d^2z_\lambda\equiv dx_\lambda dy_\lambda=r_\lambda dr_\lambda d\theta_\lambda$ and the integration extends over the whole complex z plane. Any operator and in particular the density operator $\hat{\rho}$, which has the following expansion in Fock representation:

$$\hat{\rho}=\sum_{\{m\}}\sum_{\{n\}}\rho_{\{m\},\{n\}}|\{m\}\rangle\langle\{n\}|,\quad(1.12)$$

where

$$\rho_{\{m\},\{n\}}=\langle\{m\}|\hat{\rho}|\{n\}\rangle,\quad(1.13)$$

can also be expressed in terms of the overcomplete family of states $|\{z\}\rangle$ in a "diagonal" form

$$\hat{\rho}=\int\cdots\int d^2\{z\}\phi(\{z\})|\{z\}\rangle\langle\{z\}|,\quad(1.14)$$

where $\phi(\{z\})$ is a suitable distribution over the complex

* Supported in part by the U. S. Army Research Office (Durham).

A preliminary account of this paper was presented at the conference on Quantum Electrodynamics of High Intensity Photon Beams held at Durham, North Carolina in August 1964.

† Present address: Department of Physics, Syracuse University, Syracuse, New York.

[1] R. J. Glauber, Phys. Rev. 130, 2529 (1963). Glauber considered only even order coherence functions of the type $G^{(n,n)}$, which he calls nth-order coherence functions.

[2] In this paper all operators are denoted by caret signatures.

variables $\{z\}$. A formal expression for this distribution has been given by Sudarshan.[3]

In the semiclassical description of optical coherence, on the other hand, one describes the statistical state of the radiation field by an ensemble probability distribution and the $N=n+m$ order coherence functons may be defined as[4]

$$\Gamma_{j_1,j_2,\cdots j_n;k_1,k_2,\cdots k_n}{}^{(n,m)}(x_1,x_2,\cdots x_n;y_1,y_2\cdots y_n)$$
$$=\langle E_{j_1}{}^*(x_1)\cdots E_{j_n}{}^*(x_n)E_{k_1}(y_1)\cdots E_{k_n}(y_n)\rangle_e, \quad (1.15)$$

where $\langle\ \rangle_e$ denotes the statistical ensemble average. It is convenient to introduce the linear functional

$$F[u]=\left\langle \exp\left\{\int \sum_{j=1}^{3}[E_j{}^*(x)u_j(x)-E_j(x)u_j{}^*(x)]d^3x\right\}\right\rangle_e \quad (1.16)$$

in terms of which the coherence functions may be defined as

$$\Gamma_{j_1,j_2,\cdots j_n;k_1,k_2\cdots k_m}{}^{(n,m)}(x_1,x_2,\cdots x_n;y_1,y_2\cdots y_m)$$
$$=(-1)^m\frac{\delta}{\delta u_{j_1}(x_1)}\cdots\frac{\delta}{\delta u_{j_n}(x_n)}\frac{\delta}{\delta u_{k_1}{}^*(y_1)}\cdots$$
$$\times\frac{\delta}{\delta u_{k_m}{}^*(y_m)}F[u]|_{u=0}, \quad (1.15')$$

where $\delta F[u]/\delta u(x)$ denotes the variational derivative of the linear functional $F[u]$ with respect to $u(x)$.[5] We may think of $F[u]$ as the characteristic functional and the $\Gamma^{(n,m)}$ as the (polynomial) moments.

It is of interest to associate the quantum density operator $\hat{\rho}$ and the (semiclassical) ensemble probability distribution $\rho(\{z\})$ in such a way that the coherence functions defined in the two schemes are identical. We may, instead, require that the characteristic functionals in the semiclassical and quantum descriptions be the same.

For convenience of discussion we shall restrict our treatment here to a system having a single degree of freedom (one mode); the essential part of the discussion can, however, be extended to systems having a finite number of modes. When the number of modes become infinite, new mathematical problems arise in connection with the quantum-mechanical specification of the state of the system.[6] But we shall not enter into these questions here.

We seek, then, a relation between the ensemble probability distribution $\rho(z)$ and the quantum density operator $\hat{\rho}$ such that

$$\mathrm{tr}\{\hat{\rho}\hat{G}(\hat{a},\hat{a}^\dagger)\}=\int\rho(z)G(z,z^*)d^2z \quad (1.17)$$

for suitable operators $\hat{G}$. However, since the operators $\hat{a}$ and $\hat{a}^\dagger$ do not commute, the association of $\hat{G}(\hat{a},\hat{a}^\dagger)$ with $G(z,z^*)$ is not unique, and will depend on the rule of association between operators and functions. It is advantageous to choose the set of operators $^\dagger\hat{G}(\hat{a},\hat{a}^+)$ such that it consists of all bounded operators including the identity operator. Since the expectation-value mapping is a linear functional it is sufficient to consider the set of unitary operators

$$\hat{G}(\hat{a},\hat{a}^\dagger)=\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a}). \quad (1.18)$$

Using the commutation relations between $\hat{a}$ and $\hat{a}^\dagger$ we may rewrite the right-hand side in the two alternative forms

$$\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})=\exp(-\tfrac{1}{2}\alpha\alpha^*)\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a}) \quad (1.19)$$
$$=\exp(+\tfrac{1}{2}\alpha\alpha^*)\exp(-\alpha^*\hat{a})\times\exp(\alpha\hat{a}^\dagger). \quad (1.19')$$

The statistical state is specified by the linear functional

$$F[\alpha]=\langle\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})\rangle, \quad (1.20)$$

or equivalently, by the linear functionals

$$F_N[\alpha]=\langle\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})\rangle=\exp(\tfrac{1}{2}\alpha\alpha^*)F[\alpha], \quad (1.21)$$

$$F_A[\alpha]=\langle\exp(-\alpha^*\hat{a})\exp(\alpha\hat{a}^\dagger)\rangle=\exp(-\tfrac{1}{2}\alpha\alpha^*)F[\alpha]. \quad (1.22)$$

If we consider the functional $F_N[\alpha]$ to be the characteristic functional, we get a distribution $\rho_N(z)$ which satisfies the relation

$$\int\rho_N(z)\exp(\alpha z^*-\alpha^*z)d^2z=\mathrm{tr}\{\hat{\rho}\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})\}, \quad (1.23)$$

so that it is identical to the distribution $\phi(z)$ discussed by Sudarshan[3] (see also Mehta, Ref. 7). In this paper we shall be mainly interested in the correspondence based on antinormal ordering and in considering $F_A[\alpha]$ to be the characteristic functional. Thus we seek a distribution $\rho_A(z)$ satisfying the relation

$$\int\rho_A(z)\exp(\alpha z^*-\alpha^*z)d^2z=\mathrm{tr}\{\hat{\rho}\exp(-\alpha^*\hat{a})\exp(\alpha\hat{a}^\dagger)\}. \quad (1.24)$$

In the following section we derive an explicit form for $\rho_A(z)$ and in Sec. III we discuss some of the properties and show that it is in fact a continuous function. Section IV includes a discussion about the nature of $\rho_N(z)$ and also the relation between the function $\rho_A(z)$ and the distribution $\rho_N(z)$. It is shown that $\rho_N(z)$ can

[3] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963); also in *Proceedings of the Symposium on Optical Masers* (Polytechnic Press, Brooklyn, New York, and John Wiley & Sons, Inc., New York, 1963), p. 45.

[4] E. Wolf, in *Proceedings of the Symposium on Optical Masers* (Polytechnic Press, Brooklyn, New York, and John Wiley & Sons, Inc., New York, 1963), p. 29.

[5] Compare E. Hopf, J. Rational Mech. and Anal. **1**, 87 (1952); I. E. Segal, Canad. J. Math. **13**, 1 (1961); R. M. Lewis and R. H. Kraichnan, Commun. Pure and Appl. Math. **15**, 397 (1962).

[6] I. E. Segal, Illinois J. Math. **6**, 500 (1962); see also E. C. G. Sudarshan, J. Math. Phys. **4**, 1029 (1963).

[7] C. L. Mehta, J. Math. Phys. **5**, 677 (1964).

be regarded as a limit of a sequence of tempered distributions. It then readily follows that the quantum and the semiclassical descriptions of statistical light beams are equivalent; and we thus obtain a rigorous formulation of a theorem established heuristically in an earlier paper.[3]

II. THE DISTRIBUTION FOR ANTINORMAL ORDERING

We are interested in obtaining a distribution $\rho_A(z)$ such that the quantum generating functional $F_A[\alpha]$ satisfies the relation

$$F_A[\alpha] \equiv \mathrm{tr}\{\hat{\rho}\exp(-\alpha^*\hat{a})\exp(\alpha\hat{a}^\dagger)\} \tag{2.1}$$

$$= \int \rho_A(z)\exp(\alpha z^* - \alpha^* z)d^2z, \tag{2.2}$$

where, as before, $d^2z = dxdy$ and the integration extends over the whole complex z plane. Using the resolution of the identity

$$\frac{1}{\pi}\int |z\rangle\langle z| d^2z = \hat{1}, \tag{2.3}$$

we may rewrite (2.1) in the form

$$F_A[\alpha] = \frac{1}{\pi}\int \exp(\alpha z^* - \alpha^* z)\,\mathrm{tr}\{\hat{\rho}|z\rangle\langle z|\}d^2z. \tag{2.4}$$

Comparing (2.2) and (2.4) we see that

$$\rho_A(z) = \frac{1}{\pi}\langle z|\hat{\rho}|z\rangle. \tag{2.5}$$

Thus we note that the function $(1/\pi)\langle z|\hat{\rho}|z\rangle$ can be regarded as a probability distribution function so long as the rule of association between operators and functions is that based on antinormal ordering.[8]

III. PROPERTIES OF $\rho_A(z)$

Since $\hat{\rho}$ is a density operator, which is necessarily Hermitian and positive definite, its matrix elements in the Fock representation satisfy the inequalities

$$\rho_{j,j} \geqslant 0, \tag{3.1}$$

$$|\rho_{j,k}|^2 \leqslant \rho_{j,j}\rho_{k,k}, \quad \text{for all } j \text{ and } k. \tag{3.2}$$

Normalization of the density operator implies

$$\textstyle\sum_j \rho_{j,j} = 1, \tag{3.3}$$

which gives, on using (3.1), the inequality

$$0 \leqslant \rho_{j,j} \leqslant 1, \quad \text{for all } j. \tag{3.4}$$

We shall use these properties of $\hat{\rho}$ to derive the following properties of $\rho_A(z)$.

(a) $0 \leqslant \rho_A(z) \leqslant (1/\pi)$, for all z.

(b) For all α, $|F_A[\alpha]| = |\int \rho_A(z)\exp(\alpha z^* - \alpha^* z)d^2z| \leqslant \exp(-\frac{1}{2}|\alpha|^2)$.

Hence, in particular, ρ_A is integrable and also square integrable.

(c) $0 \leqslant \int \rho_A(x,y)dx \leqslant (2/\pi)^{1/2}$, for all y.

(d) $\rho_A(z) \equiv \rho_A(x,y)$ is the boundary value of an entire analytic function of two variables, and also satisfies certain constraints connected with self-reproducing properties.

That property (a) is satisfied, can easily be seen if one expresses $\hat{\rho}$ in the form

$$\hat{\rho} = \textstyle\sum_n \lambda_n |\psi_n\rangle\langle\psi_n|, \tag{3.5}$$

where $|\psi_n\rangle$ are the eigenstates of $\hat{\rho}$ and λ_n are the corresponding eigenvalues. Thus, since $0 \leqslant \lambda_n \leqslant 1$, we have

$$\rho_A(z) = \frac{1}{\pi}\sum_n \lambda_n |\langle z|\psi_n\rangle|^2 \geqslant 0; \tag{3.6}$$

and also

$$\leqslant \frac{1}{\pi}\sum_n |\langle z|\psi_n\rangle|^2 = \frac{1}{\pi}. \tag{3.7}$$

To prove the assertion (b) we note that

$$\int \rho_A(z)\exp(\alpha z^* - \alpha^* z)d^2z = \langle\exp(-\alpha^*\hat{a})\exp(\alpha\hat{a}^\dagger)\rangle$$
$$= \exp(-\tfrac{1}{2}|\alpha|^2)\langle\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})\rangle.$$

But since $\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is unitary, we obtain

$$|\langle\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})\rangle| = |\mathrm{tr}\{\hat{\rho}\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})\}| \leqslant 1. \tag{3.8}$$

Hence,

$$\left|\int \rho_A(z)\exp(\alpha z^* - \alpha^* z)d^2z\right| \leqslant \exp(-\tfrac{1}{2}|\alpha|^2), \tag{3.9}$$

as required. In particular when $\alpha = 0$ we obtain the normalization condition

$$\int \rho_A(z)d^2z = 1. \tag{3.10}$$

If $F_A(s,t)$ denotes the Fourier transform of $\rho_A(x,y)$:

$$F_A(s,t) = \iint_{-\infty}^{\infty} \rho_A(x,y)\exp[i(sx+ty)]dxdy, \tag{3.11}$$

then using (3.9) with $\alpha = \frac{1}{2}i(s+it)$ we obtain

$$|F_A(s,t)| \leqslant \exp[-\tfrac{1}{8}(s^2+t^2)]. \tag{3.12}$$

Now since $\rho_A(z)$ is bounded and integrable, it is square integrable. In fact

$$\iint_{-\infty}^{\infty} \{\rho_A(x,y)\}^2 dxdy$$
$$= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} |F_A(s,t)|^2 dsdt$$
$$\leqslant \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \exp[-\tfrac{1}{4}(s^2+t^2)]dsdt = \frac{1}{\pi}. \tag{3.13}$$

[8] While this paper was being written, a similar probability distribution function was being considered by Y. Kano (to be published).

Fourier inversion of (3.11) gives

$$\rho_A(x,y)=\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty} F_A(s,t)\exp[-i(sx+ty)]dsdt. \quad (3.14)$$

Hence, integration over x gives

$$\int_{-\infty}^{\infty}\rho_A(x,y)dx$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}F_A(0,t)\exp(-ity)dt$$

$$\leqslant\frac{1}{2\pi}\int_{-\infty}^{\infty}|F_A(0,t)|dt\leqslant\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-\tfrac{1}{8}t^2)dt$$

$$=(2/\pi)^{1/2}, \quad (3.15)$$

which proves assertion (c). Similarly for all x,

$$0\leqslant\int\rho_A(x,y)dy\leqslant(2/\pi)^{1/2}. \quad (3.16)$$

To show the last property (d), we define a function $\rho(\zeta,\eta)$ as

$$\rho(\zeta,\eta)=\frac{1}{\pi}\langle\zeta^*|\hat{\rho}|\eta\rangle\exp(\tfrac{1}{2}|\zeta|^2+\tfrac{1}{2}|\eta|^2) \quad (3.17)$$

$$=\frac{1}{\pi}\sum_m\sum_n\rho_{m,n}\frac{\zeta^m\eta^n}{(m!n!)^{1/2}} \quad (3.18)$$

so that

$$\rho_A(z)=\rho(z^*,z)e^{-|z|^2}. \quad (3.19)$$

It may easily be seen using (3.2) and (3.4) that the series on the right-hand side of (3.18) is absolutely convergent for all finite values of $|\zeta|$ and $|\eta|$, and hence $\rho(\zeta,\eta)$ is an entire analytic function of the two complex variables ζ and η. It also follows, therefore, that ρ regarded as a function of two new independent variables $\alpha=\frac{1}{2}(\zeta+\eta)$ and $\beta=\frac{1}{2}i(\zeta-\eta)$ is an entire function of both variables. When both α and β are real, $\zeta^*=\eta$ and $\rho(\zeta,\eta)\equiv\rho_A(\eta)e^{|\eta|^2}$. Further in this case α and β are the real and imaginary parts, respectively, of η. Hence, we conclude that $\rho_A(z)\equiv\rho_A(x,y)$, $(z=x+iy)$, regarded as a function of x and y is the boundary value of an entire analytic function in α and β for their real values. [The exponential factor $\exp(|z|^2)=\exp(x^2+y^2)$ is already the boundary value of the entire function $\exp(\alpha^2+\beta^2)$ of α and β.]

To make our statement clear and unambiguous, we stress that what is meant here is that if one formally replaces the real variable x and y in $\rho_A(x,y)$ by two complex variables α and β then $\rho_A(\alpha,\beta)$ is an entire analytic function of α and β.

This property has a very interesting consequence which may be expressed by the following

Theorem[9]: If for any bounded operator $\hat{A}$,

$$A(z)=\langle z|\hat{A}|z\rangle=0$$

in any finite area over the complex plane $z=x+iy$, then $A(z)\equiv0$ over the whole complex z plane and further the operator $\hat{A}$ itself is identically zero.

Proof: In proving property (d) we only utilized the boundedness of $\hat{\rho}$ and hence the final result is valid for any bounded operator. Thus

$$A(z)=A(x,y)=\langle x+iy|\hat{A}|x+iy\rangle, \quad (3.20)$$

regarded as a function of x and y, is also a boundary value of an entire analytic function of two variables α and β for their real values. We are given that $A(\alpha,\beta)$ is analytic in α and β and that

$$A(x,y)=0, \quad \text{for} \quad a_1\leqslant x\leqslant a_2 \quad \text{and} \quad b_1\leqslant y\leqslant b_2. \quad (3.21)$$

Hence according to a well-known theorem of complex-variable theory $A(\alpha,\beta)\equiv0$ for all α and β, and hence certainly for α and β real, i.e., $A(x,y)\equiv0$ for all real x and y. We can go a step further and say that $A(\alpha,\beta)$ regarded as a function of ζ and η where $\zeta=\alpha-i\beta$ and $\eta=\alpha+i\beta$ is also identically zero for all ζ and η. This shows that

$$\langle\zeta^*|\hat{A}|\eta\rangle\equiv0, \quad \text{for all } \zeta \text{ and } \eta, \quad (3.22)$$

which holds only if $\hat{A}=0$.

Self-Reproducing Property

Since the "over-complete" representation $|z\rangle$ furnishes the resolution of the identity [Eq. (2.3)]

$$\frac{1}{\pi}\int|z\rangle\langle z|d^2z=\hat{1}, \quad (3.23)$$

we obtain from Eq. (3.17)

$$\rho(\zeta,\eta)=\frac{1}{\pi^2}\int\langle\zeta^*|\hat{\rho}|z\rangle\langle z|\eta\rangle\exp(\tfrac{1}{2}|\zeta|^2+\tfrac{1}{2}|\eta|^2)d^2z$$

$$=\frac{1}{\pi}\int\rho(\zeta,z)\exp(-|z|^2+\eta z^*)d^2z. \quad (3.24)$$

In the above derivation, use has been made of Eq. (1.10) and its Hermitian adjoint. Thus we see that $\rho(\zeta,\eta)$ possesses the self-reproducing property[10]

$$\rho(\zeta,\eta)=\int K(\eta,z)\rho(\zeta,z)d^2z, \quad (3.25)$$

[9] The last part of this theorem, namely that $\langle z|\hat{A}|z\rangle=0$ for *all* z implies the vanishing of $\hat{A}$ has also been formulated and proved for any continuous representation of the canonical variables by J. R. Klauder, J. Math. Phys. **5**, 177, 184 (1964); see also T. F. Jordan, Phys. Letters **11**, 289 (1964).

[10] N. Aronszajn, Trans. Am. Math. Soc. **68**, 337 (1950).

with the kernel[11]

$$K(\eta,z)=(1/\pi)\exp(-|z|^2+\eta z^*). \qquad (3.26)$$

In a strictly similar manner, one can also show that

$$\rho(\zeta,\eta)=\frac{1}{\pi^2}\int\langle\zeta^*|z^*\rangle\langle z^*|\hat{\rho}|\eta\rangle\exp(\tfrac{1}{2}|\eta|^2+\tfrac{1}{2}|\zeta|^2)d^2z$$

$$=\int K(\zeta,z)\rho(z,\eta)d^2z. \qquad (3.27)$$

It may be noted that Eqs. (3.25) and (3.27) are not independent and are in fact complex conjugates of each other.

The distribution $\rho_A(z)$ does not satisfy the self-reproducing property of the type given above; however, it satisfies the following constraints:

$$\rho_A(z)=\frac{1}{\pi}\int\rho_A\left(\frac{z^*+\eta}{2},i\frac{z^*-\eta}{2}\right)\times\exp(-|\eta-z|^2)d^2\eta \qquad (3.28a)$$

$$=\frac{1}{\pi}\int\rho_A\left(\frac{\eta+z}{2},i\frac{\eta-z}{2}\right)\times\exp(-|\eta-z^*|^2)d^2\eta, \qquad (3.28b)$$

where $\rho_A(\alpha,\beta)$ on the right hand side is obtained by replacing the real variables x and y in $\rho_A(x,y)$ by the complex variables α and β. These relations may be obtained by making use of Eq. (2.3) and the analytic properties of $\rho_A(x,y)$ [cf., Eq. (4.19) below].

IV. RELATION BETWEEN THE DISTRIBUTIONS $\rho_A(z)$ AND $\rho_N(z)$ AND THE EQUIVALENCE OF THE QUANTUM AND THE SEMICLASSICAL DESCRIPTION OF STATISTICAL OPTICAL FIELDS

In the quantum description of optical coherence, one usually defines the coherence functions as expectation values of normal-ordered operators. While it is possible to redefine these coherence functions as expectation values of antinormal ordered products, it is of interest to consider in what sense the distribution $\rho_N(z)$ is defined and also to examine the relation between $\rho_A(z)$ and $\rho_N(z)$. This relation then naturally leads to a rigorous formulation of the equivalence between quantum and semiclassical descriptions of statistical-optical fields which was derived heuristically in an earlier paper.[3]

[11] It may be noted that all entire functions are solutions of the integral equation

$$f(\zeta)=\int K(\zeta,\eta)f(\eta)d^2\eta,$$

where the kernel K is given by (3.26).

For this purpose consider the integral equation

$$\iint\phi(x,y)\exp[i(sx+ty)]dxdy$$

$$=\iint\rho_A(x,y)\exp[i(sx+ty)]\times\exp\left(\frac{s^2+t^2}{4}\right)dxdy, \qquad (4.1)$$

where, as in the previous section,

$$\rho_A(x,y)=(1/\pi)\langle x+iy|\hat{\rho}|x+iy\rangle,$$

so that

$$F_N[\alpha]\equiv F_N(s,t)$$

$$=\iint\phi(x,y)\exp[i(sx+ty)]dxdy$$

$$=\exp[\tfrac{1}{4}(s^2+t^2)]F_A(s,t)=e^{|\alpha|^2}F_A[\alpha];$$

$$\alpha=\tfrac{1}{2}i(s+it). \qquad (4.2)$$

Hence, if a solution exists for $\phi(x,y)$, it could be identified with $\rho_N(x+iy)$, the distribution for normal ordering. Given such a distribution $\phi(z)=\phi(x,y)$ we can express the density operator in the "diagonal" form

$$\hat{\rho}=\int\phi(z)|z\rangle\langle z|d^2z, \qquad (4.3)$$

since then

$$\mathrm{tr}\{\hat{\rho}\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})\}=\int\phi(z)\exp(\alpha z^*-\alpha^*z)d^2z.$$

Using this diagonal form we may rewrite the relation between $\rho_N(z)=\phi(z)$ and $\rho_A(z)=\rho_A(x,y)$ in the form

$$\rho_A(z)=\frac{1}{\pi}\int\rho_N(\eta)|\langle\eta|z\rangle|^2d^2\eta. \qquad (4.4)$$

Thus given ρ_N, the computation of ρ_A is immediate. The problem of defining $\rho_N(z)$ is the problem of inversion of the integral equations (4.1) or (4.4). We have, of course, no guarantee that $\rho_N(z)$ will be a continuous function.

We note, however, that (4.2) may be considered as the defining equation for a distribution ϕ which maps the test function $\exp[i(sx+ty)]$ according to[12]

$$\exp[i(sx+ty)]\xrightarrow{\phi}F_N(s,t). \qquad (4.5)$$

If it turns out that $F_N(s,t)$ has appropriate behavior, we may identify the linear functional ϕ with the linear

[12] Distributions which are defined as the Fourier transforms of entire analytic functions have been considered in the literature; see for example L. Ehrenpreis, Ann. Math. 63, 129 (1956). We are thankful to C. P. Gupta for drawing our attention to this paper.

functional associated with a function $\phi(z)=\rho_N(z)$. If $F_N(s,t)$ does not increase faster than a polynomial in s, t, we may identify ϕ with a tempered distribution. But in general neither of these conditions can be guaranteed. It is easy to construct examples in which $F_N(s,t)$ increases like an exponential or even like the exponential of a quadratic form [compare the Appendix B, Ref. (13)]. In fact, the only requirements on the distribution ϕ are normalization and a positivity condition. The normalization condition can be written simply as

$$F_N[0]=1. \tag{4.6}$$

The positivity condition is more involved, and may be written in the form

$$\left\{\left|\sum_{n=1}^{N} a_n\left(\frac{\partial}{\partial s}-i\frac{\partial}{\partial t}\right)^n\right|^2 F(s,t)\right\}_{s=t=0} \geqslant 0 \tag{4.7}$$

for all sets of numbers a_n and for all N.

While it is not possible to construct ϕ as a tempered distribution in all cases, it is possible to exhibit a sequence of tempered distributions $\phi_{(\nu)}$ such that the sequence of operators

$$\hat{\rho}_{(\nu)}=\int \phi_{(\nu)}(z)|z\rangle\langle z|d^2z, \tag{4.8}$$

converges to the operator

$$\hat{\rho}=\int \phi(z)|z\rangle\langle z|d^2z. \tag{4.9}$$

Weak convergence implies that for any two fixed vectors u and v, the sequence of matrix elements

$$\int \phi_{(\nu)}(z)\langle z|vu^\dagger|z\rangle d^2z, \tag{4.10}$$

converges to

$$\mathrm{tr}(\hat{\rho}vu^\dagger)=\int \phi(z)\langle z|vu^\dagger|z\rangle d^2z. \tag{4.11}$$

The operator integrals (4.8), (4.9) involving the distributions are to be understood in terms of the linear functions (4.10), (4.11) defined over the matrix elements. Since every *bounded* operator can be expressed as a linear combination of the unitary operators given by (1.18), it follows that the correspondence (4.5) defines the density matrix integral (4.9).

To demonstrate this assertion we note first of all that since

$$4vu^\dagger=(u+v)(u+v)^\dagger-(u-v)(u-v)^\dagger +i(u-iv)(u-iv)^\dagger-i(u+iv)(u+iv)^\dagger,$$

it is sufficient to show that the sequence of matrix elements

$$\int \phi_{(\nu)}(z)\langle z|ww^\dagger|z\rangle d^2z,$$

converges for every vector w, to

$$\int \phi(z)\langle z|ww^\dagger|z\rangle d^2z.$$

For this purpose consider the operators $\hat{\rho}$ and $\hat{\sigma}=ww^\dagger$ in terms of their Fock representation matrix elements $\rho_{n,n'}$ and $\sigma_{n,n'}$, respectively. Without any loss of generality we may choose w also to be normalized so that $\mathrm{tr}(ww^\dagger)=w^\dagger w=1$. If

$$\rho_{n,n'}=0, \qquad n,\, n'>M,$$

then it is easy to show that $F_N[\alpha]$, $\{\alpha=\frac{1}{2}i(s+it)\}$ increases only as a polynomial of degree not greater than 2M. Consequently, $\rho_N(z)$ will be a tempered distribution. It is thus sufficient to show that we could construct a sequence of *density matrices* $\hat{\rho}_{(\nu)}$ with

$$\rho_{(\nu)n,n'}=0, \qquad n,\, n'>\nu, \tag{4.12}$$

such that the sequence of numbers

$$\mathrm{tr}(\hat{\rho}_{(\nu)}\hat{\sigma})=\sum_{n,n'=0}^{\infty}\rho_{(\nu)n,n'}\sigma_{n',n}, \tag{4.13}$$

converges to

$$\mathrm{tr}(\hat{\rho}\hat{\sigma})=\sum_{n,n'=0}^{\infty}\rho_{n,n'}\sigma_{n,n'}. \tag{4.14}$$

Let us define $\rho_{(\nu)n,n'}$ as

$$\begin{aligned}\rho_{(\nu)n,n'}&=\rho_{n,n'},\quad 0\leqslant n\leqslant\nu,\quad 0\leqslant n'\leqslant\nu,\quad \text{but not } n=n'=\nu;\\ &=\sum_{\mu=\nu}^{\infty}\rho_{\mu,\mu},\quad n=n'=\nu;\\ &=0,\quad \text{otherwise.}\end{aligned} \tag{4.15}$$

The operator $\hat{\rho}_{(\nu)}$ so defined is positive definite and has unit trace. To prove the weak convergence we note that since both $\hat{\rho}$ and $\hat{\sigma}$ are Hermitian and since

$$\mathrm{tr}(\hat{\rho}^2)\leqslant 1,\quad \mathrm{tr}(\hat{\sigma}^2)=1,$$

[13] If we are only interested in weak-operator convergence, we can follow J. R. Klauder, J. McKenna, and D. G. Currie (to be published) and exhibit a series of density operators which converge weakly, for which the associated distributions $\phi_{(\nu)}$ can be identified with square integrable functions. The essential point of the construction is to define the distributions $\phi_{(\nu)}$ by the mapping

$$\exp[i(sx+ty)]\xrightarrow{\phi_{(\nu)}}\begin{cases}F_N(s,t), & -\nu\leqslant s,t\leqslant\nu\\ 0, & \text{otherwise.}\end{cases}$$

In this case we can associate with $\phi_{(\nu)}$ a function whose double Fourier transform vanishes outside a rectangle and is bounded inside it. *Note added in proof.* Dr. J. R. Klauder has informed us that this sequence of density operators also converges in norm. It is therefore possible to construct a sequence $\phi_{(\nu)}$ of distributions which is not only tempered but also square integrable which converges in norm with the distribution ϕ associated with an arbitrary density operator $\hat{\rho}$.

it follows from Schwarz' inequality that not only is tr $(\hat{\rho}\hat{\sigma})$ defined and less than unity, but the series

$$\sum_{n,n'=0}^{\infty} \rho_{n,n'}\sigma_{n',n}$$

is absolutely convergent. But this in turn implies that the sequence of numbers

$$\sum_{n,n'=0}^{\nu} \rho_{n,n'}\sigma_{n',n}$$

converges to $\mathrm{tr}(\hat{\rho}\hat{\sigma})$. On the other hand, the sequence of numbers $\sum_{\mu=\nu+1}^{\infty} \rho_{\mu,\mu}$ converges to zero. Hence the sequence (4.13) converges to (4.14) and our assertion is proved.

It is to be noted that in view of the nature of the operators $\hat{\rho}_{(\nu)}$, we have succeeded in exhibiting a sequence of density operators which have associated tempered distributions $\phi_{(\nu)}$ and which weakly converges to the density operator $\hat{\rho}$. While the sequence chosen here is in some ways the most immediate, it is by no means unique.[13]

Actually the convergence of the operator sequence is much stronger. We can, in fact, show that the operator sequence converges in norm, i.e., the sequence of numbers

$$\|\hat{\rho}-\hat{\rho}_{(\nu)}\|^2 = \sum_{n,n'=0}^{\infty} |\rho_{n,n'}-\rho_{(\nu)n,n'}|^2 \qquad (4.16)$$

converges to zero. Now

$$\sum_{n,n'=0}^{\infty} |\rho_{n,n'}-\rho_{(\nu)n,n'}|^2 = \sum_{n,n'=\nu}^{\infty} |\rho_{n,n'}|^2 - \rho_{\nu,\nu}^2 + \{\sum_{n=\nu+1}^{\infty} \rho_{n,n}\}^2 \leqslant 2\{\sum_{n=\nu}^{\infty} \rho_{n,n}\}^2,$$

and since the series

$$\sum_{n=0}^{\infty} \rho_{n,n}$$

is absolutely convergent, it follows that the sequence (4.16) also converges to zero. Since convergence in the norm implies strong convergence, the sequence of operators $\hat{\rho}_{(\nu)}$ converges strongly.

We have thus been able to exhibit a sequence of density operators which converges in norm (and hence, strongly) to any given density operator, for each member of which the associated distribution $\phi_{(\nu)}$ is tempered. But in general, the distribution associated with the limit is not tempered.

We conclude this section by pointing out some formal relations. The integral equation (4.1) has the formal solution

$$\phi(z) \equiv \rho_N(z) = \exp(-\tfrac{1}{4}\nabla^2)\rho_A(z), \qquad (4.17)$$

and we can write formally

$$\hat{\rho} = \int |z\rangle\langle z| \exp(-\tfrac{1}{4}\nabla^2)\rho_A(z) d^2z. \qquad (4.18)$$

We can also obtain an alternative expression for $\hat{\rho}$ in terms of ρ_A in the following manner by using the analytic properties of $\rho(\zeta,\eta)$, which is defined by (3.17). If we formally replace x and y in $\rho_A(x,y)\exp(x^2+y^2)$ by $\frac{1}{2}(\zeta+\eta)$ and $\frac{1}{2}i(\zeta-\eta)$ respectively, we should essentially obtain $\rho(\zeta,\eta)$. In fact[14]

$$\rho(\zeta,\eta) = \exp(\zeta\eta)\rho_A(\tfrac{1}{2}(\zeta+\eta), \tfrac{1}{2}i(\zeta-\eta)). \qquad (4.19)$$

Hence

$$\hat{\rho} = \frac{1}{\pi^2}\int\int |\zeta\rangle\langle\zeta|\hat{\rho}|\eta\rangle\langle\eta| d^2\zeta d^2\eta$$

$$= \frac{1}{\pi}\int\int \rho(\zeta^*,\eta)\exp[-\tfrac{1}{2}(|\zeta|^2+|\eta|^2)]|\zeta\rangle\langle\eta| d^2\zeta d^2\eta$$

$$= \frac{1}{\pi}\int\int |\zeta\rangle\langle\eta| \exp(\zeta^*\eta - \tfrac{1}{2}|\zeta|^2 - \tfrac{1}{2}|\eta|^2) \times \rho_A\left(\frac{\zeta^*+\eta}{2}, i\frac{\zeta^*-\eta}{2}\right) d^2\zeta d^2\eta. \qquad (4.20)$$

Finally one may note that, if the given function $\rho_A(x,y)$ is not the boundary value of an entire analytic function then $(1/\pi)\langle z|\hat{\rho}|z\rangle$ obtained by using the expression (4.20) for $\hat{\rho}$ will not agree with the original expression for $\rho_A(z)$. The reason for this is that, in general, any arbitrary function cannot be identified with the matrix element of a bounded operator in the over-complete $|z\rangle$ states. In fact $(1/\pi)\langle z|\hat{\rho}|z\rangle$ thus obtained will only be a projection of $\rho_A(z)$ which can be identified in this way.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor E. Wolf for many helpful discussions. We are indebted to Dr. Y. Kano, Dr. J. R. Klauder, Dr. J. McKenna, and Dr. D. G. Currie for making available to us copies of their papers prior to publication.

[14] An alternative algorithm for determining the operator from its diagonal matrix elements in the $|z\rangle$ representation has also been given by J. R. Klauder, J. Math. Phys. **5**, 177 (1964), Eq. (23).