ABHYANKAR, S. S.
Math. Annalen 170, 87—144 (1967)

Nonsplitting of Valuations in Extensions of Two Dimensional Regular Local Domains*

SHREERAM SHANKAR ABHYANKAR

Introduction

Let R be a two dimensional regular local domain such that the residue field of R is an algebraically closed field of characteristic $p \neq 0$. In connection with our proof of the theorem of resolution of singularities of arithmetical surfaces, in [4] and [6] we developed an algorithm dealing with a monic polynomial of degree p in an indeterminate Z with coefficients in R. As a consequence of this algorithm, in Theorems 4.23, 4.24 and 4.25 of § 4 we prove some results concerning the nonsplitting of valuations in extensions of two dimensional regular local domains which include the following: Let L be a normal extension of the quotient field K of R such that $[L:K]$ is a power of p, let w be a real valuation of K dominating R such that the residue field of w coincides with the residue field of R and w has only one extension to L, and let R_j be the j^{th} quadratic transform of R along w. Assume that if K is of characteristic zero then K contains a primitive p^{th} root of 1 and a $(p-1)^{\text{th}}$ root of p. Then there exists a positive integer e such that for all $j \geqq e$ we have that the exceptional divisor in R_j does not split in L, i.e., upon letting S_j be the quotient ring of R_j with respect to $M_{j-1}R_j$ where M_{j-1} is the maximal ideal in R_{j-1}, we have that the integral closure of S_j in L has only one maximal ideal.

The results on the nonsplitting of valuations proved in § 4 should be compared with [1]. In turn, using these results on the nonsplitting of valuations, in § 6 to § 9 we develop an algorithm dealing with a monic polynomial $f(Z)$ of degree p^n in Z with coefficients in R where n is a positive integer. Some of the main results of this algorithm are summarized in § 5. The manner in which the nonsplitting is used in the algorithm is something like this: In [4] and [6] where $n=1$, in addition to paying attention to the constant term $f(0)$, from time to time we had to compare it with the coefficients of the positive powers of Z in $f(Z)$ so as to insure that they do not interfere too much with the constant term when making a translation in Z. In the case of general n, the nonsplitting automatically guarantees the lack of such interference. On the other hand, the analysis of $f(0)$ for general n, although analogous to the corresponding analysis for $n=1$, is somewhat more involved.

* This work was supported by the National Science Foundation under NSF-G-3640-50-395 at Purdue University.

The algorithm developed here plays an important role in our recently obtained solutions of the following problems in the birational geometry over ground fields of nonzero characteristic: (1) resolution of singularities of embedded surfaces; (2) birational invariance of the arithmetic genus of three dimensional nonsingular algebraic varieties; (3) resolution of singularities of three dimensional algebraic varieties.

In this paper we have included the mixed characteristic case, as far as this was possible without much special discussion. We have done so with an eye on possible applications to the birational geometry in the arithmetical case, i.e., over the ring of integers.

§ 1. Terminology

The letter Z will denote an indeterminate.

For integers $a_1, \ldots, a_n$, b we write $(a_1, \ldots, a_n) \equiv 0(b)$ to mean that a_i is divisible by b for all i, and we write $(a_1, \ldots, a_n) \not\equiv 0(b)$ to mean that a_i is not divisible by b for some i.

Let K be a field. By a *normal extension* of K we mean an algebraic extension L of K having the following property: if $f(Z)$ is any monic polynomial in Z with coefficients in K such that $f(Z)$ is irreducible in $K[Z]$ and $f(z)=0$ for some $z \in L$ then there exist elements $z_1, \ldots, z_m$ in I such that $f(Z)=(Z-z_1)\ldots$ $\ldots(Z-z_m)$. Given a prime number p, by a *p-extension* of K we mean a normal extension L of K such that $[L:K]=p^n$ for some nonnegative integer n, and by a *p-cyclic extension* of K we mean a normal extension L of K such that $[L:K]=p$.

By a *ring* we mean a commutative ring with identity. By a *domain* we mean an integral domain. A domain is said to be *normal* if it is integrally closed in its quotient field. For an ideal P in a ring R, by $\mathrm{rad}_R P$ we denote the radical of P in R. By a *prime ideal* (resp: a *maximal ideal*) in a ring R we mean an ideal P in R such that R/P is a domain (resp: a field); note that then $P \neq R$. By a *minimal prime ideal* in a domain R we mean a nonzero prime ideal P in R such that there does not exist any nonzero prime ideal Q in R for which $Q \subset P$ and $Q \neq P$. For a prime ideal P in a domain R, the quotient ring of R with respect to P is denoted by R_P, i.e., R_P is the set of all elements in the quotient field of R which can be expressed in the form x/y with $x \in R$, $y \in R$, $y \notin P$. Given domains R and S we say that S is a *spot* over R if R is a subring of S and there exists a finite number of elements $x_1, \ldots, x_n$ in S and a prime ideal P in $R[x_1, \ldots, x_n]$ such that $S=(R[x_1, \ldots, x_n])_P$. By a *pseudogeometric ring* we mean a noetherian ring R such that for every prime ideal P in R and every finite algebraic extension L of the quotient field of R/P we have that the integral closure of R/P in L is a finite (R/P)-module.

Given polynomials $f(Z)$ and $g(Z)$ in Z with coefficients in a ring S and given a subring R of S, we say that $g(Z)$ is an *R-translate* of $f(Z)$ if $g(Z)=f(Z+r)$ for some $r \in R$. Given elements $X_1, \ldots, X_n, Y$ in a ring R, we say that Y is an *R-monomial* in $(X_1, \ldots, X_n)$ if there exists a unit D in R and nonnegative integers $m(1), \ldots, m(n)$ such that $Y=DX_1^{m(1)} \ldots X_n^{m(n)}$.

By a *quasilocal ring* we mean a ring having exactly one maximal ideal. A subset J of a quasilocal ring R is said to be a *coefficient set* for R if J contains 0 and 1 and for every $x \in R$ there exists a unique $x' \in J$ such that $x - x' \in M$ where M is the maximal ideal in R. Given quasilocal rings R and S we say that S *dominates* R if R is a subring of S and $N \cap R = M$ where M and N are the maximal ideals in R and S respectively. Given quasilocal domains R and S such that S dominates R, we say that S is *residually algebraic* (resp: *residually rational*) over R, if upon letting h be the canonical epimorphism of S onto S/N where N is the maximal ideal in S, we have that $h(S)$ is algebraic over $h(R)$ (resp: $h(S) = h(R)$).

By a *local ring* we mean a noetherian quasilocal ring. The dimension of a local ring R is denoted by $\dim R$, i.e., $\dim R$ is the greatest integer n such that there exists a sequence $P_0 \subset P_1 \subset \cdots \subset P_n$ of distinct prime ideals in R. A local ring R is said to be *regular* if the maximal ideal in R is generated by n elements where $n = \dim R$. The completion of a local ring R is regarded to be an overring of R.

Let w be a valuation of a field K. The valuation ring of w is denoted by R_w and the maximal ideal in R_w is denoted by M_w, i.e., R_w (resp: M_w) is the set of all elements x in K such that $w(x) \geqq 0$ (resp: $w(x) > 0$). Given a quasilocal ring R, we say that w *dominates* R if R_w dominates R. Given a quasilocal ring R such that w dominates R, we say that w is *residually algebraic* (resp: *residually rational*) over R if R_w is residually algebraic (resp: residually rational) over R. w is said to be *discrete* if the value group of w is an infinite cyclic group. Note that w is discrete if and only if R_w is a one dimensional regular local domain. w is said to be *real* (resp: *rational*) if the value group of w is order isomorphic to a subgroup of the additive group of real (resp: rational) numbers. w is said to be *irrational* if w is real but not rational. Note that the following four conditions are equivalent: (1) w is real; (2) $M_w \neq \{0\}$ and M_w is the only nonzero prime ideal in R_w; (3) $R_w \neq K$ and there does not exist any subring of K containing R_w which is different from K and different from R_w; (4) given any nonzero elements x and y in M_w there exists a positive integer n such that $nw(y) \geqq w(x)$, i.e., $y^n/x \in R_w$. Elements $a_1, \ldots, a_q$ in an additive abelian group are said to be *rationally dependent* if there exist integers $m_1, \ldots, m_q$ such that $m_1 a_1 + \cdots + m_q a_q = 0$ and $m_i \neq 0$ for some i; $a_1, \ldots, a_q$ are said to be *rationally independent* if they are not rationally dependent. Note that the following three conditions are equivalent: (1′) w is rational; (2′) if a and b are any elements in the value group of w then a and b are rationally dependent; (3′) if x and y are any nonzero elements in M_w then there exist positive integers m and n such that $w(x^m) = w(y^n)$.

Let R be a regular local domain with maximal ideal M and quotient field K. For any $0 \neq z \in K$, upon taking nonzero elements x and y in R for which $z = x/y$, we define: $\operatorname{ord}_R z = a - b$ where a and b are the greatest integers such that $x \in M^a$ and $y \in M^b$ (note that since R is regular, $\operatorname{ord}_R z$ is uniquely determined by R and z); we also define: $\operatorname{ord}_R 0 = \infty$. Note that if $\dim R > 0$ then ord_R is a discrete valuation of K dominating R.

Let x be an element in a domain R such that xR is a prime ideal in R, and S is a one dimensional regular local domain where $S = R_{xR}$; note that then $x \neq 0$ and $\mathrm{ord}_S x = 1$; also note that for any $0 \neq z \in R$ if b is the greatest integer such that $z/x^b \in R$ then $z/x^b \notin xR$ and hence z/x^b is a unit in S and hence $\mathrm{ord}_S z = b$; we define: $\mathrm{ord}_{xR} = \mathrm{ord}_S$; (the more suggestive and typographically more convenient notation ord_{xR} should be understood to stand for a more logical notation like $\mathrm{ord}_{x,R}$).

Let R be a ring and let $x \in R$ such that R/xR is a regular local domain; for any $z \in R$ we define: $\mathrm{ord}_{R/x} z = \mathrm{ord}_{h(R)} h(z)$ where h is the canonical epimorphism of R onto R/xR.

Note that if R is a regular local domain then for any $x \in R$ with $\mathrm{ord}_R x = 1$ we have that R/xR is a regular local domain. Also note that if R is a normal noetherian domain and x is a nonzero element in R such that xR is a prime ideal in R then R_{xR} is a one dimensional regular local domain.

Let R be a two dimensional regular local domain, let J be a coefficient set for R, and let (x, y) be a basis of the maximal ideal in R. Given $F \in R$ there exist unique elements $F(i,j)$ in J for all nonnegative integers i, j such that $F = \Sigma F(i,j) \times x^i y^j$ in the completion of R where the sum is over all nonnegative integers i, j; the expression $\Sigma F(i,j) x^i y^j$ is called the *expansion of* F *in* $J[[x, y]]$, and the element $F(i,j)$ is called the coefficient of $x^i y^j$ in the expansion of F in $J[[x, y]]$. In § 7 and § 8 we shall tacitly use the elementary observations concerning expansions made in [6: Lemma 1].

§ 2. Nonsplitting

Definition 2.1. Let R be a normal quasilocal domain with maximal ideal M and quotient field K. Let L be an algebraic extension of K and let S be the integral closure of R in L. Recall that S contains at least one maximal ideal, and if N is any maximal ideal in S then $N \cap R = M$ and $h(S)$ is an algebraic extension of $h(R)$ where h is the canonical epimorphism of S onto S/N. Also recall that if $[L:K]$ is finite then S contains at most a finite number of maximal ideals. We say that R *splits* in L if S is not quasilocal. We say that R is *totally ramified* in L if R does not split in L and $h(S)$ is purely inseparable over $h(R)$ where N is the maximal ideal in S and h is the canonical epimorphism of S onto S/N; note that for a field H of characteristic zero, H is the only overfield of H which is regarded to be purely inseparable over H. Note that if L is purely inseparable over K then automatically R is totally ramified in L.

Given a valuation w of a field K and given an algebraic extension L of K we say that w *splits* in L if R_w *splits* in L, and we say that w is *totally ramified* in L if R_w is totally ramified in L.

Definition 2.2. Let w be a valuation of a field K and let

$$f(Z) = Z^m + F + \sum_{i=1}^{m-1} f_i Z^{m-i}, \quad f_i \in K, \quad F \in K, \quad m > 0.$$

We say that $f(Z)$ is of *prenonsplitting-type* relative to w provided there exists $t_i \in R_w$ such that $f_i^m = t_i F^i$ for $0 < i < m$. We say that $f(Z)$ is of *preramified-type* relative to w provided the following three conditions hold: (1) $f(Z)$ is of prenonsplitting-type relative to w; (2) R_w/M_w is of characteristic $p \neq 0$ and m is power of p; (3) if there exists $G \in K$ and a unit G' in R_w such that $F = G'G^m$ then there exists $t_i \in M_w$ such that $f_i^m = t_i F^i$ for $0 < i < m$. We say that $f(Z)$ is of *nonsplitting-type* relative to w provided every K-translate of $f(Z)$ is of prenonsplitting-type relative to w. We say that $f(Z)$ is of *ramified-type* relative to w provided every K-translate of $f(Z)$ is of preramified-type relative to w.

Lemma 2.3. *Let K be a field, let L be a normal extension of K, let w be a valuation of K such that w does not split in L, and let $f(Z)$ be a monic polynomial of degree $m > 0$ in Z with coefficients in K such that $f(Z)$ is irreducible in $K[Z]$ and $f(z) = 0$ for some $z \in L$. Then $f(Z)$ is of nonsplitting-type relative to w.*

Proof. Let S be the integral closure of R_w in L and let N be the maximal ideal in S. For any K-automorphism g of L we clearly have $g(S) = S$ and hence $g(N) = N$. Given any K-automorphism g of L and any nonzero element y in L let

$$\delta_j = \prod_{k=0}^{j-1} g^k(g(y)/y).$$

By induction we see that $g^j(y) = y\delta_j$ for all $j > 0$. Now $g^n(y) = y$ for some $n > 0$ and then $y = y\delta_n$, i.e., $\delta_n = 1$. Since $g(N) = N$ we get that if $g(y)/y \in N$ then $\delta_j \in N$ for all $j > 0$; since $\delta_n = 1$, we must have $g(y)/y \notin N$. Upon replacing g by g^{-1} we get that $g^{-1}(y)/y \notin N$. Since $g^{-1}(N) = N$ we get that if $y/g(y) \in N$ then $g^{-1}(y)/y = g^{-1}(y/g(y)) \in N$ which is a contradiction; therefore $y/g(y) \notin N$. Thus $g(y)/y \notin N$ and $y/g(y) \notin N$; now S is the valuation ring of a valuation of L and hence we conclude that $g(y)/y$ is a unit in S. Thus we have shown that *for every $0 \neq y \in L$ and every K-automorphism g of L, $g(y)/y$ is a unit in S.*

We want to show that for any K-translate $f'(Z)$ of $f(Z)$ we have that $f'(Z)$ is of prenonsplitting-type relative to w. Now L is a normal extension of K, $f'(Z)$ is a monic polynomial of degree $m > 0$ in Z with coefficients in K, $f'(Z)$ is irreducible in $K[Z]$, and $f'(x) = 0$ for some $x \in L$. Therefore there exist elements $x_1, \ldots, x_m$ in L such that $f'(Z) = (Z - x_1) \ldots (Z - x_m)$, and there exist K-automorphisms $g_1, \ldots, g_m$ of L such that $g_i(x) = x_i$ for $1 \leqq i \leqq m$. If $m = 1$ then $f'(Z)$ is obviously of prenonsplitting-type relative to w. So now assume that $m > 1$. Then $x \neq 0 \neq f'(0)$. By the above italicized remark we get that $x_i = s_i x$ where s_i is a unit in S for $1 \leqq i \leqq m$. Let $F = f'(0)$ and let f_i be the coefficient of Z^{m-i} in $f'(Z)$. Then $F = (-1)^m x_1 \ldots x_m$, and f_i is a symmetric function in $x_1, \ldots, x_m$ of degree i for $0 < i < m$. Therefore $F = rx^m$ where r is a unit in S, and $f_i = r_i x^i$ where $r_i \in S$ for $0 < i < m$. Therefore $f_i^m = t_i F^i$ where $t_i \in S$ for $0 < i < m$. Now $t_i = f_i^m / F^i \in K$ and $K \cap S = R_w$; therefore $t_i \in R_w$ for $0 < i < m$. Consequently $f'(Z)$ is of prenonsplitting-type relative to w.

Lemma 2.4. *Let R be a normal quasilocal domain with quotient field K and maximal ideal M such that R/M is of characteristic $p \neq 0$. Let L be a normal extension of K such that R is totally ramified in L. Let $f(Z)$ be a monic polynomial of degree $m = p^n$ in Z with coefficients in R, where n is a positive integer, such that $f(Z)$ is irreducible in $K[Z]$ and $f(z) = 0$ for some $z \in L$. Then $f(Z) - Z^m - - f(0) \in M[Z]$.*

Proof. Let S be the integral closure of R in L, let N be the maximal ideal in S, and let h be the canonical epimorphism of S onto S/N. If g is any K-automorphism of L then $g(S)=S$ and $g(N)=N$, and we get an $h(R)$-automorphism g' of $h(S)$ by taking $g'(h(y))=h(g(y))$ for all $y\in S$; since $h(S)$ is purely inseparable over $h(R)$ we get that g' is the identity map of $h(S)$ and hence $h(y)=h(g(y))$ for all $y\in S$. Now L is a normal extension of K, $f(Z)$ is irreduzible in $K[Z]$, and $f(z)=0$ with $z\in L$. Therefore there exist elements $z_1,\dots,z_m$ in L such that $f(Z)=(Z-z_1)\dots(Z-z_m)$, and there exist K-automorphisms $g_1,\dots,g_m$ of L auch that $g_i(z)=z_i$ for $1\leqq i\leqq m$. Since $f(Z)\in R[Z]$ we get that $z\in S$, and hence $h(z)=h(z_i)$ for $1\leqq i\leqq m$. Therefore upon letting $f'(Z)$ be the polynomial in Z obtained from $f(Z)$ by applying h to the coefficients of $f(Z)$ we get that $f'(Z)=(Z-h(z_1))\dots(Z-h(z_m))=(Z-h(z))^m=Z^m-h(z)^m$. Therefore $f(Z)-$ $-Z^m-f(0)\in M[Z]$.

Lemma 2.5. *Let K be a field, let L be a normal extension of K, let w be a valuation of K such that R_w/M_w is of characteristic $p\neq 0$ and w is totally ramified in L, and let $f(Z)$ be a monic polynomial of degree $m=p^n$ in Z with coefficients in K, where n is a positive integer, such that $f(Z)$ is irreducible in $K[Z]$ and $f(z)=0$ for some $z\in L$. Then $f(Z)$ is of ramified-type relative to w.*

Proof. We want to show that for any K-translate $f'(Z)$ of $f(Z)$ we have that $f'(Z)$ is of preramified-type relative to w. Now $f'(Z)=Z^m+f_1Z^{m-1}+\dots+$ $+f_{m-1}Z+F$ where $f_1,\dots,f_{m-1}$, F are elements in K, $f'(Z)$ is irreducible in $K[Z]$, and $f'(x)=0$ for some $x\in L$. By Lemma 2.3, $f'(Z)$ is of prenonsplitting-type relative to w and hence there exists $t_i\in R_w$ such that $f_i^m=t_iF^i$ for $0<i<m$. Therefore it suffices to show that if $F=F^*G^m$ where $G\in K$ and F^* is a unit in R_w then $t_i\in M_w$ for $0<i<m$. Now $G\neq 0$ because $f'(Z)$ is irreducible in $K[Z]$ and $m>1$. Let $x^*=x/G$, $f_i^*=f_i/G^i$ for $0<i<m$, and $f^*(Z)=Z^m+$ $+f_1^*Z^{m-1}+\dots+f_{m-1}^*Z+F^*$. Then $f^*(Z)\in R_w[Z]$, $f^*(Z)$ is irreducible in $K[Z]$, $x^*\in L$, and $f^*(x^*)=0$. Therefore by Lemma 2.4 we get that $f_i^*\in M_w$ for $0<i<m$. Now $t_i=f_i{}^m/F^i=f_i^{*m}/F^{*i}$ and hence $t_i\in M_w$ for $0<i<m$.

In the proofs of Lemma 2.7, 2.8, 2.9 we shall use the following well known result; for a proof see for instance [11: § 7 and § 8].

Lemma 2.6. *Let R be a one dimensional regular local domain with quotient field K, let L be a finite algebraic extension of K, let T be the integral closure of R in K, let $P_1,\dots,P_n$ be the distinct maximal ideals in T, let $S_i=T_{P_i}$, let $N_i=P_iS_i$, and let h_i be the canonical epimorphism of S_i onto S_i/N_i. Then for $1\leqq i\leqq n$ we have that S_i is a one dimensional regular local domain and there exists a unique positive integer e_i such that $\operatorname{ord}_{S_i}x=e_i\operatorname{ord}_R x$ for all $x\in K$*: (e_i is called the reduced ramification index of S_i over R). *Furthermore*

$$\sum_{i=1}^{n} e_i[h_i(S_i):h_i(R)]\leqq[L:K],$$

and equality holds if and only if T is a finite R-module.

Lemma 2.7. *Let R be a one dimensional regular local domain with quotient field K, and let $f(Z)=Z^p+f_1Z^{p-1}+\dots+f_{p-1}Z+F$ where $f_1,\dots,f_{p-1}$, F are elements in K and p is a positive integer. Let z be an element in an overfield* *of K such that $f(z)=0$ and let $L=K(z)$. Assume that $F \neq 0$, the greatest common divisor of p and $\operatorname{ord}_R F$ is one, and $\operatorname{ord}_R f_i \geqq (i/p)\operatorname{ord}_R F$ for $0<i<p$. Then $[L:K]=p$ and R is totally ramified in L.*

Proof. Clearly $[L:K] \leqq p$. Let T be the integral closure of R in L, let P be a maximal ideal in T, let $S=T_P$, and let e be the reduced ramification index of S over R. Since $\operatorname{ord}_R f_i \geqq (i/p)\operatorname{ord}_R F$ we get that $\operatorname{ord}_S f_i \geqq (i/p)\operatorname{ord}_S F$ for $0<i<p$. Therefore, if $\operatorname{ord}_S z<(1/p)\operatorname{ord}_S F$ then $\operatorname{ord}_S F>\operatorname{ord}_S z^p$ and $\operatorname{ord}_S f_i z^{p-i}>$ $>\operatorname{ord}_S z^p$ for $0<i<p$, and hence $\operatorname{ord}_S f(z)=\operatorname{ord}_S(z^p+f_1 z^{p-1}+\cdots+f_{p-1}z+F)$ $=\operatorname{ord}_S z^p<\operatorname{ord}_S F$ which is a contradiction because $f(z)=0 \neq F$. Again, if $\operatorname{ord}_S z>(1/p)\operatorname{ord}_S F$ then $\operatorname{ord}_S z^p>\operatorname{ord}_S F$ and $\operatorname{ord}_S f_i z^{p-i}>\operatorname{ord}_S F$ for $0<i<p$, and hence $\operatorname{ord}_S f(z)=\operatorname{ord}_S(z^p+f_1 z^{p-1}+\cdots+f_{p-1}z+F)=\operatorname{ord}_S F$ which is a contradiction because $f(z)=0 \neq F$. Therefore $\operatorname{ord}_S z=(1/p)\operatorname{ord}_S F$ and hence $\operatorname{ord}_S z=(e/p)\operatorname{ord}_R F$. Since the greatest common divisor of p and $\operatorname{ord}_R F$ is one we get that $e \geqq p$. Since $[L:K] \leqq p$, by Lemma 2.6 we conclude that R is totally ramified in L and $[L:K]=p$.

Lemma 2.8. *Let R be a two dimensional regular local domain with quotient field K, let (x, y) be a basis of the maximal ideal in R, and let $f(Z)=Z^p+$ $+f_1 Z^{p-1}+\cdots+f_{p-1}Z+F$ where p is a prime number and $f_1, \ldots, f_{p-1}, F$ are elements in R. Let $a=\operatorname{ord}_{xR}F$. Let z be an element in an overfield of K such that $f(z)=0$ and let $L=K(z)$. Assume that $p \in xR$, $F \neq 0$, $\operatorname{ord}_{xR} f_i>ai/p$ for $0<i<p$, and $(a, \operatorname{ord}_{R/x}F/x^a) \not\equiv 0(p)$. Then $[L:K]=p$ and ord_{xR} is totally ramified in L.*

Proof. If $a \not\equiv 0(p)$ then our assertion follows from Lemma 2.7. Now assume that $a \equiv 0(p)$. Let R' be the valuation ring of ord_{xR}, i.e., $R'=R_{xR}$. Let T be the integral closure of R' in L, let P be a maximal ideal in T, let $S=T_P$, let $N=PS$, and let h be the canonical epimorphism of S onto S/N. Let $z'=z/x^{a/p}$, $F'=F/x^a$, and $f_i'=f_i/x^{ai/p}$ for $0<i<p$. Then $z' \in L$, $F' \in R$, $f_i' \in R$ for $0<i<p$, and $z'^p+f_1' z'^{p-1}+\cdots+f_{p-1}' z'+F'=0$. Therefore $z' \in T$ and hence $z' \in S$. Now $\operatorname{ord}_{xR} f_i'>0$ and hence $h(f_i')=0$ for $0<i<p$. Therefore $h(z')^p+h(F')=0$. Clearly $h(R)$ is a one dimensional regular local domain with quotient field $h(R')$ and $\operatorname{ord}_{h(R)} h(F')=\operatorname{ord}_{R/x} F/x^a \not\equiv 0(p)$. Therefore $h(z') \notin h(R')$ and $h(z')^p \in h(R')$. Since $p \in xR$ we get that $h(R')(h(z'))$ is purely inseparable over $h(R')$ and $[h(R')(h(z'))$ $:h(R')]=p$. Now $[L:K] \leqq p$ and hence by Lemma 2.6 we conclude that $[L:K]=p$, P is the only maximal ideal in T, and $h(S)=h(R')(h(z'))$. Therefore ord_{xR} is totally ramified in L.

Lemma 2.9. *Let R be a two dimensional regular local domain with quotient field K, let (x, y) be a basis of the maximal ideal in R, and let $f(Z)=Z^p+$ $+f_1 Z^{p-1}+\cdots+f_{p-1}Z+F$ where p is a prime number and $f_1, \ldots, f_{p-1}, F$ are elements in R. Let z be an element in an overfield of K such that $f(z)=0$ and let $L=K(z)$. Assume that there exist nonnegative integers a and b such that $F/(x^a y^b)$ is a unit in R and $(a, b) \not\equiv 0(p)$. Then we have the following.* (1) *If $\operatorname{ord}_{xR} f_i \geqq ai/p$ and $\operatorname{ord}_{yR} f_i \geqq bi/p$ for $0<i<p$ then $[L:K]=p$, ord_{xR} does not split in L and ord_{yR} does not split in L.* (2) *If $p \in xR$ and $\operatorname{ord}_{xR} f_i>ai/p$ for $0<i<p$ then $[L:K]=p$ and ord_{xR} is totally ramified in L.* (3) *If $p \in yR$ and $\operatorname{ord}_{yR} f_i>bi/p$ for $0<i<p$ then $[L:K]=p$ and ord_{yR} is totally ramified in L.*

Proof. Clearly $F \neq 0$, $\operatorname{ord}_{xR} F = a$, and $\operatorname{ord}_{R/x} F/x^a = b$. Therefore (2) follows from Lemma 2.8, and by symmetry (3) follows from (2). To prove (1) assume that $\operatorname{ord}_{xR} f_i \geqq ai/p$ and $\operatorname{ord}_{yR} f_i \geqq bi/p$ for $0 < i < p$. By symmetry it suffices to show that $[L:K] = p$ and ord_{xR} does not split in L. If $a \not\equiv 0(p)$ then by Lemma 2.7 we get that $[L:K] = p$ and ord_{xR} is totally ramified in L. So now assume that $a \equiv 0(p)$. Then $b \not\equiv 0(p)$. Let R' be the valuation ring of ord_{xR}, i.e., $R' = R_{xR}$. Let T be the integral closure of R' in L, let P be a maximal ideal in T, let $S = T_P$, let $N = PS$, and let h be the canonical epimorphism of S onto S/N. Let $z' = z/x^{a/p}$, $F' = F/x^a$, and $f_i' = f_i/x^{ai/p}$ for $0 < i < p$. Then $z' \in L$, $F' \in R$, F'/y^b is a unit in R, and $f_i' \in R$ and $\operatorname{ord}_{yR} f_i' \geqq bi/p$ for $0 < i < p$. Also $z'^p + f_1' z'^{p-1} + \cdots + f_{p-1}' z' + F' = 0$ and hence $z' \in T$. Now $h(R)$ is a one dimensional regular local domain with quotient field $h(R')$. Also $\operatorname{ord}_{h(R)} h(y) = 1$ and hence $\operatorname{ord}_{h(R)} h(F') = b$ and $\operatorname{ord}_{h(R)} h(f_i') \geqq bi/p$ for $0 < i < p$. Now $h(z')^p + h(f_1')\, h(z')^{p-1} + \cdots + h(f_{p-1}')\, h(z') + h(F') = 0$ and hence by Lemma 2.7 we get that $[h(R')(h(z')) : h(R')] = p$ and hence $[h(S) : h(R')] \geqq p$. However $[L:K] \leqq p$ and hence by Lemma 2.6 we conclude that $[L:K] = p$ and P is the only maximal ideal in T, i.e., ord_{xR} does not split in L.

Lemma 2.10. *Let R be a normal quasilocal domain with quotient field K and let L be a p-extension of K where p is some prime number. Then R does not split (resp; R is totally ramified) in L if and only if for every subfield K^* of L which is a separable p-cyclic extension of K we have that R does not split (resp; R is totally ramified) in K^*.*

Proof. The "only if" part is obvious. To prove the "if" part let L^* be the maximal separable extension of K in L and let H be the set of all subfields K^* of L^* such that K^* is a p-cyclic extension of K. Then L^* is a separable p-extension of K, and L is a purely inseparable extension of L^*. Now every normal quasilocal domain with quotient field L^* is totally ramified in L and hence it suffices to show that if for every $K^* \in H$ we have that R does not split (resp: R is totally ramified) in K^* then R does not split (resp: R is totally ramified) in L^*. Let G be the group of all K-automorphisms of L^*. Then G is a p-group, i.e., a finite group whose order is a power of p. Let M be the maximal ideal in R, let S be the integral closure of R in L^*, let N be a maximal ideal in S, and let G_s be splitting group of N over M, i.e., G_s is the set of all elements g in G such that $g(N) = N$. Assume that for every $K^* \in H$ we have that R does not split in K^*. Suppose if possible that $G_s \neq G$. Since G is a p-group there exists a normal subgroup G' of G such that $G_s \subset G'$ and G/G' is of order p (for instance see [12: pp. 110–111]). Let K' be the fixed field of G' and let R' be the integral closure of R in K'. Then $K' \in H$ and hence R' is quasilocal. Now $G \neq G'$ and hence we can take $g_1 \in G$ such that $g_1 \notin G'$. Let $N_1 = g_1(N)$. Then N_1 is a maximal ideal in S. Now R' is a normal quasilocal domain with quotient field K', L^* is a finite normal extension of K', G' is the group of all K'-automorphisms of L^*, S is the integral closure of R' in L^*, and N and N_1 are maximal ideals in S; therefore there exists $g_2 \in G'$ such that $g_2(N_1) = N$ (for instance see [3: Proposition 1.25]). Now $g_2 g_1(N) = N$ and hence $g_2 g_1 \in G_s$. Since $G_s \subset G'$ we get that $g_2 g_1 \in G'$. This is a contradiction because G' is a subgroup of G, $g_1 \notin G'$, and $g_2 \in G'$. Therefore

$G_s = G$ and hence by [3: Proposition 1.46] we get that N is the only maximal ideal in S, i.e., R does not split in L^*. Now assume that R is totally ramified in K^* for all $K^* \in H$. Let G_i be the inertia group of N over M, i.e., G_i is the set of all g in G such that $x - g(x) \in N$ for all $x \in S$. Let K_i be the fixed field of G_i, let $R_i = S \cap K_i$, and let h be the canonical epimorphism of S onto S/N. By [3: Theorem 1.48] we get that $h(S)$ is purely inseparable over $h(R_i)$, $h(R_i)$ is separable over $h(R)$, and $[h(R_i):h(R)] = [K_i:K]$. Suppose if possible that $h(S)$ is not purely inseparable over $h(R)$. Then $G_i \neq G$. Since G is a p-group there exists a normal subgroup G'' of G such that $G_i \subset G''$ and G/G'' is of order p (for instance see [12: pp. 110–111]). Let K'' be the fixed field of G'' and let $R'' = S \cap K''$. Then $K \subset K'' \subset K_i$ and hence $h(R) \subset h(R'') \subset h(R_i)$; since $h(R_i)$ is separable over $h(R)$ we get that $h(R_i)$ is separable over $h(R'')$ and $h(R'')$ is separable over $h(R)$; therefore by [3: Theorem 1.45] we get that $[h(R_i):h(R'')] \leqq \leqq [K_i:K'']$ and $[h(R''):h(R)] \leqq [K'':K]$; since $[h(R_i):h(R)] = [K_i:K]$ we must have $[h(R''):h(R)] = [K'':K] = p$ and hence $h(R'')$ is not purely inseparable over $h(R)$. This is a contradiction because $K'' \in H$. Therefore $h(S)$ is purely inseparable over $h(R)$ and hence R is totally ramified in L^*.

§ 3. Quadratic transforms

For the definition and properties of quadratic transforms see [2: § 2] and [7: § 3]. Let R be a two dimensional regular local domain, let (x, y) be a basis of the maximal ideal M in R, and let w be a valuation of the quotient field K of R such that w dominates R and w is residually algebraic over R. Note the following.

3.1. Let R' be a quadratic transform of R. Then R' is a regular local domain with quotient field K, R' dominates R, $\dim R' = 1$ or 2, and $\dim R' = 2$ if and only if R' is residually algebraic over R. If $\dim R' = 2$ then: R' is an n^{th} quadratic transform of R for a unique nonnegative integer n, R' contains a unique i^{th} quadratic transform R_i of R for $0 \leqq i \leqq n$, $R_0 = R$, $R_n = R'$, and R_i is an immediate quadratic transform of R_{i-1} for $0 < i \leqq n$.

3.2. For any nonnegative integer n there is a unique n^{th} quadratic transform R' of R such that w dominates R'; R' is called the n^{th} quadratic transform of R along w. By a quadratic transform of R along w is meant a quadratic transform R' of R such that w dominates R'.

3.3. Let R' be a two dimensional local domain which is an immediate quadratic transform of R and let M' be the maximal ideal in R'. Then either $x/y \in R'$ or $y/x \in R'$. If $y/x \in R'$ then there exists a monic polynomial $f(Z)$ in Z with coefficients in R such that $(x, f(y/x))R' = M'$. If $y/x \in R'$ then $R'_{xR'}$ is the valuation ring of ord_R (for instance see the proof of [7: Proposition 9]) and hence $\operatorname{ord}_R = \operatorname{ord}_{xR'}$. Now suppose that $y/x \in M'$. Let $y' = y/x$, $A = R[y']$, $P = (x, y')A$. Then $R' = A_P$ and $M' = (x, y')R'$. We claim that $\operatorname{ord}_{yR} = \operatorname{ord}_{y'R'}$. Since K is the quotient field of R it suffices to show that $\operatorname{ord}_{yR} z = \operatorname{ord}_{y'R'} z$ for all $z \in R$. Clearly $yR \subset y'R'$ and hence in turn it suffices to show that if $t \in (y'R') \cap R$ then $t \in yR$. Since $t \in y'R'$ and $R' = A_P$ we can write $rt = y's$ where $r \in A$, $r \notin P$,

$s \in A$. Since r and s are in A we can write

$$r = \sum_{i=0}^{n} r_i y'^i \quad \text{and} \quad y's = \sum_{i=1}^{n} s_i y'^i \quad \text{where} \quad r_i \in R, \quad s_i \in R,$$

and n is a positive integer. Now $r - r_0 \in y'A \subset P$, $M \subset P$, and $r \notin P$; therefore we must have $r_0 \notin M$ and hence $r_0 \notin yR$; also $x^n \notin yR$ and hence $x^n r_0 \notin yR$; now

$$x^n r - x^n r_0 = \sum_{i=1}^{n} x^{n-i} r_i y^i \in yR$$

and hence $x^n r \notin yR$. However

$$x^n rt = x^n y' s = \sum_{i=1}^{n} x^{n-i} r_i y^i \in yR$$

and hence $t \in yR$.

Definition 3.4. By a *canonical* n^{th} *quadratic transform* of (R, x, y) we mean a triple (R', x', y') where R' is a two dimensional local domain which is an n^{th} quadratic transform of R and (x', y') is a basis of the maximal ideal in R' such that upon letting R_i to be the i^{th} quadratic transform of R contained in R' there exists a basis (x_i, y_i) of the maximal ideal in R_i for $0 \leqq i \leqq n$ such that: x and y are R_0-monomials in (x_0, y_0), $M_{i-1}R_i = x_i R_i$ or $M_{i-1}R_i = y_i R_i$ for $0 < i \leqq n$, x_{i-1} and y_{i-1} are R_i-monomials in (x_i, y_i) for $0 < i \leqq n$, and $x' = x_n$ and $y' = y_n$. By a *canonical quadratic transform* of (R, x, y) we mean a triple (R', x', y') which is a canonical n^{th} quadratic transform of (R, x, y) for some nonnegative integer n. By a *canonical* n^{th} *quadratic transform* of (R, x, y) *along* w we mean a canonical n^{th} quadratic transform (R', x', y') of (R, x, y) such that w dominates R'. By a *canonical quadratic transform* of (R, x, y) *along* w we mean a canonical quadratic transform (R', x', y) of (R, x, y) such that w dominates R'. Note the following: (1) If R' is a two dimensional local domain which is a quadratic transform of R then there exists a basis (x', y') of the maximal ideal in R' such that (R', x', y') is a canonical quadratic transform of (R, x, y). (2) If (R', x', y') is a canonical quadratic transform of (R, x, y) then x and y are R'-monomials in (x', y'). (3) If (R', x', y') is a canonical quadratic transform of (R, x, y) and (R'', x'', y'') is a canonical quadratic transform of (R', x', y') then (R'', x'', y'') is a canonical quadratic transform of (R, x, y).

Definition 3.5. Now assume that R/M is algebraically closed and let J be a coefficient set for R. Given a two dimensional local domain R' which is an n^{th} quadratic transform of R there then exist unique nonzero elements $x_0, y_0, x_1, y_1, \ldots, x_n, y_n$ in the maximal ideal of R' such that $x_0 = x$ and $y_0 = y$ and such that for $0 < i \leqq n$ we have: if $y_{i-1}/x_{i-1} \in R'$ then $x_{i-1} = x_i$ and $y_{i-1} = x_i(y_i + t_i)$ with $t_i \in J$, and if $y_{i-1}/x_{i-1} \notin R'$ then $x_{i-1} = x_i y_i$ and $y_{i-1} = y_i$; note that the elements t_i are also uniquely determined, and (x_i, y_i) is a basis of the maximal ideal in R_i for $0 \leqq i \leqq n$ where R_i is the i^{th} quadratic transform of R contained in R'. By a *canonical* n^{th} *quadratic transform* of (R, x, y, J) we mean a triple (R', x', y') where R' is a two dimensional local domain which is an n^{th} quadratic transform of R and $x' = x_n$ and $y' = y_n$ where x_n and y_n are as defined above.

By a *canonical quadratic transform* of (R, x, y, J) we mean a triple (R', x', y') which is a canonical n^{th} quadratic transform of (R, x, y, J) for some nonnegative integer n. For any nonnegative integer n clearly there exists a unique canonical n^{th} quadratic transform (R', x', y') of (R, x, y, J) such that w dominates R'; (R', x', y') is called *the canonical n^{th} quadratic transform* of (R, x, y, J) *along* w. By a *canonical quadratic transform* of (R, x, y, J) *along* w we mean a triple (R', x', y') which is the canonical n^{th} quadratic transform of (R, x, y, J) along w for some nonnegative integer n. Note the following: (1) If (R', x', y') is a canonical quadratic transform of (R, x, y, J) then (R', x', y') is a canonical quadratic transform of (R, x, y). (2) If (R', x', y') and (R'', x'', y'') are canonical quadratic transforms of (R, x, y, J) such that $R' \subset R''$ then (R'', x'', y'') is a canonical quadratic transform of (R', x', y', J).

Lemma 3.6. *Let R_n be the n^{th} quadratic transform of R along w. Then* $\bigcup_{n=0}^{\infty} R_n = R_w$.

Proof. See [2: Lemma 12].

We shall now give a slightly sharper version of [2: Theorem 2]. (We take this opportunity to make the following correction to [2]. Line 21 on page 342 of [2] which reads "let $P_i \dots$ and" should be replaced by "let $P_i = M_w \cap R_{i-1}$ where w is the real discrete valuation of K with which u is composed. Since w is nontrivial, $P_i \neq (0)$. Since $x_i \notin M_w \cap R_i$ and").

Lemma 3.7. *Assume that w is real and let $f_1, \dots, f_q$ be any finite number of nonzero elements in R_w. Then there exists a nonnegative integer n_0 such that for any $n \geqq n_0$ and any canonical n^{th} quadratic transform (R', x', y') of (R, x, y) along w we have that $f_1, \dots, f_q$ are R'-monomials in (x', y').*

Proof. Let R_n be the n^{th} quadratic transform of R along w and let M_n be the maximal ideal in R_n. For $n > 0$ we can take $z_n \in R_n$ such that $M_{n-1} R_n = z_n R_n$. By Lemma 3.6 there exists a nonnegative integer m such that $f_i \in R_m$ for $i = 1, \dots, q$. Let $g_m = f_1$ and by induction define $g_n \in R_n$ for all $n > m$ by the equation: $g_{n-1} = g_n z_n^{a(n)}$ where $a(n) = \operatorname{ord}_{z_n R_n} g_{n-1}$. For $n \geqq m$ let V_n be the set of all discrete valuations v of K such that $R_n \subset R_v$ and $g_n \in M_v \cap R_n \neq M_n$. Then V_n is the set of all valuations v of K such that $R_v = (R_n)_P$ for some minimal prime ideal P in R_n containing g_n. Therefore V_n is a finite set, and $V_n = \emptyset$ if and only if g_n is a unit in R_n. Suppose if possible that $\bigcap_{n=m}^{\infty} V_n \neq \emptyset$ and take $v \in \bigcap_{n=m}^{\infty} V_n$; since $R_n \subset R_v$ for all $n \geqq m$, by Lemma 3.6 we get that $R_w \subset R_v$; since w is real we must then have $R_w = R_v$ and hence v dominates R_n for all n; this is a contradiction because $M_v \cap R_n \neq M_n$ for all $n \geqq m$. Therefore $\bigcap_{n=m}^{\infty} V_n = \emptyset$. For any $n \geqq m$ let v be any element in V_n; then $M_v \cap R_n$ and $z_n R_n$ are minimal prime ideals in R_n, $g_n \in M_v \cap R_n$, and $g_n \notin z_n R_n$; therefore $z_n \notin M_v \cap R_n$; consequently $M_v \cap R_{n-1} \neq M_{n-1}$ and hence $v \in V_{n-1}$. Thus $V_n \subset V_{n-1}$ for all $n > m$. Since V_n is a finite set for all $n \geqq m$ and $\bigcap_{n=m}^{\infty} V_n = \emptyset$, we conclude that there exists an integer $n_1 > m$ such that $V_n = \emptyset$ for all $n \geqq n_1$. It follows that f_1 is an R_n-monomial in $(z_{m+1}, \ldots, z_n)$ for all $n \geqq n_1$. Similarly there exists an integer $n_i > m$ such that f_i is an R_n-monomial in $(z_{m+1}, \ldots, z_n)$ for all $n \geqq n_i$. Let $n_0 = \max(n_1, n_2, \ldots, n_q)$. For any $n \geqq n_0$ let (R', x', y') be any canonical n^{th} quadratic transform of (R, x, y) along w. Then there exists a basis (x_i, y_i) of M_i for $0 \leqq i \leqq n$ such that x and y are R_0-monomials in (x_0, y_0), $M_{i-1}R_i = x_iR_i$ or $M_{i-1}R_i = y_iR_i$ for $0 < i \leqq n$, x_{i-1} and y_{i-1} are R_i-monomials in (x_i, y_i) for $0 < i \leqq n$, and $x' = x_n$ and $y' = y_n$. For $0 < i \leqq n$, since $M_{i-1}R_i = z_iR_i$ it follows that z_i/x_i or z_i/y_i is a unit in R_i according as $M_{i-1}R_i = x_iR_i$ or $M_{i-1}R_i = y_iR_i$. Consequently $z_1, \ldots, z_n$ are R'-monomials in (x', y'). Therefore $f_1, \ldots, f_q$ are R'-monomials in (x', y').

Lemma 3.8. *Assume that $w(x)$ and $w(y)$ are rationally dependent. Let R_i be the i^{th} quadratic transform of R along w. Let W be the set of all positive integers i such that x and y are R_i-monomials in a nonzero nonunit in R_i. Then W is nonempty. Let n be the smallest integer in W. If x' and y' are any elements in R_n such that (R_n, x', y') is a canonical quadratic transform of (R, x, y) then either x and y are R_n-monomials in x', or x and y are R_n-monomials in y'. If R/M is algebraically closed, J is a coefficient set for R, and x_n and y_n are the elements in R_n such that (R_n, x_n, y_n) is a canonical quadratic transform of (R, x, y, J), then x and y are R_n-monomials in x_n.*

Proof. Since $w(x)$ and $w(y)$ are rationally dependent, there exists a positive integer n and nonzero elements $x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}$ in K such that $x_0 = x$, $y_0 = y$, $w(x_i) \neq w(y_i)$ for $0 \leqq i < n-1$, $w(x_{n-1}) = w(y_{n-1})$, and for $0 < i \leqq n-1$: if $w(y_{i-1}) > w(x_{i-1})$ then $x_{i-1} = x_i$ and $y_{i-1} = x_iy_i$, and if $w(y_{i-1}) < w(x_{i-1})$ then $x_{i-1} = x_iy_i$ and $y_{i-1} = y_i$. By induction we see that for $0 \leqq i \leqq n-1$: (x_i, y_i) is a basis of the maximal ideal in R_i and $x = x_i^{a(i)}y_i^{b(i)}$ and $y = x_i^{c(i)}y_i^{d(i)}$ where $a(i)$, $b(i)$, $c(i)$, $d(i)$ are nonnegative integers such that $a(i)d(i) - b(i)c(i) = 1$. It follows that $i \notin W$ for $0 < i \leqq n-1$. Let $x_n = x_{n-1}$. Then $M_{n-1}R_n = x_nR_n$ and x and y are R_n-monomials in x_n. Therefore W is nonempty and n is the smallest integer in W. If x' and y' are any elements in R_n such that (R_n, x', y') is a canonical quadratic transform of (R, x, y) then either $M_{n-1}R_n = x'R$ or $M_{n-1}R_n = y'R_n$; since $M_{n-1}R_n = x_nR_n$ we get that either x_n/x' is a unit in R_n or x_n/y' is a unit in R_n; therefore either x and y are R_n-monomials in x', or x and y are R_n-monomials in y'. If R/M is algebraically closed and J is a coefficient set for R then clearly (R_n, x_n, y_n) is a canonical quadratic transform of (R, x, y, J) for some $y_n \in R_n$.

Lemma 3.9. *Assume that w is rational and let $f_1, \ldots, f_q$ be any finite number of nonzero elements in R_w. Then there exists a canonical quadratic transform (R', x', y') of (R, x, y) along w such that $f_1, \ldots, f_q$ are R'-monomials in x'. If R/M is algebraically closed and J is a coefficient set for R then there exists a canonical quadratic transform (R'', x'', y'') of (R, x, y, J) along w such that $f_1, \ldots, f_q$ are R''-monomials in x''.*

Proof. Follows from Lemmas 3.7 and 3.8.

Lemma 3.10. *Assume that w is discrete and R/M is algebraically closed. Let J be a coefficient set for R and let (R_n, x_n, y_n) be the canonical n^{th} quadratic transform of (R, x, y, J) along w. Then there exists a nonnegative integer m such* *that for all $n \geqq m$ we have that $x_n = x_m$ and $x_n R_w = M_w$. Given any finite number of nonzero elements $f_1, \ldots, f_q$ in R_w there exists an integer $m' \geqq m$ such that $f_1, \ldots, f_q$ are R_n-monomials in x_n for all $n \geqq m'$.*

Proof. We can take $z \in R_w$ such that $zR_w = M_w$. By Lemma 3.6, $z \in R_d$ for some nonnegative integer d. Then $x_d R_w = M_w^a$ and $y_d R_w = M_w^b$ where a and b are positive integers such that either $a = 1$ or $b = 1$. If $a = 1$ then take $m = d$, and if $a \neq 1$ then take $m = d + a - 1$. In both cases $x_m R_w = M_w$ and hence $x_n = x_m$ for all $n \geqq m$. Now $f_i = D_i x_m^{a(i)}$ where D_i is a unit in R_w and $a(i)$ is a nonnegative integer. By Lemma 3.6 there exists an integer $m' \geqq m$ such that $D_i \in R_{m'}$ for $i = 1, \ldots, q$. For all $n \geqq m'$ then D_i is a unit in R_n and hence f_i is an R_n-monomial in x_n for $i = 1, \ldots, q$.

Lemma 3.11. *Assume that w is irrational. Then there exists a nonnegative integer m and a canonical n^{th} quadratic transform (R_n, x_n, y_n) of (R, x, y) along w for all $n \geqq m$ such that we have the following.* (1) *$(w(x_n), w(y_n))$ is a free basis of the value group of w for all $n \geqq m$.* (2) *For any $n > m$, if $w(y_{n-1}) > w(x_{n-1})$ then $x_{n-1} = x_n$ and $y_{n-1} = x_n y_n$, and if $w(y_{n-1}) < w(x_{n-1})$ then $x_{n-1} = x_n y_n$ and $y_{n-1} = y_n$.* (3) *For any $n \geqq m$, if x' and y' are any elements in R_n such that (R_n, x', y') is a canonical quadratic transform of (R, x, y) then either x_n/x' and y_n/y' are units in R_n or x_n/y' and y_n/x' are units in R_n.*

Proof. By [2: Theorem 1] there exist nonzero nonunits r and s in R_w such that $(w(r), w(s))$ is a free basis of the value group of w. Let R_n be the n^{th} quadratic transform of R along w. By Lemma 3.7 there exists a nonnegative integer m such that for any $n \geqq m$, if x' and y' are any elements in R_n such that (R_n, x', y') is a canonical quadratic transform of (R, x, y) then r and s are R_n-monomials in (x', y'). Fix any elements x_m and y_m in R_m such that (R_m, x_m, y_m) is a canonical quadratic transform of (R, x, y). Then r and s are R_m-monomials in (x_m, y_m); since $(w(r), w(s))$ is a free basis of the value group of w it follows that $(w(x_m), w(y_m))$ is a free basis of the value group of w. In particular $w(x_m)$ and $w(y_m)$ are rationally independent and hence there exist unique elements $x_{m+1}, y_{m+1}, x_{m+2}, y_{m+2}, \ldots$ in K such that for all $n > m$ we have that: $w(y_{n-1}) \neq w(x_{n-1})$, if $w(y_{n-1}) > w(x_{n-1})$ then $x_{n-1} = x_n$ and $y_{n-1} = x_n y_n$, and if $w(y_{n-1}) < w(x_{n-1})$ then $x_{n-1} = x_n y_n$ and $y_{n-1} = y_n$. Clearly for all $n \geqq m$, (R_n, x_n, y_n) is a canonical quadratic transform of (R, x, y) and $(w(x_n), w(y_n))$ is a free basis of the value group of w. To prove (3), for any $n \geqq m$ let x' and y' be any elements in R_n such that (R_n, x', y') is a canonical quadratic transform of (R, x, y). Then r and s are R_n-monomials in (x', y'). Also $r = D' x_n^{a'} y_n^{b'}$ and $s = E' x_n^{c'} y_n^{d'}$ where D' and E' are units in R_n and a', b', c', d' are nonnegative integers. Since $(w(r), w(s))$ is a free basis of the value group of w we get that either $a'd' - b'c' = 1$ or $a'd' - b'c' = -1$. If $a'd' - b'c' = 1$ then let $D'' = E'^{b'}/D'^{d'}$, $E'' = D'^{c'}/E'^{a'}$, $a'' = d'$, $b'' = -b'$, $c'' = -c'$, $d'' = a'$; and if $a'd' - b'c' = -1$ then let $D'' = D'^{d'}/E'^{b'}$, $E'' = E'^{a'}/D'^{c'}$, $a'' = -d'$, $b'' = b'$, $c'' = c'$, $d'' = -a'$. Then D'' and E'' are units in R_n and a'', b'', c'', d'' are integers such that $x_n = D'' r^{a''} s^{b''}$ and $y_n = E'' r^{c''} s^{d''}$. Since r and s are R_n-monomials in (x', y') we get that $x_n = D x'^a y'^b$ and $y_n = E x'^c y'^d$ where D and E are units in R_n and a, b, c, d are integers. Let M_n be the maximal ideal in R_n. Since (x', y') is a basis of M_n we get that

$\mathrm{ord}_{x'R_n} x' = 1$, $\mathrm{ord}_{x'R_n} y' = 0$, $\mathrm{ord}_{x'R_n} D = 0$, and $\mathrm{ord}_{x'R_n} x_n \geqq 0$; since $x_n = Dx'^a y'^b$ we get that $a' \geqq 0$. Similarly $b' \geqq 0$, $c' \geqq 0$, $d' \geqq 0$. Since (x_n, y_n) and (x', y') are bases of M_n, D is a unit in R_n, and $x_n = Dx'^a y'^b$, we get that $1 = \mathrm{ord}_{R_n} x_n = a(\mathrm{ord}_{R_n} x') + b(\mathrm{ord}_{R_n} y') = a + b$. Similarly $c + d = 1$. Also $(x_n, y_n)R_n = M_n \not\subset x'R_n$ and hence if $b = 0$ then $d \neq 0$. Similarly if $a = 0$ then $c \neq 0$. Therefore either $(a, b, c, d) = (1, 0, 0, 1)$ or $(a, b, c, d) = (0, 1, 1, 0)$. If $(a, b, c, d) = (1, 0, 0, 1)$ then x_n/x' and y_n/y' are units in R_n, and if $(a, b, c, d) = (0, 1, 1, 0)$ then x_n/y' and y_n/x' are units in R_n.

Lemma 3.12. *Assume that R/M is algebraically closed. Let J be a coefficient set for R and let (R_n, x_n, y_n) be the canonical n^{th} quadratic transform of (R, x, y, J) along w. Given any $p \in M_w$ we have the following. If w is real then there exists a nonnegative integer m such that $p \in x_n R_n$ for all $n \geqq m$. If w is irrational then there exists a nonnegative integer m' such that $p \in x_n y_n R_n$ for all $n \geqq m'$.*

Proof. First suppose that w is real. By Lemma 3.6 there exists a nonnegative integer d such that $p \in R_d$. Since w is real there exists an integer $m > d$ such that $w(x_{m-1}) \leqq w(y_{m-1})$. Then $M_{m-1}R_m = x_m R_m$ where M_{m-1} is the maximal ideal in R_{m-1}. Since $p \in M_w$ we get that $p \in M_{m-1}$ and hence $p \in x_m R_m$. Clearly $x_{n-1} \in x_n R_n$ for all $n > 0$, and hence $p \in x_n R_n$ for all $n \geqq m$. Now suppose that w is irrational. By Lemma 3.11 there exists an integer $e \geqq m$ such that $w(x_i) \neq w(y_i)$ for all $i \geqq e$. Since w is real there exists an integer $m' > e$ such that $w(x_{m'-1}) > w(y_{m'-1})$. Then $x_{m'-1} = x_{m'} y_{m'}$ and hence $p \in y_{m'} R_{m'}$. Clearly $y_{n-1} \in y_n R_n$ for all $n > e$ and hence $p \in y_n R_n$ for all $n \geqq m'$. Therefore $p \in x_n y_n R_n$ for all $n \geqq m'$.

Lemma 3.13. *Assume that R/M is algebraically closed. Let J be a coefficient set for R and let (R_n, x_n, y_n) be the canonical n^{th} quadratic transform of (R, x, y, J) along w. If w is real nondiscrete then there exists a unique nonnegative integer m such that $w(y_i) \geqq w(x_i)$ for all $i < m$ and $w(y_m) < w(x_m)$.*

Proof. See [4: (1.3)].

Lemma 3.14. (1) *Assume that w is nonreal. Then there exists a unique nonzero nonmaximal prime ideal P in R_w, and we have the following: $(R_w)_P$ is the only subring of K containing R_w which is different from K and different from R_w; $P(R_w)_P = P$, i.e., P is the unique maximal ideal in $(R_w)_P$; $(R_w)_P$ and R_w/P are one dimensional regular local domains; and for any nonzero elements X' and Y' in M_w we have that $Y'/X'^n \in M_w$ for every positive integer n if and only if $Y' \in P$ and $X' \notin P$. In particular there exist nonzero elements X and Y in R_w such that $Y(R_w)_P = P$ and $XR_w = M_w$, and for any such elements X and Y we have the following: given any $0 \neq f \in K$ there exist unique integers a and b such that $w(f) = aw(X) + bw(Y)$, i.e., $f/(X^a Y^b)$ is a unit in R_w; moreover $f \in R_w$ if and only if either $b > 0$, or $b = 0$ and $a \geqq 0$; in particular $(w(X), w(Y))$ is a free basis of the value group of w.*

(2) *If $y/x^n \in M_w$ for every positive integer n then w is nonreal and $P \cap R = yR$ where P is the unique nonzero nonmaximal prime ideal in R_w. If w is nonreal and $P \cap R = yR$ where P is the unique nonzero nonmaximal prime ideal in R_w then: $y/x^n \in M_w$ for every positive integer n, $y(R_w)_P = P$, $xR_w = M_w$, $R_{yR} = (R_w)_P$, and $h(R) = h(R_w)$ where h is the canonical epimorphism of R_w onto R_w/P.*

(3) *Assume that* $y/x^n \in M_w$ *for every positive integer* n. *Let* R_n *be the* n^{th} *quadratic transform of* R *along* w *and let* $y_n = y/x^n$. *Then* (R_n, x, y_n) *is a canonical quadratic transform of* (R, x, y) *for all* $n \geqq 0$. *Given* $0 \neq f \in R$ *let* $b = \operatorname{ord}_{yR} f$ *and* $a = \operatorname{ord}_{R/y} f/y^b$; *then* $f/(x^a y^b)$ *is a unit in* R_a *and hence a unit in* R_w. *Given any finite number of nonzero elements* $f_1, \ldots, f_q$ *in* R_w *there exists a nonnegative integer* m *such that* $f_1, \ldots, f_q$ *are* R_n*-monomials in* (x, y_n) *for all* $n \geqq m$.

(4) *Assume that* w *is nonreal and* R *is a spot over a pseudogeometric domain, and let* $f_1, \ldots, f_q$ *be any finite number of nonzero elements in* R_w. *Then there exists a canonical quadratic transform* (R', x', y') *of* (R, x, y) *along* w *such that* $y'/x'^n \in M_w$ *for every positive integer* n, *and* $f_1, \ldots, f_q$ *are* R'*-monomials in* (x', y').

Proof of (1). Actually (1) is true for any two dimensional local domain R, i.e., without assuming R to be regular. The proof follows from [2: Theorem 1] and well known elementary properties of valuation rings.

Proof of (2). If $y/x^n \in M_w$ for every positive integer n then clearly w is nonreal and by (1) we get that $y \in P$ and $x \notin P$ where P is the unique nonzero nonmaximal prime ideal in R_w, and hence $0 \neq yR \subset P \cap R \neq M$; since yR and $P \cap R$ are prime ideals in R and $\dim R = 2$, we must have $P \cap R = yR$. Now assume that w is nonreal and $P \cap R = yR$ where P is the unique nonzero nonmaximal prime ideal in R_w. Since $P \cap R = yR$ we get that $R_{yR} \subset (R_w)_P$; since R_{yR} and $(R_w)_P$ are one dimensional regular local domains with quotient field K we must have $R_{yR} = (R_w)_P$ and hence $y(R_w)_P = P$. Let h' be the canonical epimorphism of $(R_w)_P$ onto $(R_w)_P/P$. Then $h'(R)$ and $h'(R_w)$ are one dimensional regular local domains with quotient field $h'((R_w)_P)$ and $h'(R) \subset h'(R_w)$; therefore $h'(R) = h'(R_w)$ and hence $h(R) = h(R_w)$ where h is the canonical epimorphism of R_w onto R_w/P; consequently $h(x)\,h(R_w) = h(M_w)$. Now $x \in M_w$ and $x \notin P$; therefore $z/x \in P(R_w)_P = P \subset R_w$ for all $z \in P$, and hence $xR_w = M_w$. Since $y \in P$, $x \in M_w$, and $x \notin P$, we also get that $y/x^n \in M_w$ for every positive integer n.

Proof of (3). Clearly (R_n, x, y_n) is a canonical quadratic transform of (R, x, y) for all $n \geqq 0$. Now $h(f/y^b) = Dh(x)^a$ where h is the canonical epimorphism of R onto R/yR and D is a unit in $h(R)$. Therefore $f/y^b = Ex^a + ry$ where E is a unit in R and r is an element in R. Let $E' = E + ry_a$. Then E' is a unit in R_a and $f/(x^a y^b) = E'$. The last assertion now follows from Lemma 3.6.

Proof of (4). In view of (3) it suffices to find a canonical quadratic transform (R', x', y') of (R, x, y) along w such that $y'/x'^n \in M_w$ for every positive integer n. By [5: (VI) on page 15] there exists a quadratic transform R'' of R along w and a basis (x^*, y^*) of the maximal ideal M'' in R'' such that $y^*/x^{*n} \in M_w$ for every positive integer n. We can take a basis (x'', y'') of M'' such that (R'', x'', y'') is a canonical quadratic transform of (R, x, y). Now either $(x'', y^*)\,R'' = M''$ or $(y'', y^*) = M''$. Upon relabelling x'' and y'' we may assume that $(x'', y^*)\,R'' = M''$. Then by (2) we get that $y^*/x''^n \in M_w$ for every positive integer n. If $y'' \in y^*R''$ then $y''/x''^n \in M_w$ for every positive integer n and hence it suffices to take (R'', x'', y'') for (R', x', y'). Now assume that $y'' \notin y^*R''$ and let $b = \operatorname{ord}_{R''/y^*} y''$. Then b is a positive integer and by (3) we get that $w(y'') = bw(x'')$. Let R' be the b^{th} quadratic transform of R'' along w, let $x' = x''$, and let $y' = y^*/x''^b$. Then (x', y') is a basis of the maximal ideal in R' and $y'/x'^n \in M_w$ for every positive integer n. Let R_1 be the $(b-1)^{\text{th}}$ quadratic transform of R'' along w and let $y_1 = y''/x''^{b-1}$. Then (R_1, x'', y_1) is a canonical $(b-1)^{\text{th}}$ quadratic transform of (R'', x'', y'') along w and $w(y_1) = w(x'')$. It follows that (R', x', y') is a canonical first quadratic transform of (R_1, x'', y_1), and hence (R', x', y') is a canonical quadratic transform of (R, x, y) along w.

§ 4. Nonsplitting and quadratic transforms

Let R be a two dimensional regular local domain with maximal ideal M such that R/M is an algebraically closed field of characteristic $p \neq 0$. Let (x, y) be a basis of M. Let J be a coefficient set for R. Let w be a valuation of the quotient field K of R such that w dominates R and w is residually algebraic over R. Let (R_n, x_n, y_n) be the canonical n^{th} quadratic transform of (R, x, y, J) along w, and let M_n be the maximal ideal in R_n.

Definition 4.1. Let $f(Z) \in K[Z]$. $f(Z)$ is said to be *R-standard* if

$$f(Z) = Z^p + F + \sum_{i=1}^{p-1} f_i Z^{p-i}$$

where $F, f_1, \ldots, f_{p-1}$ are elements in R such that: $f_i \in f_{i+1} M$ for $1 \leqq i \leqq p-2$, $p \in f_1 M$, and $f_{p-1} = g' g^{p-1}$ where g' is a unit in R and g is a nonzero element in R. $f(Z)$ is said to be of *$[R, x, y]$-standard-type (u, v)* if $f(Z)$ is R-standard and u and v are nonnegative integers such that $f_{p-1}/(x^u y^v)^{p-1}$ is a unit in R where f_{p-1} is the coefficient of Z in $f(Z)$. $f(Z)$ is said to be of *$[R, x, y]$-standard-type zero* if there exist nonnegative integers u and v such that: $f(Z)$ is of $[R, x, y]$-standard-type (u, v), and $f(0) \in x^{up} y^{vp} R$. $f(Z)$ is said to be of *$[R, x, y]$-standard-type one* if there exist nonnegative integers u, v, a, b such that: $f(Z)$ is of $[R, x, y]$-standard-type (u, v), $f(0)/(x^a y^b)$ is a unit in R, $(a, b) \not\equiv 0(p)$, $a < up$, and $b \leqq vp$. $f(Z)$ is said to be of *$[R, x, y]$-standard-type two* if there exist nonnegative integers u, v, a, b such that: $f(Z)$ is of $[R, x, y]$-standard-type (u, v), $f(0) \in x^a y^b R$, $a < up$, $b \leqq vp$, $b \equiv 0(p)$, and $\operatorname{ord}_{R/x} f(0)/(x^a y^b) = 1$.

In [6: Theorem 8(3)] we proved the following:

Theorem 4.2. *Assume that w is real nondiscrete. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is of $[R, x, y]$-standard-type (u, v) for some nonnegative integers u and v. Also assume that there exists a coefficient set J' for R and an element z in $M \cap (\operatorname{rad}_R f_{p-1} R)$, where f_{p-1} is the coefficient of Z in $f(Z)$, such that if i is any positive integer and r and r' are any elements in J such that $r^i - r' \in M$ then $r^i - r' \in zR$. Then there exists a nonnegative integer m, a basis (x', y') of M_m, an element s' in R_m, and an R_m-monomial s in (x', y') such that for $f'(Z) = s^{-p} \times$ $\times f(sZ + s')$ we have that $f'(Z)$ is of $[R_m, x_m, y_m]$-standard-type (u', v') where u' and v' are nonnegative integers such that if $u' + v' \neq 0$ then; $u' > 0$ and there exist nonnegative integers a' and b' such that $f(0)/(x'^{a'} y'^{b'})$ is a unit in R_m, $a' < p$, $b' < p$, $(a', b') \not\equiv 0(p)$, and if $v' = 0$ then $b' = 0$.*

We shall now deduce the following slight refinement of the above theorem.

Theorem 4.3. *Assume that w is real nondiscrete. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is R-standard. Then there exists a nonnegative integer m and an R_m-translate $f'(Z)$ of $f(Z)$ such that either $f'(Z)$ is of $[R_m, x_m, y_m]$-standard-type* *zero, or $f'(Z)$ is of $[R_m, x_m, y_m]$-standard-type one, or $f'(Z)$ is of $[R_m, y_m, x_m]$-standard-type one.*

Proof. Let J' be any coefficient set for R. Then J' is a coefficient set for R_n for all $n \geqq 0$. Let $z = x$ if $w(y) \geqq w(x)$, and $z = y$ if $w(y) < w(x)$. Then $z \in M_n$ for all $n \geqq 0$. Also $MR_1 = zR_1$ and hence $MR_n = zR_n$ for all $n > 0$. It follows that if n and i are any positive integers and r and r' are any elements in J' such that $r^i - r' \in M_n$ then $r^i - r' \in M$ and hence $r^i - r' \in zR_n$. Let f_{p-1} be the coefficient of Z in $f(Z)$. Now w is real, $z \in M_w$, and $0 \neq f_{p-1} \in R_w$; therefore there exists a positive integer q such that $z^q/f_{p-1} \in R_w$. By Lemma 3.6 there exists a positive integer n' such that $z^q/f_{p-1} \in R_{n'}$. It follows that $z \in \operatorname{rad}_{R_n} f_{p-1} R_n$ for all $n \geqq n'$. By assumption there exists a nonzero element g in R such that f_{p-1}/g^{p-1} is a unit in R. By Lemma 3.7 there exists an integer $n \geqq n'$ such that g is an R_n-monomial in (x_n, y_n). Consequently there exist nonnegative integers u^* and v^* such that $f_{p-1}/(x_n^{u^*} y_n^{v^*})^{p-1}$ is a unit in R_n. It follows that $f(Z)$ is of $[R_n, x_n, y_n]$-standard-type (u^*, v^*). Upon taking $(R_n, x_n, y_n, u^*, v^*)$ for (R, x, y, u, v) in Theorem 4.2 we find an integer $m \geqq n$, a basis (x', y') of M_m, an element s' in R_m, and an R_m-monomial s in (x', y') such that for $f''(Z) = s^{-p} f(sZ + s')$ we have that $f''(Z)$ is of $[R_m, x_m, y_m]$-standard-type (u'', v'') where u'' and v'' are nonnegative integers such that if $u'' + v'' \neq 0$ then: $u'' > 0$ and there exist nonnegative integers a'' and b'' such that $f''(0)/(x'^{a''} y'^{b''})$ is a unit in R_m, $a'' < p$, $b'' < p$, $(a'', b'') \not\equiv 0(p)$, and if $v'' = 0$ then $b'' = 0$. Let f''_{p-1} be the coefficient of Z in $f''(Z)$. Then $f''_{p-1}/(x'^{u''} y'^{v''})^{p-1}$ is a unit in R_m. Also there exist nonnegative integers d and e such that $s/(x'^d y'^e)$ is a unit in R_m. Let $u' = u'' + d$ and $v' = v'' + e$. Then u' and v' are nonnegative integers. Let $f'(Z) = f(Z + s')$ and let f'_{p-1} be the coefficient of Z in $f'(Z)$. Then $f''(Z) = s^{-p} f'(sZ)$ and hence $f'_{p-1} = s^{p-1} f''_{p-1}$ and $f'(0) = s^p f''(0)$. Therefore we get the following: (1) $f'_{p-1}/(x'^{u'} y'^{v'})^{p-1}$ is a unit in R_m; (2) if $u'' + v'' = 0$ then $(f'(0))^{p-1} \in (f'_{p-1})^p R_m$; (3) if $u'' + v'' \neq 0$ then upon letting $a' = a'' + dp$ and $b' = b'' + ep$ we have that a' and b' are nonnegative integers, $f'(0)/(x'^{a'} y'^{b'})$ is a unit in R_m, $(a', b') \not\equiv 0(p)$, $a' < u'p$, and $b' \leqq v'p$. Now $f_{p-1}/(x_n^{u^*} y_n^{v^*})^{p-1}$ is a unit in R_n, and x_n and y_n are R_m-monomials in (x_m, y_m); consequently there exist nonnegative integers u and v such that $f_{p-1}/(x_m^u y_m^v)^{p-1}$ is a unit in R_m; since $f(Z)$ is R-standard we get that $f(Z)$ is R_m-standard; it follows that $f(Z)$ is of $[R_m, x_m, y_m]$-standard-type (u, v). Since $f'(Z) = f(Z + s')$, by [4: (1.6)] we then get that $f'(Z)$ is of $[R_m, x_m, y_m]$-standard-type (u, v) and hence in particular: (4) $f'(Z)$ is R_m-standard, and (5) $f'_{p-1}/(x_m^u y_m^v)^{p-1}$ is a unit in R_m. By (4) and (5) it follows that if $(f'(0))^{p-1} \in (f'_{p-1})^p R_m$ then $f'(Z)$ is of $[R_m, x_m, y_m]$-standard-type zero. So now assume that $(f'(0))^{p-1} \notin (f'_{p-1})^p R_m$. Then by (1), (2) and (3) we get that there exist nonnegative integer u', v', a', b' such that: (6) $f'_{p-1}/(x'^{u'} y'^{v'})^{p-1}$ is a unit in R_m; and (7) $f'(0)/(x'^{a'} y'^{b'})$ is a unit in R_m, $(a', b') \not\equiv 0(p)$, $a' < u'p$, and $b' \leqq v'p$. First suppose that $v' > 0$; now $u' > 0$ by (7); since (x_m, y_m) and (x', y') are bases of M_m, by (5) and (6) we get that either x'/x_m and y'/y_m are units in R_m or y'/x_m and x'/y_m are units in R_m; by (4), (6) and (7) it follows that if x'/x_m and y'/y_m are units in R_m then $f'(Z)$ is of $[R_m, x_m, y_m]$-standard-type one and if y'/x_m and x'/y_m are units in R_m then $f'(Z)$ is of $[R_m, y_m, x_m]$-standard-type one. Next suppose that $v' = 0$; now $u' > 0$ by (7);

since (x_m, y_m) and (x', y') are bases of M_m, by (5) and (6) it follows that either x'/x_m is a unit in R_m or x'/y_m is a unit in R_m; by (4), (6) and (7) it follows that if x'/x_m is a unit in R_m then $f'(Z)$ is of $[R_m, x_m, y_m]$-standard-type one, and if x'/y_m is a unit in R_m then $f'(Z)$ is of $[R_m, y_m, x_m]$-standard-type one.

Lemma 4.4. *Assume that $w(y)=w(x)$. Let $f(Z)\in K[Z]$. Assume that $f(Z)$ is of $[R, x, y]$-standard-type two. Then there exists an R_1-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R_1, x_1, y_1]$-standard-type either zero or one or two.*

Proof. Now there exist nonnegative integers u, v, a, b such that: $f(Z)$ is of $[R, x, y]$-standard-type (u, v), $f(0)\in x^a y^b R$, $a<up$, $b\leqq vp$, $b\equiv 0(p)$, and $\operatorname{ord}_{R/x} f(0)/(x^a y^b)=1$. Also $x=x_1$ and $y=x_1(y_1+t)$ where $0\neq t\in J$. It follows that $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type $(u', 0)$ where $u'=u+v$. If $f(0)\in x_1^{u'p}R_1$ then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type zero and it suffices to take $f'(Z)=f(Z)$. Now assume that $f(0)\notin x_1^{u'p}R_1$. Since $\operatorname{ord}_{R/x} f(0)/(x^a y^b)=1$ we get that $f(0)/(x^a y^b)=Ay+Bx+D$ where $0\neq A\in J$, $B\in J$, $D\in M^2$. Let $a'=a+b+1$, $B'=At+B$, and $D'=D/x_1^2$. Then $B'\in R$, $D'\in R_1$, and

$$f(0)=x_1^{a'}(y_1+t)^b(B'+Ay_1+D'x_1).$$

In particular $f(0)\in x_1^{a'}R_1$ and hence $a'<u'p$. Since $b\equiv 0(p)$ we get that $b\in pR\subset M\subset M_1$ and hence $(y_1+t)^b-t^b\in M_1^2$. Therefore

$$f(0)=x_1^{a'}(B't^b+At^b y_1+D''x_1+E) \quad \text{with} \quad D''\in R_1,\ E\in M_1^2.$$

Now $B't^b\in R$ and $At^b\in R$, and hence there exist unique elements B^* and A^* in J such that $B't^b-B^*\in M$ and $At^b-A^*\in M$. Since $MR_1=x_1R_1$ we get that $B't^b-B^*\in x_1R_1$ and $At^b-A^*\in x_1R_1$. Therefore

$$f(0)=x_1^{a'}(B^*+A^*y_1+D^*x_1+E) \quad \text{with} \quad D^*\in R_1.$$

Since $0\neq A\in J$, $0\neq t\in J$, $A^*\in J$, and $At^b-A^*\in M$, we must have $A^*\neq 0$. It follows that if $B^*\neq 0$ then $f(0)/x_1^{a'}$ is a unit in R_1, and if $B^*=0$ then $\operatorname{ord}_{R_1/x_1} f(0)/x_1^{a'}=1$. Therefore if $a'\not\equiv 0(p)$ then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one or two according as $B^*\neq 0$ or $B^*=0$ and hence we may again take $f'(Z)=f(Z)$. So now assume that $a'\equiv 0(p)$. Since R/M is algebraically closed, there exists $r'\in J$ such that $r'^p+B^*\in M$. Let $r=r'x_1^{a'/p}$. Then $r\in R_1$. Since $MR_1=x_1R_1$ we get that

$$r^p+B^*x_1^{a'}\in x_1^{a'+1}R_1.$$

Let $f'(Z)=f(Z+r)$. By [4: (1.6)] we get that $f'(Z)$ is of $[R_1, x_1, y_1]$-standard-type $(u', 0)$. Since $a'<u'p$ and $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type $(u', 0)$ we get that $f'(0)-r^p-f(0)\in x_1^{a'+1}R_1$. Therefore

$$f'(0)=x_1^{a'}(A^*y_1+D_1x_1+E) \quad \text{with} \quad D_1\in R_1.$$

Since $0\neq A^*\in J$ and $E\in M_1^2$ we conclude that $\operatorname{ord}_{R_1/x_1} f'(0)/x_1^{a'}=1$ and hence $f'(Z)$ is of $[R_1, x_1, y_1]$-standard-type two.

Lemma 4.5. *Assume that $w(y)>w(x)$. Let $f(Z)\in K[Z]$. Assume that $f(Z)$ is of $[R, x, y]$-standard-type two. Then there exists an R_1-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R_1, x_1, y_1]$-standard-type zero or one or two.*

Proof. Now there exist nonnegative integers u, v, a, b such that: $f(Z)$ is of $[R, x, y]$-standard-type (u, v), $f(0) \in x^a y^b R$, $a < up$, $b \leqq vp$, $b \equiv 0(p)$, and

$$\operatorname{ord}_{R/x} f(0)/(x^a y^b) = 1 .$$

Also $x = x_1$ and $y = x_1 y_1$. It follows that $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type (u', v) where $u' = u + v$. Since $\operatorname{ord}_{R/x} f(0)/(x^a y^b) = 1$ we get that $f(0)/(x^a y^b) = Ay + Bx + D$ where $0 \neq A \in J$, $B \in J$, $D \in M^2$. Let $a' = a + b + 1$ and $D' = D/x_1^2$. Then $D' \in R_1$ and

$$f(0) = x_1^{a'} y_1^b (B + Ay_1 + D'x_1) \in x_1^{a'} y_1^b R_1 .$$

For a moment suppose that $a' \geqq u'p$; since $a < up$ and $b \leqq vp$ we must then have $a' = u'p$ and $b = vp$; consequently $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type zero and it suffices to take $f'(Z) = f(Z)$. Now assume that $a' < u'p$. If $B \neq 0$ then $f(0)/(x_1^{a'} y_1^b)$ is a unit in R_1, and if $B \neq 0$ then $\operatorname{ord}_{R_1/x_1} f(0)/(x_1^{a'} y_1^b) = 1$. Therefore if $a' \not\equiv 0(p)$ then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one or two according as $B \neq 0$ or $B = 0$ and hence we may again take $f'(Z) = f(Z)$. So now also assume that $a' \equiv 0(p)$. Since R/M is algebraically closed there exists $r' \in J$ such that $r'^p + B \in M$. Let $r = r' x_1^{a'/p} y_1^{b/p}$. Then $r \in R_1$. Since $MR_1 = x_1 R_1$ we get that

$$r^p + Bx_1^{a'} y_1^b \in x_1^{a'+1} y_1^b R_1 .$$

Let $f'(Z) = f(Z + r)$. By [4: (1.6)] we get that $f'(Z)$ is of $[R_1, x_1, y_1]$-standard-type (u', v). Since $a' < u'p$, $b \leqq vp$, and $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type (u', v) we get that $f'(0) - r^p - f(0) \in x_1^{a'+1} y_1^b R_1$. Therefore

$$f'(0) = x_1^{a'} y_1^b (Ay_1 + D^* x_1) \quad \text{with} \quad D^* \in R_1 .$$

Since $0 \neq A \in J$ we conclude that $\operatorname{ord}_{R_1/x_1} f'(0)/(x_1^{a'} y_1^b) = 1$ and hence $f'(Z)$ is of $[R_1, x_1, y_1]$-standard-type two.

Lemma 4.6. *Assume that $w(y) < w(x)$. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is of $[R, x, y]$-standard-type two. Then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one.*

Proof. Now there exist nonnegative integers u, v, a, b such that: $f(Z)$ is of $[R, x, y]$-standard-type (u, v), $f(0) \in x^a y^b R$, $b \equiv 0(p)$, $a < up$, $b \leqq vp$, and $\operatorname{ord}_{R/x} f(0)/(x^a y^b) = 1$. Also $x_1 = x_1 y_1$ and $y = y_1$. It follows that $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type (u, v') where $v' = u + v$. Let $b' = a + b + 1$. Since $a < up$ and $b \leqq vp$ we get that $b' \leqq v'p$. Since $b \equiv 0(p)$ we also get that $(a, b') \not\equiv 0(p)$. Since $\operatorname{ord}_{R/x} f(0)/(x^a y^b) = 1$ we get that $f(0)/(x^a y^b) = Ay + Bx + D$ where $0 \neq A \in J$, $B \in J$, and $D \in M^2$. Let $D' = D/y_1^2$. Then $D' \in R_1$ and

$$f(0) = x_1^a y_1^{b'} (A + Bx_1 + D'y_1) .$$

Since $0 \neq A \in J$ we conclude that $f(0)/(x_1^a y_1^{b'})$ is a unit in R_1. Therefore $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one.

Lemma 4.7. *Assume that $w(y) = w(x)$. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is either of $[R, x, y]$-standard-type one or of $[R, y, x]$-standard-type one. Then either $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one or there exists an R_1-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R_1, x_1, y_1]$-standard-type two.*

Proof. Now there exist nonnegative integers u, v, a, b such that: $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type (u, v), $f(0)/(x^a y^b)$ is a unit in R, $(a, b) \not\equiv 0(p)$, $a \leqq up$,

$b \leqq vp$, and either $a < up$ or $b < vp$. Also $x = x_1$ and $y = x_1(y_1 + t)$ where $0 \neq t \in J$. Let $u' = u + v$ and $a' = a + b$. Then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type $(u', 0)$, $f(0)/x_1^{a'}$ is a unit in R_1, and $a' < u'p$. Therefore if $a' \not\equiv 0(p)$ then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one. Now assume that $a' \equiv 0(p)$. Since $(a, b) \not\equiv 0(p)$ we must then have $b \not\equiv 0(p)$. Since $f(0)/(x^a y^b)$ is a unit in R we get that $f(0)/(x^a y^b) = A + D$ where $0 \neq A \in J$ and $D \in M$. Let $D' = D/x_1$. Then $D' \in R_1$ and

$$\begin{aligned} f(0) &= x_1^{a'}(y_1 + t)^b (A + D'x_1) \\ &= x_1^{a'}(At^b + Abt^{b-1}y_1 + D''x_1 + E) \quad \text{where} \quad D'' \in R_1,\ E \in M_1^2 . \end{aligned}$$

Now $Abt^{b-1} \in R$ and hence there exists a unique element $A' \in J$ such that $A' - Abt^{b-1} \in M$. Since $b \not\equiv 0(p)$, $0 \neq A \in J$, and $0 \neq t \in J$ we get that $A' \neq 0$. Let $E^* = E + (Abt^{b-1} - A')y_1$ and $B = At^b$. Then

$$f(0) = x_1^{a'}(B + A'y_1 + D''x_1 + E^*), \quad B \in R, \quad E^* \in M_1^2 .$$

Since R/M is algebraically closed there exists $r' \in J$ such that $r'^p + B \in M$. Let $r = r' x_1^{a'/p}$. Then $r \in R_1$. Since $MR_1 = x_1 R_1$ we get that

$$r^p + Bx_1^{a'} \in x_1^{a'+1} R_1 .$$

Let $f'(Z) = f(Z + r)$. By [4: (1.6)] we get that $f'(Z)$ is of $[R_1, x_1, y_1]$-standard-type $(u', 0)$. Since $a' < u'p$ and $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type $(u', 0)$ we get that $f'(0) - r^p - f(0) \in x_1^{a'+1} R_1$. Therefore

$$f'(0) = x_1^{a'}(A'y_1 + D^*x_1 + E^*) \quad \text{where} \quad D^* \in R_1 .$$

Since $0 \neq A' \in J$ we conclude that $\operatorname{ord}_{R_1/x_1} f'(0)/x_1^{a'} = 1$. Therefore $f'(Z)$ is of $[R_1, x_1, y_1]$-standard-type two.

Lemma 4.8. *Assume that $w(y) > w(x)$. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is of $[R, x, y]$-standard-type one. Then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one.*

Proof. Now there exist nonnegative integers u, v, a, b such that: $f(Z)$ is of $[R, x, y]$-standard-type (u, v), $f(0)/(x^a y^b)$ is a unit in R, $(a, b) \not\equiv 0(p)$, $a < up$, and $b \leqq vp$. Also $x = x_1$ and $y = x_1 y_1$. Let $u' = u + v$ and $a' = a + b$. Then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type (u', v), $f(0)/(x_1^{a'} y_1^b)$ is a unit in R_1, $(a', b) \not\equiv 0(p)$, and $a' < up$. Therefore $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one.

Lemma 4.9. *Assume that $w(y) < w(x)$. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is of $[R, x, y]$-standard-type one. Then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one and of $[R_1, y_1, x_1]$-standard-type one.*

Proof. Now there exist nonnegative integers u, v, a, b such that: $f(Z)$ is of $[R, x, y]$-standard-type (u, v), $f(0)/(x^a y^b)$ is a unit in R, $(a, b) \not\equiv 0(p)$, $a < up$, and $b \leqq vp$. Also $x = x_1 y_1$ and $y = y_1$. Let $v' = u + v$ and $b' = a + b$. Then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type (u, v'), $f(0)/(x_1^a y_1^{b'})$ is a unit in R_1, $(a, b') \not\equiv 0(p)$, and $b' < vp$. Therefore $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one and of $[R_1, y_1, x_1]$-standard-type one.

Lemma 4.10. *Assume that $w(y) \neq w(x)$. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is of $[R, x, y]$-standard-type one. Then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one.*

Proof. Follows from Lemmas 4.8 and 4.9.

Lemma 4.11. *If $w(y) \neq w(x)$ then (R_1, y_1, x_1) is the canonical first quadratic transform of (R, y, x).*

Proof. Obvious.

Lemma 4.12. *Assume that $w(y) \neq w(x)$. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is either of $[R, x, y]$-standard-type one or of $[R, y, x]$-standard-type one. Then $f(Z)$ is either of $[R, x_1, y_1]$-standard-type one or of $[R_1, y_1, x_1]$-standard-type one.*

Proof. Follows from Lemmas 4.10 and 4.11.

Lemma 4.13. *Assume that $w(y) \neq w(x)$. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is of $[R, x, y]$-standard-type one and of $[R, y, x]$-standard-type one. Then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one and of $[R_1, y_1, x_1]$-standard-type one.*

Proof. Follows from Lemmas 4.10 and 4.11.

Lemma 4.14. *Assume that $w(y) < w(x)$. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is of $[R, y, x]$-standard-type one. Then $f(Z)$ is of $[R_1, y_1, x_1]$-standard-type one.*

Proof. Follows from Lemmas 4.8 and 4.11.

Lemma 4.15. *Assume that $w(y) > w(x)$. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is of $[R, y, x]$-standard-type one. Then $f(Z)$ is of $[R_1, x_1, y_1]$-standard-type one and of $[R_1, y_1, x_1]$-standard-type one.*

Proof. Follows from Lemmas 4.9 and 4.11.

Theorem 4.16. *Assume that w is irrational. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is R-standard. Also assume that for each nonnegative integer n it is true that there does not exist any R_n-translate of $f(Z)$ which is of $[R_n, x_n, y_n]$-standard-type zero. Then there exists a nonnegative integer m and an R_m-translate $f'(Z)$ of $f(Z)$ such that for all $n \geqq m$ we have that $f'(Z)$ is of $[R_n, x_n, y_n]$-standard-type one and of $[R_n, y_n, x_n]$-standard-type one.*

Proof. By Lemma 3.11 there exists a nonnegative integer d such that $w(x_n) \neq w(y_n)$ for all $n \geqq d$. Clearly $f(Z)$ is R_d-standard and hence by Theorem 4.3 there exists an integer $e \geqq d$ and an R_e-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ ist either of $[R_e, x_e, y_e]$-standard-type one or of $[R_e, y_e, x_e]$-standard-type one. First suppose that $f'(Z)$ is of $[R_e, x_e, y_e]$-standard-type one; since w is real there exists an integer $m > e$ such that $w(y_i) > w(x_i)$ for $e \leqq i < m-1$ and $w(y_{m-1}) < w(x_{m-1})$; applying Lemma 4.8 successively $m-e-1$ times we see that $f'(Z)$ if of $[R_{m-1}, x_{m-1}, y_{m-1}]$-standard-type one; by Lemmas 4.9 and 4.13 we then get that for all $n \geqq m$: $f'(Z)$ is of $[R_n, x_n, y_n]$-standard-type one and $f'(Z)$ is of $[R_n, y_n, x_n]$-standard-type one. Next suppose that $f'(Z)$ is of $[R_e, y_e, x_e]$-standard-type one; since w is real there exists an integer $m > e$ such that $w(y_i) < w(x_i)$ for $e \leqq i < m-1$ and $w(y_{m-1}) > w(x_{m-1})$; applying Lemma 4.14 successively $m-e-1$ times we see that $f'(Z)$ is of $[R_{m-1}, y_{m-1}, x_{m-1}]$-standard-type one; by Lemmas 4.15 and 4.13 we then see that for all $n \geqq m$: $f'(Z)$ is of $[R_n, x_n, y_n]$-standard-type one and $f'(Z)$ is of $[R_n, y_n, x_n]$-standard-type one.

Theorem 4.17. *Assume that w is real nondiscrete. Let $f(Z) \in K[Z]$. Assume that $f(Z)$ is R-standard. Also assume that for every nonnegative integer n it is true that there does not exist any R_n-translate of $f(Z)$ which is of $[R_n, x_n, y_n]$-standard-type zero. Then there exists a positive integer m such that for every integer $n \geqq m$ there exists an R_n-translate $f^{(n)}(Z)$ of $f(Z)$ such that for all $n \geqq m$*

we have the following; if $w(y_{n-1}) \geqq w(x_{n-1})$ *then* $f^{(n)}(Z)$ *is of* $[R_n, x_n, y_n]$-*standard-type either one or two, and if* $w(y_{n-1}) < w(x_{n-1})$ *then* $f^{(n)}(Z)$ *is of* $[R_n, x_n, y_n]$-*standard-type one.*

Proof. If w is not rational then our assertion follows from Theorem 4.16. Now assume that w is rational. By Theorem 4.3 there exists a nonnegative integer d and an R_d-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is either of $[R_d, x_d, y_d]$-standard-type one or of $[R_d, y_d, x_d]$-standard-type one. Since w is rational there exists an integer $m > d$ such that $w(x_i) \neq w(y_i)$ for $d \leqq i < m-1$ and $w(x_{m-1}) = w(y_{m-1})$. By applying Lemma 4.12 successively $m-d-1$ times we see that $f'(Z)$ is either of $[R_{m-1}, x_{m-1}, y_{m-1}]$-standard-type one or of $[R_{m-1}, y_{m-1}, x_{m-1}]$-standard-type one. By Lemma 4.7 there exists an R_m-translate $f^{(m)}(Z)$ of $f'(Z)$ such that $f^{(m)}(Z)$ is of $[R_m, x_m, y_m]$-standard-type either one or two. Repeatedly applying Lemmas 4.4, 4.5, 4.6, 4.7, 4.10 we find an R_n-translate $f^{(n)}(Z)$ of $f^{(m)}(Z)$ for all $n > m$ such that for all $n > m$ we have the following: if $w(y_{n-1}) \geqq w(x_{n-1})$ then $f^{(n)}$ is of $[R_n, x_n, y_n]$-standard-type either one or two, and if $w(y_{n-1}) < w(x_{n-1})$ then $f^{(n)}(Z)$ is of $[R_n, x_n, y_n]$-standard-type one.

Lemma 4.18. *Let* $f(Z) \in K[Z]$, *let* z *be an element in an overfield of* K *such that* $f(z)=0$, *and let* $L = K(z)$. *Assume that* $f(Z)$ *is of* $[R, x, y]$-*standard-type zero and* $f(Z)$ *is irreducible in* $K[Z]$. *Then* R *splits in* L *and* w *splits in* L.

Proof. Now there exist nonnegative integers u and v such that $f(Z)$ is of $[R, x, y]$-standard-type (u, v), and $f(0) \in x^{up} y^{vp} R$. Let $z' = z/(x^u y^v)$ and $f'(z) = (x^u y^v)^{-p} f(x^u y^v Z)$. Then $L = K(z')$, $f'(z') = 0$, $f'(Z)$ is irreducible in $K[Z]$, and $f'(Z)$ is of $[R, x, y]$-standard-type $(0, 0)$. In particular $f'(Z) = Z^p + f'_1 Z^{p-1} + \cdots + f'_{p-1} Z + F'$ where $F' \in R$, $f'_{p-1} \in R$, $f'_{p-1} \notin M$, and $f'_i \in M$ for $1 \leqq i \leqq p-2$. Therefore by [6: Lemma 11 (1)], R splits in L and w splits in L.

Theorem 4.19. *Let* $f(Z) \in K[Z]$, *let* z *be an element in an overfield of* K *such that* $f(z)=0$, *and let* $L = K(z)$. *Assume that* $f(Z)$ *is* R-*standard,* $f(Z)$ *is irreducible in* $K[Z]$, *and* w *does not split in* L. *Then we have the following.* (1) *If* w *is real nondiscrete then there exists a positive integer* m *such that for every integer* $n \geqq m$ *there exists an* R_n-*translate* $f^{(n)}(Z)$ *of* $f(Z)$ *such that for all* $n \geqq m$ *we have the following: if* $w(y_{n-1}) \geqq w(x_{n-1})$ *then* $f^{(n)}(Z)$ *is of* $[R_n, x_n, y_n]$-*standard-type either one or two, and if* $w(y_{n-1}) < w(x_{n-1})$ *then* $f^{(n)}(Z)$ *is of* $[R_n, x_n, y_n]$-*standard-type one.* (2) *If* w *is irrational then there exists a nonnegative integer* m *and an* R_m-*translate* $f'(Z)$ *of* $f(Z)$ *such that for all* $n \geqq m$ *we have that* $f'(Z)$ *is of* $[R_n, x_n, y_n]$-*standard-type one and of* $[R_n, y_n, x_n]$-*standard-type one.*

Proof. If n is any nonnegative integer and $f^*(Z)$ is any R_n-translate of $f(Z)$ then $f^*(Z)$ is irreducible in $K[Z]$ and there exists $z^* \in L$ such that $L = K(z^*)$ and $f^*(z^*) = 0$. Therefore our assertion follows from Theorem 4.16, Theorem 4.17, and Lemma 4.18.

Lemma 4.20. *Let* L *be a separable* p-*cyclic extension of* K. *Assume that* K *is of nonzero characteristic. Then there exists a primitive element* z *of* L *over* K *such that the minimal monic polynomial of* z *over* K *is* R-*standard.*

Proof. Now K is of characteristic p and hence there exists a primitive element z' of L over K such that $z'^p - z' \in K$ (for instance see [8: Chapter IX]). Since K is the quotient field of R, there exist elements G and H in R such that

$G \neq 0$ and $z'^p - z' = H/G$. Let $z = z'G$, $F = -HG^{p-1}$, and $f(Z) = Z^p - G^{p-1}Z + F$. Then z is a primitive element of L over K, $f(Z)$ is the minimal monic polynomial of z over K, and clearly $f(Z)$ is R-standard.

Lemma 4.21. *Let L be a p-cyclic extension of K. Assume that K is of characteristic zero, K contains a primitive p^{th} root of 1, K contains a $(p-1)^{\text{th}}$ root of p, and w is real nondiscrete. Then either* (1) *there exists a primitive element z' of L over K, a nonnegative integer m', a unit D_n in R_n for all $n \geqq m'$, and nonnegative integers $a(n)$ and $b(n)$ for all $n \geqq m'$, such that for all $n \geqq m'$ we have that $Z^p + D_n x_n^{a(n)} y_n^{b(n)}$ is the minimal monic polynomial of z' over K and $(a(n), b(n)) \not\equiv 0(p)$; or* (2) *there exists a primitive element z of L over K and a nonnegative integer m such that the minimal monic polynomial of z over K is R_m-standard.*

Proof. Since K contains a primitive p^{th} root of 1, there exists a primitive element z' of L over K such that the minimal monic polynomial of z' over K is of the form $Z^p + F$ with $0 \neq F \in R$. By Lemma 3.7 there exists a nonnegative integer m' such that for all $n \geqq m'$ we have that $F = D_n x_n^{a(n)} y_n^{b(n)}$ where D_n is a unit in R_n and $a(n)$ and $b(n)$ are nonnegative integers. If $(a(n), b(n)) \not\equiv 0(p)$ for all $n \geqq m'$ then we have nothing more to show. So now assume that $(a(e), b(e)) \equiv 0(p)$ for some $e \geqq m'$. Let $z^* = z'/(x_e^{a(e)/p} y_e^{b(e)/p})$ and $f^*(Z) = Z^p + D_e$. Then z^* is a primitive element of L over K and $f^*(Z)$ is the minimal monic polynomial of z^* over K. By [6: Lemma 31] there exists an integer $m \geqq e$, and elements s and t in R_m with $t \neq 0$ such that for $f(Z) = t^{-p} f^*(tZ + s)$ we have that $f(Z)$ is R_m-standard. Let $z = (z^* - s)/t$. Then z is a primitive element of L over K and $f(Z)$ is the minimal monic polynomial of z over K.

Lemma 4.22. *Let $f(Z) \in K[Z]$, let z be an element in an overfield of K such that $f(z) = 0$, and let $L = K(z)$. Then we have the following.* (1) *If $f(Z)$ is of $[R, x, y]$-standard-type two then $[L:K] = p$ and ord_{xR} is totally ramified in L.* (2) *If $f(Z)$ is of $[R, x, y]$-standard-type one then $[L:K] = p$, ord_{xR} is totally ramified in L, and ord_{yR} does not split in L.* (3) *If $f(Z)$ is of $[R, x, y]$-standard-type one and $f(Z)$ is of $[R, y, x]$-standard-type one then $[L:K] = p$, ord_{xR} is totally ramified in L, and ord_{yR} is totally ramified in L.* (4) *If $p \in xR$ and $f(Z) = Z^p + Dx^a y^b$ where D is a unit in R and a and b are nonnegative integers such that $(a, b) \not\equiv 0(p)$ then $[L:K] = p$, ord_{xR} is totally ramified in L, and ord_{yR} does not split in L.* (5) *If $p \in xyR$ and $f(Z) = Z^p + Dx^a y^b$ where D is a unit in R and a and b are nonnegative integers such that $(a, b) \not\equiv 0(p)$ then $[L:K] = p$, ord_{xR} is totally ramified in L, and ord_{yR} is totally ramified in L.*

Proof. (1) follows from Lemma 2.8. (2), (3), (4), (5) follow from Lemma 2.9.

Theorem 4.23. *Let L be a p-extension of K such that w does not split in L. Assume that if K is of characteristic zero then K contains a primitive p^{th} root of 1 and K contains a $(p-1)^{\text{th}}$ root of p. Then we have the following.* (1) *If w is real nondiscrete then there exists a nonnegative integer m such that for all $n \geqq m$: ord_{R_n} does not split in L and $\operatorname{ord}_{x_n R_n}$ is totally ramified in L.* (2) *If w is irrational then there exists a nonnegative integer m such that for all $n \geqq m$: ord_{R_n} is totally ramified in L, $\operatorname{ord}_{x_n R_n}$ is totally ramified in L, and $\operatorname{ord}_{y_n R_n}$ is totally ramified in L.*

Proof. The case when L is a separable p-cyclic extension of K follows from Lemmas 3.12, 4.20, 4.21, 4.22, Theorem 4.19, and the observation that for any

$n>0$: if $w(y_{n-1}) \geqq w(x_{n-1})$ then $\operatorname{ord}_{R_{n-1}} = \operatorname{ord}_{x_n R_n}$, and if $w(y_{n-1}) < w(x_{n-1})$ then $\operatorname{ord}_{R_{n-1}} = \operatorname{ord}_{y_n R_n}$. In the general case let H be the set of all subfields of L which are separable p-cyclic extensions of K. Note that H is a finite set and for each K' in H we have that w does not split in K'. To prove (1) suppose that w is real nondiscrete; then for each $K' \in H$ there exists a nonnegative integer $m(K')$ such that for all $n \geqq m(K')$: ord_{R_n} does not split in K' and $\operatorname{ord}_{x_n R_n}$ is totally ramified in K'; since H is a finite set we can take a nonnegative integer m such that $m \geqq m(K')$ for all $K' \in H$; by Lemma 2.10 it follows that for all $n \geqq m$: ord_{R_n} does not split in L and $\operatorname{ord}_{x_n R_n}$ is totally ramified in L. To prove (2) suppose that w is irrational; then for each $K' \in H$ there exists a nonnegative integer $m(K')$ such that for all $n \geqq m(K')$: ord_{R_n}, $\operatorname{ord}_{x_n R_n}$, and $\operatorname{ord}_{y_n R_n}$ are totally ramified in K'; since H is a finite set we can take a nonnegative integer m such that $m \geqq m(K')$ for all $K' \in H$; by Lemma 2.10 it follows that for all $n \geqq m$: ord_{R_n}, $\operatorname{ord}_{x_n R_n}$, and $\operatorname{ord}_{y_n R_n}$ are totally ramified in L.

For the sake of completeness we shall now prove analogues of the above result for discrete valuations and nonreal valuations.

Theorem 4.24. *Let L be a finite algebraic extension of K. Assume that w is discrete and w does not split in L. Then there exists a nonnegative integer m such that for all $n \geqq m$:* ord_{R_n} *is totally ramified in L and* $\operatorname{ord}_{x_n R_n}$ *is totally ramified in L.*

Proof. Let L^* be the maximal separable extension of K in L. Let $e = [L^* : K]$. Let S be the integral closure of R_w in L^*. Then S is a one dimensional regular local domain. Take $z \in S$ such that $\operatorname{ord}_S z = 1$, and let $f(Z)$ be the minimal monic polynomial of z over K. By Lemma 2.6 we get that $L^* = K(z)$ and the reduced ramification index of S over R_w is e. By well known properties of Dedekind domains it follows that $f(Z) - Z^e \in M_w[Z]$ and $f(0) R_w = M_w$ (for instance see [11: Lemma 2 on page 305 and the formula in the middle of page 300]). Hence by Lemma 3.10 there exists a nonnegative integer m such that for all $n \geqq m$ we have that: $x_n = x_m$, $x_n R_w = M_w$, and $f(Z) - Z^e \in (x_n R_n)[Z]$. Since $x_n R_w = M_w$ $= f(0) R_w$ we get that $\operatorname{ord}_{x_n R_n} f(0) = 1$ for all $n \geqq m$. Therefore by Lemma 2.7, $\operatorname{ord}_{x_n R_n}$ is totally ramified in L^* for all $n \geqq m$. Since L is purely inseparable over L^* we conclude that $\operatorname{ord}_{x_n R_n}$ is totally ramified in L for all $n \geqq m$. Since $x_n = x_m$ for all $n \geqq m$ we get that $\operatorname{ord}_{R_n} = \operatorname{ord}_{x_{n+1} R_{n+1}}$ for all $n \geqq m$.

Theorem 4.25. *Let L be a p-extension of K. Assume that R is a spot over a pseudogeometric domain, w is nonreal, and w does not split in L. Then there exists a nonnegative integer m such that for all $n \geqq m$:* ord_{R_n} *is totally ramified in L.*

Proof. As in the proof of Theorem 4.23, in view of Lemma 2.10, without loss of generality we may assume that L is a separable p-cyclic extension of K. Then by [6: Theorem 1] there exists a nonnegative integer m, and a basis (X, Y) of M_m such that $Y/X^e \in M_w$ for all $e > 0$, and a primitive element z of L over K such that upon letting $f(Z) = Z^p + f_1 Z^{p-1} + \cdots + f_{p-1} Z + F$ with $f_1, \ldots, f_{p-1}, F$ in K be the minimal monic polynomial of z over K we have that either $F = X$ and $f_i \in X R_m$ for $0 < i < p$, or $F = Y$ and $f_i \in Y R_m$ for $0 < i < p$. Let $Y_n = Y X^{m-n}$. Then (X, Y_n) is a basis of M_n for all $n \geqq m$. Also $\operatorname{ord}_{R_n} = \operatorname{ord}_{X R_{n+1}}$ for all $n \geqq m$ and hence it suffices to show that $\operatorname{ord}_{X R_n}$ is totally ramified in L for all $n>m$. If $F=X$ and $f_i \in XR_m$ for $0<i<p$ then for all $n \geqq m$ we get that $f_i \in XR_n$ for $0<i<p$ and hence ord_{XR_n} is totally ramified in L by Lemma 2.7. So now assume that $F=Y$ and $f_i \in YR_m$ for $0<i<p$, and let $n>m$ be given. Then $p \in XR_n$, $F=Y_nX^{n-m}$, and $f_i \in Y_nX^{n-m}R_n$ for $0<i<p$. Clearly $\mathrm{ord}_{XR_n}F=n-m \leqq \mathrm{ord}_{XR_n}f_i$ for $0<i<p$ and $\mathrm{ord}_{R_n/X}F/X^{n-m}=1 \not\equiv 0(p)$. Since $n>m$ we get that $\mathrm{ord}_{XR_n}f_i>(i/p)\,\mathrm{ord}_{XR_n}F$ for $0<i<p$. Therefore by Lemma 2.8 it follows that ord_{XR_n} is totally ramified in L.

§ 5. Permissible and stable polynomials

Let R be a two dimensional regular local domain with maximal ideal M such that R/M is an algebraically closed field of characteristic $p \neq 0$. Let (x, y) be a basis of M and let J be a coefficient set for R. Let $X \subset M$. Let w be a valuation of the quotient field K of R such that w dominates R and w is residually algebraic over R. Let (R_j, x_j, y_j) be the canonical j^{th} quadratic transform of (R, x, y, J) along w.

Definition 5.1. Let $f(Z)$ be a monic polynomial of positive degree in Z with coefficients in K. $f(Z)$ is said to be $[R, x, y, J, w]$-*permissible* if for every canonical quadratic transform (R', x', y') of (R, x, y, J) along w we have that $f(Z)$ is of nonsplitting-type relative to $\mathrm{ord}_{R'}$ and $f(Z)$ is of ramified-type relative to $\mathrm{ord}_{x'R'}$. Note that if $f(Z)$ is $[R, x, y, J, w]$-permissible, $f'(Z)$ is a K-translate of $f(Z)$, and (R', x', y') is a canonical quadratic transform of (R, x, y, J) along w, then $f'(Z)$ is $[R', x', y', J, w]$-permissible. This remark will be used tacitly in § 9.

Let $F \in R$. F is said to be of $[R, x, y]$-*stable-pretype* $(m;\, a, b, c)$ if $m=p^n$ where n is a positive integer and a, b, c are nonnegative integers such that: $\mathrm{ord}_{xR}F=a$, $\mathrm{ord}_{yR}F \geqq b$, $\mathrm{ord}_{R/x}F/(x^ay^b)=c$, $(a,\, b+c) \not\equiv 0(m)$, and either (1) $b \equiv 0(m)$ and $c \leqq m/p$, or (2) $b \not\equiv 0(p)$ and $c<m/p$.

Let $f(Z) \in K[Z]$. $f(Z)$ is said to be of $[R, x, y, J, X, w]$-*stable-type* $(m;\, a, b, c)$ if: $m=p^n$ where n is a positive integer, $f(Z)$ is a monic polynomial of degree m in Z with coefficients in R, $f(Z)$ is $[R, x, y, J, w]$-permissible, $f(0)$ is of $[R, x, y]$-stable-pretype $(m;\, a, b, c)$, $X \subset \mathrm{rad}_R\, y^bR$, and $\mathrm{ord}_{yR}f_i \geqq bi/m$ for $0<i<m$ where f_i is the coefficient of Z^{m-i} in $f(Z)$. $f(Z)$ is said to be $[R, x, y, J, X, w]$-*stable* if $f(Z)$ is of $[R, x, y, J, X, w]$-stable-type $(m;\, a, b, c)$ for some integers m, a, b, c.

In § 6 to § 9 we shall develop an algorithm dealing with monic polynomials of degree p^n with coefficients in R; here we shall state the following two results from that algorithm. § 6 to § 9 depend on § 2 to § 5 only in the use of Definitions 2.2, 3.5, 5.1, Observations 3.1, 3.2, 3.3, and Lemma 3.13. In Lemma 9.9 of § 9 we shall prove Theorem 5.2 which motivates the term "stable", and in Lemma 9.25 of § 9 we shall prove Theorem 5.3 which motivates the term "permissible".

Theorem 5.2. *Let* $f^{(0)}(Z) \in K[Z]$ *be of* $[R, x, y, J, X, w]$-*stable-type* $(m;\, a_0, b_0, c_0)$. *Then for each* $j>0$ *there exists an* R_j-*translate* $f^{(j)}(Z)$ *of* $f^{(0)}(Z)$ *such that* $f^{(j)}(Z)$ *is of* $[R_j, x_j, y_j, J, X, w]$-*stable-type* $(m;\, a_j, b_j, c_j)$ *where for all* $j \geqq 0$ *we have the following:* (1) *if* $w(y_j)<w(x_j)$ *then* $f^{(j+1)}(Z)=f^{(j)}(Z)$, $a_{j+1}=a_j$,

$b_{j+1} = \operatorname{ord}_{R_j} f^{(j)}(0)$, *and* $c_{j+1} = a_j + b_j + c_j - b_{j+1}$; (2) *if* $w(y_j) = w(x_j)$ *then* $b_{j+1} = 0$; (3) *if* $w(y_j) > w(x_j)$ *then* $b_{j+1} = b_j$. *In connection with* (2) *note that if w is rational then there exist infinitely many distinct nonnegative integers j for which* $w(y_j) = w(x_j)$.

Theorem 5.3. *Let $m = p^n$ where n is a positive integer and let $f(Z)$ be a monic polynomial of degree m with coefficients in R. Assume that w is real nondiscrete and $f(Z)$ is $[R, x, y, J, w]$-permissible. Also assume that either:* 1) *R is of characteristic p and $f(Z) \neq Z^m + f(0)$; or:* 2) *R is a spot over a pseudogeometric domain, $f(Z)$ is irreducible in $K[Z]$, and $h(R_w)$ does not split in $h(K[Z])$ where h is the canonical epimorphism of $K[Z]$ onto $K[Z]/f(Z)K[Z]$. Then there exists a nonnegative integer e and for each $j \geqq e$ an R_j-translate $f^{(j)}(Z)$ of $f(Z)$ such that $f^{(j)}(Z)$ is of $[R_j, x_j, y_j, J, X, w]$-stable-type $(m; a_j, b_j, c_j)$ where for all $j \geqq e$ we have the following:* (1) *if* $w(y_j) < w(x_j)$ *then* $f^{(j+1)}(Z) = f^{(j)}(Z)$, $a_{j+1} = a_j$, $b_{j+1} = \operatorname{ord}_{R_j} f^{(j)}(0)$, *and* $c_{j+1} = a_j + b_j + c_j - b_{j+1}$; (2) *if* $w(y_j) = w(x_j)$ *then* $b_{j+1} = 0$; (3) *if* $w(y_j) > w(x_j)$ *then* $b_{j+1} = b_j$. *In connection with* (2) *note that if w is rational then there exist infinitely many distinct nonnegative integers j for which* $w(y_j) = w(x_j)$.

From Lemma 2.3, Lemma 2.5, and Theorem 4.23 we get the following.

Theorem 5.4. *Let L be a p-extension of K. Assume that w is real nondiscrete and w does not split in L. Also assume that if K is of characteristic zero then K contains a primitive p^{th} root of 1 and K contains a $(p-1)^{\text{th}}$ root of p. Let z be an element in L such that $z \notin K$ and z is integral over R. Let $m = [K(z):K]$ and let $f(Z)$ be the minimal monic polynomial of z over K. Then there exists a nonnegative integer e such that $f(Z)$ is $[R_j, x_j, y_j, J, w]$-permissible for all $j \geqq e$.*

From Theorems 5.3 and 5.4 we get the following.

Theorem 5.5. *Let L be a p-extension of K. Assume that w is real nondiscrete and w does not split in L. Also assume that if K is of characteristic zero then K contains a primitive p^{th} root of 1 and K contains a $(p-1)^{\text{th}}$ root of p. Let z be an element in L such that $z \notin K$ and z is integral over R. Let $m = [K(z):K]$ and let $f(Z)$ be the minimal monic polynomial of z over K. Assume that either:* 1) *R is of characteristic p and $f(Z) \neq Z^m + f(0)$, or:* 2) *R is a spot over a pseudogeometric domain. Then there exists a nonnegative integer e and for each $j \geqq e$ an R_j-translate $f^{(j)}(Z)$ of $f(Z)$ such that $f^{(j)}(Z)$ is of $[R_j, x_j, y_j, J, X, w]$-stable-type $(m; a_j, b_j, c_j)$ where for all $j \geqq e$ we have the following:* (1) *if* $w(y_j) < w(x_j)$ *then* $f^{(j+1)}(Z) = f^{(j)}(Z)$, $a_{j+1} = a_j$, $b_{j+1} = \operatorname{ord}_{R_j} f^{(j)}(0)$, *and* $c_{j+1} = a_j + b_j + c_j - b_{j+1}$; (2) *if* $w(y_j) = w(x_j)$ *then* $b_{j+1} = 0$; (3) *if* $w(y_j) > w(x_j)$ *then* $b_{j+1} = b_j$. *In connection with* (2) *note that if w is rational then there exist infinitely many nonnegative integers j for which* $w(y_j) = w(x_j)$.

Remark 5.6. For $i = 1, 2, 3, 5$ and $k = 1, 2, \ldots, 25$ let $5.i'$ and $9.k'$ stand respectively for $5.i$ and $9.k$ when $X = \emptyset$, i.e., equivalently, with all reference to X omitted. Then 5.2′ and 5.3′ would be repetitions of 9.9′ and 9.25′ respectively, and 5.5′ would follow from 5.3′ and 5.4. It is easily seen that 5.2 follows from 5.2′, and in view of Lemma 3.13, 5.3 and 5.5 follow from 5.3′ and 5.5′ respectively. Thus, if the reader so prefers, from § 9 he may delete all reference to X.

§ 6. Lemmas on polynomials in one indeterminate

Let k be a field of characteristic $p \neq 0$. Let $m = p^n$ where n is a nonnegative integer. Let

$$A(Z) = A_0 + A_1 Z + \cdots + A_e Z^e$$

where e is a nonnegative integer, $A_0, A_1, \ldots, A_e$ are elements in k, and $A_e \neq 0$. Let b be a nonnegative integer, let $0 \neq D \in k$, and let E_j be the elements in k such that

$$(D+Z)^b A(Z) = \sum_j E_j Z^j .$$

Lemma 6.1. *Assume that $b \equiv 0(m)$ and $A(Z) \notin k[Z^m]$. Then there exists j such that $E_j \neq 0$, $j \not\equiv 0(m)$, and $j \leqq e$.*

Proof. Let V be the set of all integers i such that $0 \leqq i \leqq e$, $i \not\equiv 0(m)$, and $A_i \neq 0$. Since $A(Z) \notin k[Z^m]$ we get that V is nonempty. Let j be the smallest element in V. Then $A_j \neq 0$, $j \not\equiv 0(m)$, and $j \leqq e$. Since $b \equiv 0(m)$ we get that

$$(D+Z)^b = D^b + \sum_{i>0} D_i Z^{mi} \quad \text{with} \quad D_i \in k .$$

Since j is the smallest element in V we therefore get that $E_j = D^b A_j$ and hence $E_j \neq 0$.

Lemma 6.2. *Assume that $b + e \not\equiv 0(m)$ and $b \equiv 0(m)$. Then there exists j such that $E_j \neq 0$, $j \not\equiv 0(m)$, and $j \leqq e$.*

Proof. Now we must have $e \not\equiv 0(m)$ and hence $A(Z) \notin k[Z^m]$. Therefore our assertion follows from Lemma 6.1.

Lemma 6.3. *Assume that $b + e \not\equiv 0(mp)$, $b \equiv 0(m)$, and $A(Z) \in k[Z^m]$. Then $e \equiv 0(m)$ and there exists j such that $E_j \neq 0$ and $0 < j \leqq e + m$.*

Proof. Since $b + e \not\equiv 0(mp)$ we get that if $b = 0$ then $e \neq 0$ and $E_e = A_e \neq 0$. Hence if $b = 0$ then it suffices to take $j = e$. Now assume that $b \neq 0$. Then $b = mb'$ where b' is a positive integer. Since $A(Z) \in K[Z^m]$, we get that $e = me'$ where e' is a nonnegative integer and $A(Z) = A'(Z^m)$ where $A'(Z)$ is a polynomial of degree e' in Z with coefficients in k. Now $b + e = m(b' + e')$ and by assumption $b + e \not\equiv 0(mp)$. Therefore $b' + e' \not\equiv 0(p)$. Let $d' = D^m$. Then $0 \neq D' \in K$. Let E'_q be the elements in k such that

$$(D' + Z)^{b'} A'(Z) = \sum_q E'_q Z^q .$$

Then by [6: Lemma 27] there exists t such that $E'_t \neq 0$ and $0 < t \leqq e' + 1$. Clearly $D' + Z^m = (D+Z)^m$ and hence $(D' + Z^m)^{b'} = (D+Z)^b$. Since $A'(Z^m) = A(Z)$, upon substituting Z^m for Z in the above displayed equation we get that

$$\sum_i E_i Z^i = \sum_q E'_q Z^{mq}$$

and hence $E_{mq} = E'_q$ for all q. Let $j = mt$. Then $E_j = E'_t \neq 0$. Since $0 < t \leqq e' + 1$, we get that $0 < j \leqq e + m$.

Lemma 6.4. *Let v be an integer such that $0 \leqq v < m$ and let $B(Z) = (D+Z)^v A(Z)$. Assume that $A(-D) \neq 0$ and $B(Z) \in k[Z^m]$. Then $v = 0$.*

Proof. Since $B(Z)\in k[Z^m]$, we get that $B(Z)=B'(Z^m)$ with $B'(Z)\in k[Z]$. Then $B(-D)=B'((-D)^m)$ and clearly $(-D)^m=-D^m$. Therefore $B(-D)=B'(-D^m)$. Suppose if possible that $v\neq 0$. Then $B(-D)=0$ and hence $B'(-D^m)=0$. Consequently $B'(Z)=(D^m+Z)B^*(Z)$ with $B^*(Z)\in k[Z]$. Since $B(Z)=B'(Z^m)$ we then get that $B(Z)=(D^m+Z^m)B^*(Z^m)$, and hence $B(Z)=(D+Z)^m A^*(Z)$ where $A^*(Z)=B^*(Z^m)\in k[Z]$. This is a contradiction because by assumption $B(Z)=(D+Z)^v A(Z)$ with $0\leqq v<m$ and $A(Z)\in k[Z]$ with $A(-D)\neq 0$.

Lemma 6.5. *Assume that $b+e\not\equiv 0(m)$ and $e<m/p$. Then there exists j such that $E_j\neq 0$, $j\not\equiv 0(m)$, and $j\leqq m/p$.*

Proof. Since $b+e\not\equiv 0(m)$ and $m=p^n$, there exists an integer n' such that upon letting $m'=p^{n'}$ we have that $0\leqq n'<n$, $b+e\equiv 0(m')$, and $b+e\not\equiv 0(m'p)$. Let t be the greatest integer such that $(D+Z)^t$ divides $A(Z)$ in $k[Z]$. Let $e'=e-t$ and $A'(Z)=A(Z)(D+Z)^{-t}$. Then $0\leqq e'\leqq e<m/p$, $A'(Z)$ is a nonzero polynomial of degree e' in Z with coefficients in k, and $A'(-D)\neq 0$. Let b' and v be the unique integers such that $b'\equiv 0(m')$, $0\leqq v<m'$, and $b+t=b'+v$. Then $b'+(v+e')=b+e\equiv 0(m')$; since $b'\equiv 0(m')$ we must have $v+e'\equiv 0(m')$, i.e., $v+e'\equiv 0(p^{n'})$; since $0\leqq v<m'=p^{n'}$, $e'<m/p=p^{n-1}$, and $n'<n$, we conclude that $v+e'\leqq p^{n-1}$, i.e., $v+e'\leqq m/p$. Let $B(Z)=(D+Z)^v A'(Z)$. Then $B(Z)$ is a nonzero polynomial of degree $v+e'$ in Z with coefficients in k and

$$(D+Z)^{b'}B(Z)=\sum_i E_i Z^i.$$

Therefore if $B(Z)\notin k[Z^{m'}]$ then by Lemma 6.1 there exists j such that $E_j\neq 0$, $j\not\equiv 0(m)$, and $j\leqq v+e'$; since $v+e'\leqq m/p$ we get that $j\leqq m/p$. Now assume that $B(Z)\in k[Z^{m'}]$. Since $B(Z)=(D+Z)^v A'(Z)$, $0\leqq v<m'$, and $A'(-D)\neq 0$, by Lemma 6.4 we then get that $v=0$. Consequently $A'(Z)\in k[Z^{m'}]$ and

$$(D+Z)^{b'}A'(Z)=\sum_i E_i Z^i.$$

Therefore, since $b'\equiv 0(m')$, by Lemma 6.3 we get that $e'\equiv 0(m')$ and there exists j such that $E_j\neq 0$ and $0<j\leqq e'+m'$. Let $d=e'/m'$. Since $e'\equiv 0(m')$ and $m'=p^{n'}$, we get that d is an integer and $e'=dp^{n'}$. Since $e'<m/p=p^{n-1}$ and $e'=dp^{n'}$, we get that $dp^{n'}<p^{n-1}$, i.e., $d<p^{n-1-n'}$. Since d is an integer and $n'<n$, we must then have $d+1\leqq p^{n-1-n'}$, i.e., $d+1\leqq(m/p)/m'$. Now $e'+m'=m'(d+1)$ and hence $e'+m'\leqq m/p$. Since $0<j\leqq e'+m'$, we get that $0<j\leqq m/p$ and hence $j\not\equiv 0(m)$.

Lemma 6.6. *Assume that $b+e\not\equiv 0(m)$. Then there exists j such that $E_j\neq 0$ and $0<j\leqq e+m/p$.*

Proof. Since $b+e\not\equiv 0(m)$ and $m=p^n$, there exists a nonnegative integer n' such that upon letting $m'=p^{n'}$ we have that $b+e\equiv 0(m')$, $b+e\not\equiv 0(m'p)$, and $m'\leqq m/p$. Let t be the greatest integer such that $(D+Z)^t$ divides $A(Z)$ in $k[Z]$. Let $e'=e-t$ and $A'(Z)=A(Z)(Z+D)^{-t}$. Then $0\leqq e'\leqq e$, $A'(Z)$ is a nonzero polynomial of degree e' in Z with coefficients in k, and $A'(-D)\neq 0$. Let b' and v be the unique integers such that $b'\equiv 0(m')$, $0\leqq v<m'$, and $b+t=b'+v$. Let $B(Z)=(D+Z)^v A'(Z)$. Then $B(Z)$ is a nonzero polynomial of degree $v+e'$

in Z with coefficients in k and

$$(D+Z)^{b'} B(Z) = \sum_i E_i Z^i .$$

Therefore if $B(Z) \notin k[Z^{m'}]$ then by Lemma 6.1 there exists j such that $E_j \neq 0$, $j \not\equiv 0(m)$, and $j \leqq v + e'$; since $v < m' \leqq m/p$ and $e' \leqq e$ we get that $j \leqq e + m/p$; since $j \not\equiv 0(m)$ we get that $0 < j$. Now assume that $B(Z) \in k[Z^{m'}]$. Since $B(Z) = (D+Z)^v A'(Z)$, $0 \leqq v < m'$, and $A'(-D) \neq 0$, by Lemma 6.4 we then get that $v = 0$. Consequently $A'(Z) \in k[Z^{m'}]$ and

$$(D+Z)^{b'} A'(Z) = \sum_i E_i Z^i .$$

Therefore, since $b' \equiv 0(m')$, by Lemma 6.3 there exists j such that $E_j \neq 0$ and $0 < j \leqq e' + m'$. Since $m' \leqq m/p$ and $e' \leqq e$ we get that $0 < j \leqq e + m/p$.

Lemma 6.7. *Assume that $b + e \not\equiv 0(m)$ and $e + m/p < m$. Then there exists j such that $E_j \neq 0$, $j \not\equiv 0(m)$, and $j \leqq e + m/p$.*

Proof. By Lemma 6.6 there exists j such that $E_j \neq 0$ and $0 < j \leqq e + m/p$. Since $e + m/p < m$ we conclude that $j \not\equiv 0(m)$.

§ 7. Effect of a quadratic transformation on an element in a two dimensional regular local domain

Let R be a two dimensional regular local domain with maximal ideal M such that R/M is an algebraically closed field of characteristic $p \neq 0$. Let (x, y) be a basis of M and let J be a coefficient set for R. Let w be a valuation of the quotient field K of R such that w dominates R and w is residually algebraic over R. Let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w.

Let $F \in R$. Let $\Sigma F(i, j) x^i y^j$ be the expansion of F in $J[[x, y]]$. Let $\Sigma F'(i, j) x'^i y'^j$ be the expansion of F in $J[[x', y']]$.

Definition 7.1. F is said to be of $[R, x, y, J]$-*pretype* $(m; a, b, c)$ if: $m = p^n$ where n is a positive integer, and a, b, c are nonnegative integers such that $F \in x^a y^b R$, $F(a, b+c) \neq 0$, and $(a, b+c) \not\equiv 0(m)$; note that then $\operatorname{ord}_{xR} F = a$, $\operatorname{ord}_{yR} F \geqq b$, and $a + b \leqq \operatorname{ord}_R F \leqq a + b + c$.

F is said to be of $[R, x, y, J]$-*pretype* $(m; a, b, c)'$ if F is of $[R, x, y, J]$-pretype $(m; a, b, c)$ and $F(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j \leqq \operatorname{ord}_R F$.

F is said to be of $[R, x, y, J]$-*pretype* $(m; a, b, c)''$ if F is of $[R, x, y, J]$-pretype $(m; a, b, c)$ and $F(i, j) = 0$ whenever $(i, j) \equiv 0(m)$, $i \leqq a$, and $j \leqq b$. Note that if F is of $[R, x, y, J]$-pretype $(m; a, b, c)$ and $(a, b) \not\equiv 0(m)$ then F is of $[R, x, y, J]$-pretype $(m; a, b, c)''$. Also note that if F is of $[R, x, y, J]$-pretype $(m; a, b, c)'$ then F is of $[R, x, y, J]$-pretype $(m; a, b, c)''$.

F is said to be of $[R, x, y, J]$-*pretype* $(m; a, b, c)^*$ if F is of $[R, x, y, J]$-pretype $(m; a, b, c)$ and $F(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i \leqq a$. Note that if F is of $[R, x, y, J]$-pretype $(m; a, b, c)$ and $a \not\equiv 0(m)$ then F is of $[R, x, y, J]$-pretype $(m; a, b, c)^*$. Also note that if F is of $[R, x, y, J]$-pretype $(m; a, b, c)^*$ then F is of $[R, x, y, J]$-pretype $(m; a, b, c)''$.

If $m=p^n$ where n is a positive integer, then for any nonnegative integers b and c we define

$$[m;b,c]=\begin{cases}0 & \text{if } b\equiv 0(m) \text{ and } c\leqq m/p\\ 0 & \text{if } b\not\equiv 0(m) \text{ and } c<m/p\\ c & \text{if } b\equiv 0(m) \text{ and } c>m/p\\ c+m/p & \text{if } b\not\equiv 0(m) \text{ and } c\geqq m/p.\end{cases}$$

Note that then $[m;b,c]=0$ if and only if either 1) $b\equiv 0(m)$ and $c\leqq m/p$, or 2) $b\not\equiv 0(m)$ and $c<m/p$. Also note that $[m;b,c]<m$ if and only if either 1′) $b\equiv 0(m)$ and $c<m$, or 2′) $b\not\equiv 0(m)$ and $c<m-m/p$.

For any nonnegative integers a, b, c for which $\operatorname{ord}_{xR}F=a$ and $\operatorname{ord}_{yR}F\geqq b$ we clearly have that: $\operatorname{ord}_{R/x}F/(x^a y^b)=c$ if and only if $F(a,b+c)\neq 0$ and $F(a,j)=0$ whenever $j<b+c$. Therefore the following three conditions are equivalent: (1) F is of $[R,x,y]$-stable-pretype $(m;a,b,c)$; (2) F is of $[R,x,y,J]$-pretype $(m;a,b,c)$, $[m;b,c]=0$, and $\operatorname{ord}_{R/x}F/(x^a y^b)=c$; (3) F is of $[R,x,y,J]$-pretype $(m;a,b,c)$, $[m;b,c]=0$, and $F(a,j)=0$ whenever $j<b+c$.

Lemma 7.2. *Let $m=p^n$ where n is a positive integer. Assume that $w(y)>w(x)$, $0\neq F\in x^a y^b R$ where a and b are nonnegative integers, and $F(i,j)=0$ whenever $(i,j)\equiv 0(m)$ and $i+j\leqq \operatorname{ord}_R F$. Let $a'=\operatorname{ord}_R F$. Let q be any nonnegative integer such that $F(a'-q,q)\neq 0$ (q exists because $a'=\operatorname{ord}_R F$). Let $c'=q-b$. Then $0\leqq c'\leqq a'-a-b$ and F is of $[R',x',y',J]$-pretype $(m;a',b,c')^*$.*

Proof. Now $\operatorname{ord}_{x'R'}=\operatorname{ord}_R$, $\operatorname{ord}_{y'R}=\operatorname{ord}_{yR}$, $\operatorname{ord}_R F=a'$, and $\operatorname{ord}_{yR}F\geqq b$. Therefore $F\in x'^{a'}y'^{b}R'$. Since $F(a'-q,\ q)\neq 0$ and $F\in x^a y^b R$, we get that $b\leqq q$ and $a\leqq a'-q$; therefore $0\leqq c'\leqq a'-a-b$. Since $F(a'-q,\ q)\neq 0$ and $F(i,j)=0$ whenever $(i,j)\equiv 0(m)$ and $i+j\leqq a'$, we get that $(a'-q,\ q)\not\equiv 0(m)$ and hence $(a',\ b+c')\not\equiv 0(m)$. Now

$$F-\sum_{i+j=a'} F(i,j)\,x^i y^j\in M^{a'+1}\subset x'^{a'+1}R'$$

and

$$\sum_{i+j=a'} F(i,j)\,x^i y^j=\sum_{i+j=a'} F(i,j)\,x'^{a'}y'^{j}.$$

Therefore

$$F=\sum_{i+j=a'} F(i,j)\,x'^{a'}y'^{j} \bmod x'^{a'+1}R'$$

and hence

$$F'(a',j)=\begin{cases}F(a'-j,j) & \text{if } 0\leqq j\leqq a'\\ 0 & \text{if } j>a'.\end{cases}$$

In particular $F'(a',b+c')=F(a'-q,q)\neq 0$. Since $F(i,j)=0$ whenever $(i,j)\equiv 0(m)$ and $i+j\leqq a'$, we get that $F'(a',j)=F(a'-j,j)=0$ whenever $0\leqq j\leqq a'$ and $(a'-j,j)\equiv 0(p)$. Since $F\in x'^{a'}R'$ we conclude that $F'(i,j)=0$ whenever $(i,j)\equiv 0(m)$ and $i\leqq a'$. Therefore F is of $[R',x',y',J]$-pretype $(m;a',b,c')^*$.

Lemma 7.3. *Let $m=p^n$ where n is a positive integer. Assume that $w(y)=w(x)$, $0\neq F\in x^a y^b R$ where a and b are nonnegative integers, and $F(i,j)=0$ whenever $(i,j)\equiv 0(m)$ and $i+j\leqq \operatorname{ord}_R F$. Let $a'=\operatorname{ord}_R F$. Then we have the following.*

(1) *If either* $a' \not\equiv 0(m)$ *or* $b \equiv 0(m)$ *then* F *is of* $[R', x', y', J]$*-pretype* $(m; a', 0, c')$ *where* $c' \leqq a' - a - b$.

(2) *If* $a' \equiv 0(m)$, $b \not\equiv 0(m)$, *and* $a' - a - b < m/p$ *then* F *is of* $[R', x', y', J]$*-pretype* $(m; a', 0, c')$ *where* $c' \leqq m/p$.

(3) *If* $a' \equiv 0(m)$, $b \not\equiv 0(m)$, *and* $a' - a - b < m - m/p$ *then* F *is of* $[R', x', y', J]$*-pretype* $(m; a', 0, c')$ *where* $c' \leqq a' - a - b + m/p$.

Proof. Now $\mathrm{ord}_{x'R'} = \mathrm{ord}_R$ and hence

1) $$F \in x'^{a'} R'.$$

Let q be the greatest nonnegative integer such that $F(a'-q, q) \neq 0$, and let $e = q - b$. Since $F \in x^a y^b R$ and $F(i,j) = 0$ whenever $(i,j) \equiv 0(m)$ and $i + j \leqq a'$, we then get that $a \leqq a' - q$, $b \leqq q$, and $(a'-q, q) \not\equiv 0(m)$. Therefore

2) $$0 \leqq e \leqq a' - a - b$$

and

3) $$(a', b+e) \not\equiv 0(m).$$

Since $w(y) = w(x)$ we get that $x = x'$ and $y = x'(t + y')$ with $0 \neq t \in J$. Let G_j be the elements in R defined by the equation

4) $$(t+Z)^b \sum_{j=0}^{e} F(a'-b-j, b+j)(t+Z)^j = \sum_{j=0}^{b+e} G_j Z^j$$

in $R[Z]$. Let H_j be the unique element in J such that $G_j - H_j \in M$. Then $G_j - H_j \in x'R'$ and hence

$$\begin{aligned}\sum_{i+j=a'} F(i,j)\, x^i y^j &= x'^{a'}(t+y')^b \sum_{j=0}^{e} F(a'-b-j, b+j)(t+y')^j \\ &= \sum_{j=0}^{b+e} H_j x'^{a'} y'^j \bmod x'^{a'+1} R'.\end{aligned}$$

Also

$$F - \sum_{i+j=a'} F(i,j)\, x^i y^j \in M^{a'+1} \subset x'^{a'+1} R'$$

and hence

$$F = \sum_{j=0}^{b+e} H_j x'^{a'} y'^j \bmod x'^{a'+1} R'.$$

Since the elements H_j are in J we therefore get that

5) $$F'(a', j) = H_j \quad \text{for} \quad 0 \leqq j \leqq b + e.$$

Let $k = R/M$ and let h be the canonical epimorphism of R onto k. Upon applying h to 4) we get that

$$(h(t)+Z)^b \sum_{j=0}^{e} h(F(a'-b-j, b+j))(h(t)+Z)^j = \sum_{j=0}^{b+e} h(G_j) Z^j$$

in $k[Z]$. Let $D = h(t)$ and $E_j = h(G_j)$, and let A_j be the elements in k defined by the equation

$$\sum_{j=0}^{e} h(F(a'-b-j, b+j))(h(t)+Z)^j = \sum_{j=0}^{e} A_j Z^j$$

in $k[Z]$. Then

6) $$(D+Z)^b \sum_{j=0}^{e} A_j Z^j = \sum_{j=0}^{b+e} E_j Z^j \quad \text{in } k[Z].$$

Since $0 \neq t \in J$ and $D = h(t)$ we get that $D \neq 0$. Since $A_e = h(F(a'-q, q))$ and $0 \neq F(a'-q, q) \in J$ we get that $A_e \neq 0$. Thus

7) $$D \neq 0 \quad \text{and} \quad A_e \neq 0.$$

Now $h(H_j) = h(G_j) = E_j$ and $H_j \in J$. Therefore $H_j = 0 \Leftrightarrow E_j = 0$. Therefore by 5) we get the following:

8) $$\text{For } 0 \leqq j \leqq b+e: \quad F'(a', j) = 0 \Leftrightarrow E_j = 0.$$

Since $A_e \neq 0$ there exists an integer u such that $0 \leqq u \leqq e$, $A_u \neq 0$, and $A_j = 0$ whenever $0 \leqq j < u$; since $D \neq 0$, by 6) we get that $E_u \neq 0$ and hence by 8) we get that $F'(a', u) \neq 0$; since $u \leqq e$, by 2) we get that $u \leqq a' - a - b$. If $a' \not\equiv 0(m)$ then upon taking $c' = u$ we thus conclude that F is of $[R', x', y', J]$-pretype $(m; a', 0, c')$ where $c' \leqq a' - a - b$. So from now on assume that $a' \equiv 0(m)$. Then by 3) we get that

9) $$b + e \not\equiv 0(m).$$

If $b \equiv 0(m)$ then, in view of 6), 7), 8), 9), by Lemma 6.2 there exists c' such that $F'(a', c') \neq 0$, $c' \not\equiv 0(m)$, and $c' \leqq e$; it follows that F is of $[R', x', y', J]$-pretype $(m; a', 0, c')$, and by 2) we get that $c' \leqq a' - a - b$. If $b \not\equiv 0(m)$ and $a' - a - b < m/p$ then $e < m/p$ by 2), and hence, in view of 6), 7), 8), 9), by Lemma 6.5 there exists c' such that $F'(a', c') \neq 0$, $c' \not\equiv 0(m)$, and $c' \leqq m/p$; it follows that F is of $[R', x', y', J]$-pretype $(m; a', 0, c')$. If $b \not\equiv 0(m)$ and $a' - a - b < m - m/p$ then $e + m/p < m$ by 2), and hence, in view of 6), 7), 8), 9), by Lemma 6.7 there exists c' such that $F'(a', c') \neq 0$, $c' \not\equiv 0(m)$, and $c' \leqq e + m/p$; it follows that F is of $[R', x', y', J]$-pretype $(m; a', 0, c')$, and by 2) we get that $c' \leqq a' - a - b + m/p$.

Lemma 7.4. *Let $m = p^n$ where n is a positive integer. Assume that $w(y) \geqq w(x)$, $F \neq 0$, and $F(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j \leqq \operatorname{ord}_R F$. Let $a' = \operatorname{ord}_R F$. Then F is of $[R', x', y', J]$-pretype $(m; a', 0, c')$ where $c' \leqq a'$.*

Proof. If $w(y) > w(x)$ then our assertion follows from Lemma 7.2 by taking $a = b = 0$. If $w(y) = w(x)$ then our assertion follows from Lemma 7.3 (1) by taking $a = b = 0$.

Lemma 7.5. *Assume that $w(y) \geqq w(x)$, and F is of $[R, x, y, J]$-pretype $(m; a, b, c)'$ where $b \equiv 0(m)$. Let $a' = \operatorname{ord}_R F$. Let $b' = b$ if $w(y) > w(x)$, and $b' = 0$ if $w(y) = w(x)$. Then $b' \equiv 0(m)$, and F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq c$.*

Proof. Clearly $b' \equiv 0(m)$, $a' - a - b \leqq c$, $0 \neq F \in x^a y^b$, and $F(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j \leqq \operatorname{ord}_R F$. If $w(y) > w(x)$ then by Lemma 7.2 we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq a' - a - b$ and hence $c' \leqq c$. If $w(y) = w(x)$ then by Lemma 7.3 (1) we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq a' - a - b$ and hence $c' \leqq c$.

Lemma 7.6. *Assume that $w(y) \geqq w(x)$, and F is of $[R, x, y, J]$-pretype $(m; a, b, c)'$ where $[m; b, c] < m$. Let $a' = \operatorname{ord}_R F$. Let $b' = b$ if $w(y) > w(x)$, and*

$b'=0$ *if* $w(y)=w(x)$. *Then* F *is of* $[R', x', y', J]$*-pretype* $(m; a', b', c')$ *where* $[m; b', c'] \leqq [m; b, c]$.

Proof. Clearly $a'-a-b \leqq c$, $0 \neq F \in x^a y^b R$, and $F(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \operatorname{ord}_R F$. If $w(y)>w(x)$ then by Lemma 7.2 we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq a'-a-b$; since $b'=b$ and $a'-a-b \leqq c$ we deduce that $[m; b', c'] \leqq [m; b, c]$. If $w(y)=w(x)$ and either $a' \not\equiv 0(m)$ or $b \equiv 0(m)$ then by Lemma 7.3 (1) we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq a'-a-b$; since $b'=0$ and $a'-a-b \leqq c$ we deduce that $[m; b', c'] \leqq [m; b, c]$. If $w(y)=w(x)$, $a' \equiv 0(m)$, $b' \not\equiv 0(m)$, and $c<m/p$ then $a'-a-b<m/p$ and hence by Lemma 7.3 (2) we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq m/p$; since $b'=0$ we deduce that $[m; b', c']=0$ and hence $[m; b', c'] \leqq [m; b, c]$. Now it only remains to consider the case when $w(y)=w(x)$, $a' \equiv 0(m)$, $b \not\equiv 0(m)$, and $c' \geqq m/p$; since $[m; b, c]<m$ we must then have $[m; b, c]=c+m/p<m$ and hence $c<m-m/p$; since $a'-a-b \leqq c$ we conclude that $a'-a-b<m-m/p$; therefore by Lemma 7.3 (3) we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq a'-a-b+m/p$; since $b'=0$ and $a'-a-b \leqq c$ we deduce that $[m; b', c'] \leqq c' \leqq c+m/p$ and hence $[m; b', c'] \leqq [m; b, c]$.

Lemma 7.7. *Let* $m=p^n$ *where* n *is a positive integer. Assume that* $w(y) \geqq w(x)$, $0 \neq F \in x^a y^b R$ *where* a *and* b *are nonnegative integers, and* $F(i,j)=0$ *whenever* $(i,j) \equiv 0(m)$ *and* $i+j \leqq \operatorname{ord}_R F$. *Let* $a'=\operatorname{ord}_R F$. *Assume that* $a'-a-b<m/p$. *Let* $b'=b$ *if* $w(y)>w(x)$, *and* $b'=0$ *if* $w(y)=w(x)$. *Then* F *is of* $[R', x', y', J]$*-pretype* $(m; a', b', c')$ *where* $[m; b', c']=0$.

Proof. If $w(y)>w(x)$ then by Lemma 7.2 we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq a'-a-b$; since $a'-a-b<m/p$ we deduce that $[m; b', c']=0$. If $w(y)=w(x)$ and either $a' \not\equiv 0(m)$ or $b \equiv 0(m)$ then by Lemma 7.3 (1) we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq a'-a-b$; since $a'-a-b<m/p$ we deduce that $[m; b', c']=0$. If $w(y)=w(x)$, $a' \equiv 0(m)$, and $b \not\equiv 0(m)$, then by Lemma 7.3 (2) we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq m/p$; since $b'=0$ we deduce that $[m; b', c']=0$.

Lemma 7.8. *Assume that* $w(y) \geqq w(x)$, *and* F *is of* $[R, y, x, J]$*-pretype* $(m; b, a, c)'$ *where* $c<m/p$. *Let* $a'=\operatorname{ord}_R F$. *Let* $b'=b$ *if* $w(y)>w(x)$, *and* $b'=0$ *if* $w(y)=w(x)$. *Then* F *is of* $[R', x', y', J]$*-pretype* $(m; a', b', c')$ *where* $[m; b', c']=0$.

Proof. Clearly $0 \neq F \in x^a y^b R$, and $F(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \operatorname{ord}_R F$. Also $a'-a-b \leqq c$ and hence $a'-a-b<m/p$. Therefore by Lemma 7.7 we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $[m; b', c']=0$.

Lemma 7.9. *Assume that* $w(y) \geqq w(x)$, *and* F *is of* $[R, y, x, J]$*-pretype* $(m; b, a, c)'$ *where* $c \leqq m/p$ *and* $a+b+m/p \not\equiv 0(m)$. *Let* $a'=\operatorname{ord}_R F$. *Let* $b'=b$ *if* $w(y)>w(x)$, *and* $b'=0$ *if* $w(y)=w(x)$. *Then* F *is of* $[R', x', y', J]$*-pretype* $(m; a', b', c')$ *where* $[m; b', c']=0$.

Proof. Clearly $a'-a-b \leqq c$, $0 \neq F \in x^a y^b R$, and $F(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \operatorname{ord}_R F$. If $a'-a-b<m/p$ then by Lemma 7.7 we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $[m; b', c']=0$. So now assume

that $a'-a-b \geqq m/p$. Since $a'-a-b \leqq c \leqq m/p$ we then must have $a'-a-b = m/p = c$. Therefore $F(a'-b,\ b) = F(a+c,\ b) \neq 0$ and $a' = a+b+c \not\equiv 0(m)$. Since $F(a'-q,\ q) \neq 0$ where $q=b$, by Lemma 7.2 we get that if $w(y) > w(x)$ then F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c'=0$ and hence $[m; b', c'] = 0$. Since $a' \not\equiv 0(m)$, by Lemma 7.3 (1) we get that if $w(y) = w(x)$ then F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c' \leqq a'-a-b$; since $b'=0$ and $a'-a-b = m/p$ we again have that $[m; b', c'] = 0$.

Lemma 7.10. *Let $m = p^n$ where n is a positive integer. Assume that $w(y) < w(x)$, $0 \neq F \in y^b x^a R$ where b and a are nonnegative integers, and $F(i,j) = 0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \operatorname{ord}_R F$. Let $b' = \operatorname{ord}_R F$. Then F is of $[R', y', x', J]$-pretype $(m; b', a, c')^*$ where $c' \leqq b'-b-a$.*

Proof. Follows from Lemma 7.2 by interchanging x and y.

Lemma 7.11. *Assume that $w(y) < w(x)$, and F is of $[R, y, x, J]$-pretype $(m; b, a, c)'$. Let $b' = \operatorname{ord}_R F$. Then F is of $[R', y', x', J]$-pretype $(m; b', a, c')^*$ where $c' \leqq c$.*

Proof. Now $0 \neq F \in y^b x^a R$, and $F(i,j) = 0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \operatorname{ord}_R F$. Therefore by Lemma 7.10 we get that F is of $[R', y', x', J]$-pretype $(m; b', a, c')^*$ where $c' \leqq b'-b-a$. Clearly $b'-b-a \leqq c$ and hence $c' \leqq c$.

Lemma 7.12. *Assume that $w(y) < w(x)$, and F is of $[R, x, y, J]$-pretype $(m; a, b, c)$. Let $b' = \operatorname{ord}_R F$ and $c' = a+b+c-b'$. Then we have the following.*

(1) *F is of $[R', x', y', J]$-pretype $(m; a, b', c')$.*
(2) *If $b'-a-b \geqq m/p$ then $c' \leqq c - m/p$ and $[m; b', c'] \leqq [m; b, c]$.*
(3) *If $b'-a-b > m/p$ then $[m; b', c'] < [m; b, c]$.*
(4) *If $b'-a-b = m/p$ and $b \not\equiv 0(m)$ then $[m; b', c'] < [m; b. c]$.*
(5) *If $b'-a-b = m/p$, $[m; b, c] < m$, and $p=2$ then $[m; b', c'] = 0$.*
(6) *If $\operatorname{ord}_{R/x} F/(x^a y^b) = c$ then $\operatorname{ord}_{R'/x'} F/(x'^a y'^{b'}) = c'$.*

Proof. Now $\operatorname{ord}_{x'R'} = \operatorname{ord}_{xR}$ and $\operatorname{ord}_{xR} F = a$; hence $\operatorname{ord}_{x'R'} F = a$. Also $\operatorname{ord}_{y'R'} = \operatorname{ord}_R$ and hence $\operatorname{ord}_{y'R'} F = b'$. Therefore $F \in x'^a y'^{b'} R'$. Since $(a, b+c) \not\equiv 0(m)$ and $b'+c' = a+b+c$, we get that $(a, b'+c') \not\equiv 0(m)$. Clearly $b' \leqq a+b+c$ and hence $c' \geqq 0$. Let e be any nonnegative integer. Then

$$F - \sum_{i+j \leqq e} F(i,j)\, x^i y^j \in M^{e+1} \subset y'^{e+1} R'$$

and

$$\sum_{i+j \leqq e} F(i,j)\, x^i y^j = \sum_{i+j \leqq e} F(i,j)\, x'^i y'^{i+j}.$$

Therefore

$$F \equiv \sum_{i+j \leqq e} F(i,j)\, x'^i y'^{i+j} \bmod y'^{e+1} R'$$

and hence

$$F'(i, i+j) = F(i,j) \quad \text{whenever} \quad i \geqq 0,\ j \geqq 0, \text{ and } i+j \leqq e.$$

Since e was an arbitrary nonnegative integer, we get that

1) $$F'(i, i+j) = F(i,j) \quad \text{whenever} \quad i \geqq 0 \text{ and } j \geqq 0.$$

In particular $F'(a, a+b+c)=F(a, b+c)\neq 0$. Since $b'+c'=a+b+c$ we thus get that $F'(a, b'+c')\neq 0$. Therefore F is of $[R', x', y', J]$-pretype $(m; a, b', c')$. This proves (1). (2), (3), (4) and (5) are easily checked by using the fact that $0\leqq c'=c-(b'-a-b)$. Let $c_1=\mathrm{ord}_{R/x}F/(x^a y^b)$ and $c_1'=\mathrm{ord}_{R'/x'}F/(x'^a y'^{b'})$; then $F(a, b+c_1)\neq 0$, $F(a, j)=0$ whenever $j<b+c_1$, $F'(a, b'+c_1')\neq 0$, and $F'(a, j)=0$ whenever $j<b'+c_1'$; therefore by 1) we get that $b'+c_1'=a+b+c_1$ and hence $c_1'=a+b+c_1-b'$; consequently, if $\mathrm{ord}_{R/x}F/(x^a y^b)=c$ then $\mathrm{ord}_{R'/x'}F/(x'^a y'^{b'})=c'$; this proves (6).

Lemma 7.13. *Assume that $w(y)<w(x)$, and F is of $[R, x, y, J]$-pretype $(m; a, b, c)$. Let $d=\mathrm{ord}_R F$. Let b' be the greatest integer such that $b'\equiv 0(m)$ and $b'\leqq d$. Let a' be the greatest integer such that $a'\equiv 0(m)$ and $a'\leqq a$. Let $c'=a+b+c-b'$. Then F is of $[R', x', y', J]$-pretype $(m; a, b', c')$. Moreover, if $b\equiv 0(m)$ and $d-a'-b\geqq m$ then $c'<c$ and $[m; b', c']\leqq \max(0, [m; b, c]-1)$.*

Proof. By Lemma 7.12 (1) we get that F is of $[R', x', y', J]$-pretype $(m; a, d, a+b+c-d)$ and hence $F\in x'^a y'^d R'$, $(a, a+b+c)\not\equiv 0(m)$, and $F'(a, a+b+c)\neq 0$. Clearly $0\leqq b'\leqq d$, $c'\geqq a+b+c-d\geqq 0$, and $b'+c'=a+b+c$. Therefore F is of $[R', x', y', J]$-pretype $(m; a, b', c')$. Now assume that $b\equiv 0(m)$ and $d-a'-b\geqq m$. Then $(b'-a'-b)+(d-b')=d-a'-b\geqq m$, $(b'-a'-b)\equiv 0(m)$, and $0\leqq(d-b')<m$; consequently we must have $b'-a'-b\geqq m$. Since $b'-a-b=(b'-a'-b)-(a-a')$, we therefore get that $b'-a-b\geqq m-(a-a')$; since $a-a'<m$ we then conclude that $b'-a-b>0$, and hence $a+b+c-b'<c$. Therefore $c'<c$. Since $b'\equiv 0(m)$, $b\equiv 0(m)$, and $c'<c$, we finally deduce that $[m; b', c']\leqq \max(0, [m; b, c]-1)$.

Lemma 7.14. *Assume that $w(y)<w(x)$, and F is of $[R, x, y, J]$-pretype $(m; a, b, c)''$ where $b\equiv 0(m)$. Let $d=\mathrm{ord}_R F$. Let a' be the greatest integer such that $a'\equiv 0(m)$ and $a'\leqq a$. Then we have the following.*

(1) *If $d-a'-b\geqq m$ then F is of $[R', x', y', J]$-pretype $(m; a, b', c')$ where b' is the greatest integer such that $b'\equiv 0(m)$ and $b'\leqq d$, and where $c'=a+b+c-b'<c$.*

(2) *If $d-a'-b<m$ then F is of $[R', y', x', J]$-pretype $(m; b', a', c')^*$ where $b'=d$ and $c'<m$.*

Proof. (1) follows from Lemma 7.13. To prove (2) assume that $d-a'-b<m$ and let $b'=d$. Then $0\neq F\in y^b x^{a'} R$, and $F(i, j)=0$ whenever $(i, j)\equiv 0(m)$ and $i+j\leqq \mathrm{ord}_R F$. Therefore by Lemma 7.10 we get that F is of $[R', y', x', J]$-pretype $(m; b', a', c')^*$ where $c'\leqq b'-b-a'$. Now $b'-b-a'=d-a'-b<m$ and hence $c'<m$.

Lemma 7.15. *Assume that $w(y)<w(x)$, and F is of $[R, x, y, J]$-pretype $(m; a, b, c)'$. Let $b'=\mathrm{ord}_R F$. Then we have the following.*

(1) *If $b'-a-b\geqq m/p$ then F is of $[R', x', y', J]$-pretype $(m; a, b', c')$ where $c'\leqq c-m/p$ and $[m; b', c']\leqq [m; b, c]$.*

(2) *If $b'-a-b<m/p$ then F is of $[R', y', x', J]$-pretype $(m; b', a, c')^*$ where $c'<m/p$.*

Proof. (1) follows from parts (1) and (2) of Lemma 7.12. To prove (2) assume that $b'-a-b<m/p$. Now $0\neq F\in y^b x^a R$, and $F(i, j)=0$ whenever $(i, j)\equiv 0(m)$ and $i+j\leqq \mathrm{ord}_R F$. Therefore by Lemma 7.10 we get that F is of $[R', y', x', J]$-

pretype $(m; b', a, c')^*$ where $c' \leqq b' - b - a$. Since $b' - a - b < m/p$, we conclude that $c' < m/p$.

Lemma 7.16. *Assume that* $w(y) < w(x)$, *and* F *is of* $[R, x, y, J]$*-pretype* $(m; a, b, c)'$ *where* $[m; b, c] < m$. *Let* $d = \operatorname{ord}_R F$. *Let* a' *be the greatest integer such that* $a' \equiv 0(m)$ *and* $a' \leqq a$. *Then we have the following.*

(1) *If* $d - a - b < m/p$ *then* F *is of* $[R', y', x', J]$*-pretype* $(m; b', a, c')^*$ *where* $b' = d$ *and* $c' < m/p$.

(2) *If either* 1) $d - a - b > m/p$, *or* 2) $d - a - b = m/p$ *and* $b \not\equiv 0(m)$, *or* 3) $d - a - b = m/p$ *and* $p = 2$, *or* 4) $b \equiv 0(m)$ *and* $d - a' - b \geqq m$, *then* F *is of* $[R', x', y', J]$*-pretype* $(m; a, b', c')$ *where* $b' \leqq d$ *and* $[m; b', c'] \leqq \max(0, [m; b, c] - 1)$.

(3) *If* $d - a - b = m/p$, $b \equiv 0(m)$, $p \neq 2$, *and* $d - a' - b < m$, *then* F *is of* $[R', y', x', J]$*-pretype* $(m; b', a, c')^*$ *where* $b' = d$, $c' \leqq m/p$, *and* $a + b' + m/p \not\equiv 0(m)$.

Proof. (1) follows from Lemma 7.15 (2). (2) follows from Lemma 7.13 and parts (1), (3), (4), and (5) of Lemma 7.12. To prove (3) assume that $d - a - b = m/p$, $b \equiv 0(m)$, $p \neq 2$, and $d - a' - b < m$. Let $b' = d$. Now $0 \neq F \in y^b x^a R$, and $F(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j \leqq \operatorname{ord}_R F$. Therefore by Lemma 7.10 we get that F is of $[R', y', x', J]$-pretype $(m; b', a, c')^*$ where $c' \leqq b' - b - a$. Since $b' - b - a = d - a - b = m/p$ we get that $c' \leqq m/p$. Now $a - a' + (d - a - b) = d - a' - b$ and by assumption $d - a - b = m/p$ and $d - a' - b < m$; therefore $a - a' + m/p < m$; also clearly $a - a' \geqq 0$ and hence $a - a' + m/p > 0$. Thus $0 < a - a' + m/p < m$ and by assumption $p \neq 2$; consequently $2(a - a' + m/p) \not\equiv 0(m)$. Since $a' \equiv 0(m)$, we thus get that $2(a + m/p) \not\equiv 0(m)$. By assumption $b' = d$ and $d - a - b = m/p$, and hence $a + b' + m/p = 2(a + m/p) + b$; again by assumption $b \equiv 0(m)$ and hence $a + b' + m/p \equiv 2(a + m/p) \bmod m$. Therefore $a + b' + m/p \not\equiv 0(m)$.

Lemma 7.17. *Assume that* $w(y) = w(x)$, *and* F *is of* $[R, y, x, J]$*-pretype* $(m; b, a, c)'$ *where* $a \equiv 0(m)$ *and* $c < m$. *Let* $a' = \operatorname{ord}_R F$. *Then* F *is of* $[R', x', y', J]$*-pretype* $(m; a', 0, c')$ *where* $c' < m$.

Proof. Now $0 \neq F \in x^a y^b R$, and $F(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j \leqq \operatorname{ord}_R F$. Also $a' - a - b \leqq c$ and hence $a' - a - b < m$. If either $a' \not\equiv 0(m)$ or $b \equiv 0(m)$ then by Lemma 7.3 (1) we get that F is of $[R', x', y', J]$-pretype $(m; a', 0, c')$ where $c' \leqq a' - a - b$ and hence $c' < m$. So now assume that $a' \equiv 0(m)$ and $b \not\equiv 0(m)$. Let b' be the greatest integer such that $b' \equiv 0(m)$ and $b' \leqq b$. Then $0 < b - b' < m$ and $F \in x^a y^{b'} R$. Since $F \in x^a y^{b'} R$ and $b' \equiv 0(m)$, by Lemma 7.3 (1) we get that F is of $[R', x', y', J]$-pretype $(m; a', 0, c')$ where $c' \leqq a' - a - b'$. In particular then $(a', c') \not\equiv 0(m)$; since $a' \equiv 0(m)$ we must therefore have $c' \not\equiv 0(m)$. Now $0 < b - b' < m$, $0 \leqq a' - a - b < m$, and $(b - b') + (a' - a - b) = a' - a - b' \equiv$ $\equiv 0(m)$; consequently we must have $a' - a - b' = m$. Since $c' \leqq a' - a - b'$ we therefore get that $c' \leqq m$. However $c' \not\equiv 0(m)$ and hence $c' < m$.

Lemma 7.18. *Assume that* $w(y) > w(x)$, *and* F *is of* $[R, y, x, J]$*-pretype* $(m; b, a, c)'$ *where* $a \equiv 0(m)$ *and* $c < m$. *Let* $d = \operatorname{ord}_R F$. *Let* b' *be the greatest integer such that* $b' \equiv 0(m)$ *and* $b' \leqq b$. *Then we have the following.*

(1) *If* $d - b' - a \geqq m$ *then* F *is of* $[R', y', x', J]$*-pretype* $(m; b, a', c')^*$ *where* a' *is the greatest integer such that* $a' \equiv 0(m)$ *and* $a' \leqq d$, *and where* $c' = b + a + c -$ $- a' < c$.

(2) *If* $d-b'-a<m$ *then* F *is of* $[R', x', y', J]$*-pretype* $(m; a', b', c')$ *where* $a'=d$ *and* $c'<m$.

Proof. To prove (1) assume that $d-b'-a\geqq m$ and let a' be the greatest integer such that $a'\equiv 0(m)$ and $a'\leqq d$. Upon interchanging x and y, by Lemma 7.13 we get that F is of $[R', y', x', J]$-pretype $(m; b, a', c')$ where $c'=b+a+c-a'<c$. Clearly $d\leqq a+b+c$ and hence $d-b-a\leqq c$; since $c<m$ we thus get that $d-b-a<m$. Since $d-b-a<m$ and $d-b'-a\geqq m$ we must have $b\neq b'$. Since b' is the greatest integer such that $b'\equiv 0(m)$ and $b'\leqq b$, we conclude that $b\not\equiv 0(m)$. Since $b\not\equiv 0(m)$ and F is of $[R', y', x', J]$-pretype $(m; b, a', c')$, it follows that F is of $[R', y', x', J]$-pretype $(m; b, a', c')^*$.

To prove (2) assume that $d-b'-a<m$ and let $a'=d$. Now $0\neq F\in x^a y^{b'} R$, and $F(i,j)=0$ whenever $(i,j)\equiv 0(m)$ and $i+j\leqq \operatorname{ord}_R F$. Therefore by Lemma 7.2 we get that F is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $c'\leqq d-b'-a$. Since $d-b'-a<m$ we get that $c'<m$.

Lemma 7.19. *Assume that* $w(y)<w(x)$, *and* F *is of* $[R, x, y]$*-stable-pretype* $(m; a, b, c)$. *Let* $b'=\operatorname{ord}_R F$ *and* $c'=a+b+c-b'$. *Then* F *is of* $[R', x', y']$*-stable-pretype* $(m; a, b', c')$.

Proof. By parts (1) and (6) of Lemma 7.12 we get that F is of $[R', x', y', J]$-pretype $(m; a, b', c')$ and $\operatorname{ord}_{R'/x'} F/(x'^a y'^{b'})=c'$. Now $b'\geqq a+b$ and $c'=c-(b'-a-b)$; therefore if $b'\neq a+b$ then $c'<c$, and if $b'=a+b$ then $c'=c$. Since $[m; b, c]=0$ we get that $c\leqq m/p$. Therefore if $b'\neq a+b$ then $c'<m/p$ and hence $[m; b', c']=0$. If $b'=a+b$ then $F/(x^a y^b)$ is a unit in R and hence $c = \operatorname{ord}_{R/x} F/(x^a y^b)=0$. Therefore if $b'=a+b$ then $c'=0$ and hence $[m; b', c']=0$. Therefore in both cases $[m; b', c']=0$ and hence F is of $[R, x, y]$-stable-pretype $(m; a, b', c')$.

§ 8. Translates

Let R be a two dimensional regular local domain with maximal ideal M such that R/M is an algebraically closed field of characteristic $p\neq 0$. Let (x, y) be a basis of M and let J be a coefficient set for R.

Definition 8.1. Let $f(Z)\in K[Z]$ where K is the quotient field of R.

$f(Z)$ is said to be of $[R, x, y, J]$*-type* $(m; a, b, c)$ if $m=p^n$ where n is a positive integer and

$$f(Z)=Z^m+F+\sum_{q=1}^{m-1} f_q Z^{m-q}$$

where $F, f_1, \ldots, f_{m-1}$ are elements in R such that F is of $[R, x, y, J]$-pretype $(m; a, b, c)$ and $\operatorname{ord}_{yR} f_q\geqq bq/m$ for $0<q<m$.

$f(Z)$ is said to be of $[R, x, y, J]$*-type* $(m; a, b, c)'$ if $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$ and $f(0)$ is of $[R, x, y, J]$-pretype $(m; a, b, c)'$.

$f(Z)$ is said to be of $[R, x, y, J]$*-type* $(m; a, b, c)''$ if $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$ and $f(0)$ is of $[R, x, y, J]$-pretype $(m; a, b, c)''$.

$f(Z)$ is said to be of $[R, x, y, J]$*-type* $(m; a, b, c)^*$ if $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$ and $f(0)$ is of $[R, x, y, J]$-pretype $(m; a, b, c)^*$.

$f(Z)$ is said to be of $[R, x, y]$-*stable-type* $(m; a, b, c)$ if $m = p^n$ where n is a positive integer and

$$f(Z) = Z^m + F + \sum_{q=1}^{m-1} f_q Z^{m-q}$$

where $F, f_1, \ldots, f_{m-1}$ are elements in R such that F is of $[R, x, y]$-stable-pretype $(m; a, b, c)$ and $\operatorname{ord}_{yR} f_q \geqq bq/m$ for $0 < q < m$.

Note that the following three conditions are equivalent: (1) $f(Z)$ is of $[R, x, y]$-stable-type $(m; a, b, c)$; (2) $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, $[m; b, c] = 0$, and $\operatorname{ord}_{R/x} f(0)/(x^a y^b) = c$; (3) $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, $[m; b, c] = 0$, and $F(a, j) = 0$ whenever $j < b + c$ where $\Sigma F(i, j)\, x^i y^j$ is the expansion of $f(0)$ in $J[[x, y]]$.

Definition 8.2. Given $z \in R$ and a positive integer m, we say that J is a (z, m)-*faithful* coefficient set for R provided the following condition is satisfied: if r and s are any elements in J such that $r^m + s \in M$ then $r^m + s \in zR$. Given $z \in R$, we say that J is a z-*faithful* coefficient set for R provided J is a (z, m)-faithful coefficient set for R for every positive integer m. Note that if (R', x', y') is a canonical first quadratic transform of (R, x, y, J) and $y/x \in R'$ (resp: $y/x \notin R'$) then $MR' = x'R'$ (resp: $MR' = y'R'$) and hence J is an x'-faithful (resp: a y'-faithful) coefficient set for R'. Also note that if R' is a local domain such that R' dominates R and R' is residually algebraic over R, z is an element in R and m is a positive integer such that J is a (z, m)-faithful coefficient set for R, and z' is an element in R' such that $zR \subset z'R'$, then J is a (z', m)-faithful coefficient set for R'.

Lemma 8.3. *Let $m = p^n$ where n is a positive integer, and let*

$$f(Z) = Z^m + F + \sum_{q=1}^{m-1} f_q Z^{m-q}$$

where $F, f_1, \ldots, f_{m-1}$ are elements in R. Let e be a nonnegative integer. Let $\Sigma F(i, j)\, x^i y^j$ be the expansion of F in $J[[x, y]]$. Since R/M is algebraically closed, for any nonnegative integers u and v there exists a unique element $r(u, v)$ in J such that

$$r(u, v)^m + F(um, vm) \in M .$$

Let

$$r = \sum_{u+v=e} r(u, v)\, x^u y^v$$

and let

$$f'(Z) = f(Z + r) = Z^m + F' + \sum_{q=1}^{m-1} f'_q Z^{m-q}$$

with $F', f'_1, \ldots, f'_{m-1}$ in R. Let $\Sigma F'(i, j)\, x^i y^j$ be the expansion of F' in $J[[x, y]]$. Then we have the following.

(1) *Assume that $\operatorname{ord}_R F \geqq em$ and $f(Z)$ is of preramified-type relative to ord_R. Then $\operatorname{ord}_R F' \geqq em$, and $F'(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j = em$.*

(2) *Assume that $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, $f(Z)$ is of preramified-type relative to ord_{xR}, and J is an (x, m)-faithful coefficient set for R. Then $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, and $f'(Z)$ is of preramified-type relative to ord_{xR}.*

Proof of (1). Now

1) $$F' = r^m + F + \sum_{q=1}^{m-1} f_q r^{m-q}.$$

Since $\operatorname{ord}_R F \geqq em$ and F is of preramified-type relative to ord_R, we get that $\operatorname{ord}_R f_q > eq$ for $0 < q < m$; also by the definition of r we get that $\operatorname{ord}_R r \geqq e$. Therefore by 1) we get that $\operatorname{ord}_R F' \geqq em$ and

2) $$F' = r^m + \sum_{i+j=em} F(i,j)\, x^i y^j \bmod M^{em+1}.$$

Upon applying the multinomial theorem to the defining expression of r we get that

$$r^m = \sum_{u+v=e} r(u,v)^m x^{um} y^{vm} + p \sum_{i+j=em} B(i,j)\, x^i y^j$$

with $B(i,j) \in R$. Now R/M is of characteristic p and hence $p \in M$. Therefore by the above expression for r^m we get that

3) $$r^m \equiv \sum_{u+v=e} r(u,v)^m x^{um} y^{vm} \bmod M^{em+1}.$$

Since $r(u,v)^m + F(um, vm) \in M$, by 2) and 3) we get that

$$F' \equiv \sum_{i+j=em,\,(i,j)\not\equiv 0\,(m)} F(i,j)\, x^i y^j \bmod M^{em+1}.$$

Therefore $F'(i,j) = 0$ whenever $(i,j) \equiv 0(m)$ and $i+j = em$.

Proof of (2). Now

1) $$F' = r^m + F + \sum_{q=1}^{m-1} f_q r^{m-q}$$

and

2) $$f'_q = A(q,0)\, r^q + \sum_{k=1}^{q} A(q,k)\, f_k r^{q-k} \quad \text{for } 0 < q < m$$

where $A(q,k)$ are the elements in R defined by the equations

$$(Z+1)^{m-k} = \sum_{q=k}^{m} A(q,k)\, Z^{m-q}, \qquad 0 \leqq k < m,$$

in $R[Z]$. Since $m = p^n$ we get that

3) $$A(q,0) \in pR \quad \text{for } 0 < q < m.$$

Since $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$ we get 4), 5), 6):

4) $$\operatorname{ord}_{yR} F \geqq b \quad \text{and} \quad \operatorname{ord}_{yR} f_q \geqq bq/m \quad \text{for } 0 < q < m.$$

5) $$\operatorname{ord}_{xR} F = a.$$

6) $$(a, b+c) \not\equiv 0(m) \quad \text{and} \quad F(a, b+c) \neq 0.$$

Since $f(Z)$ is of preramified-type relative to ord_{xR} we get that

7) $$p \in xR.$$

Since $f(Z)$ is of preramified-type relative to ord_{xR}, by 5) we get 8) and 9):

8) $$\operatorname{ord}_{xR} f_q \geqq aq/m \quad \text{for} \quad 0 < q < m.$$

9) If $a \equiv 0(m)$ then $\operatorname{ord}_{xR} f_q > aq/m$ for $0 < q < m$.

By 4) we get that $F(i,j)=0$ whenever $j<b$, and hence by the definitions of $r(u,v)$ we get that $r(u,v)=0$ whenever $v<b/m$. Therefore by the definition of r we get that

10) $$\operatorname{ord}_{yR} r \geqq b.$$

Since $r(u,v)^m+F(um,vm)\in M$ and since J is an (x,m)-faithful coefficient set for R we get that

11) $$r(u,v)^m+F(um,vm)\in xR.$$

By 5) we get that $F(i,j)=0$ whenever $i<a$ and hence by the definition of $r(u,v)$ we get that

12) $$r(u,v)=0 \quad \text{whenever } u<a/m.$$

In view of 12), by the definition of r we get 13) and 14):

13) $$\operatorname{ord}_{xR} r > a/m \quad \text{if} \quad a \not\equiv 0(m).$$

14) $$\operatorname{ord}_{xR} r \geqq a/m \quad \text{if} \quad a \equiv 0(m).$$

By 1), 2), 4), 10) we get that

15) $$\operatorname{ord}_{yR} F' \geqq b \quad \text{and} \quad \operatorname{ord}_{yR} f'_q \geqq bq/m \quad \text{for} \quad 0<q<m.$$

By 2), 3), 7), 8), 9), 13), 14) we get 16) and 17):

16) If $a \not\equiv 0(m)$ then $\operatorname{ord}_{xR} f'_q \geqq aq/m$ for $0<q<m$.

17) If $a \equiv 0(m)$ then $\operatorname{ord}_{xR} f'_q > aq/m$ for $0<q<m$.

By 1), 8), 13) we get that if $a \not\equiv 0(m)$ then $F'-F\in x^{a+1}R$; therefore by 5) and 6) we get the following:

18) If $a \not\equiv 0(m)$ then $\operatorname{ord}_{xR} F' = a$ and $F'(a,b+c)\neq 0$.

In view of 12), upon applying the multinomial theorem to the defining expression of r we get that if $a\equiv 0(m)$ then

$$r^m = pDx^a + \sum_{u+v=e,\,u\geqq a/m} r(u,v)^m x^{um} y^{vm}$$

with $D\in R$. Therefore in view of 7) and 11), we get that if $a\equiv 0(m)$ then

$$r^m + \sum_{i+j=em,\,(i,j)\equiv 0(m),\,i\geqq a} F(i,j)\,x^i y^j \in x^{a+1}R.$$

Therefore, in view of 1), 9), and 14), we get that if $a\equiv 0(m)$ then

$$F' \equiv F - \sum_{i+j=em,\,(i,j)\equiv 0(m),\,i\geqq a} F(i,j)\,x^i y^j \bmod x^{a+1}R.$$

Therefore, in view of 5) and 6), we get that if $a\equiv 0(m)$ then $\operatorname{ord}_{xR} F' \geqq a$ and $F'(a,b+c)=F(a,b+c)\neq 0$. Thus we get the following:

19) If $a\equiv 0(m)$ then $\operatorname{ord}_{xR} F' = a$ and $F'(a,b+c)\neq 0$.

By 6), 7), 15), 16), 17), 18), 19) we conclude that $f'(Z)$ is of $[R,x,y,J]$-type $(m;a,b,c)$, and $f'(Z)$ is of preramified-type relative to ord_{xR}.

Lemma 8.4. *Let $f(Z)\in R[Z]$ be of $[R,x,y,J]$-type $(m;a,b,c)$. Assume that $f(Z)$ is of ramified-type relative to* ord_R, *$f(Z)$ is of preramified-type relative to*

ord_{xR}, *and* J *is an* (x, m)*-faithful coefficient set for* R. *Then there exists an* R*-translate* $f'(Z)$ *of* $f(Z)$ *such that* $f'(Z)$ *is of* $[R, x, y, J]$*-type* $(m; a, b, c)'$, *and* $f'(Z)$ *is of preramified-type relative to* ord_{xR}.

Proof. By induction on d we shall show that if d is any integer such that $d \geqq -1$ then there exists an R-translate $f^{(d)}(Z)$ of $f(Z)$ such that $f^{(d)}(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, $f^{(d)}(Z)$ is of preramified-type relative to ord_{xR}, and $F^{(d)}(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j \leqq \min(d, \mathrm{ord}_R F^{(d)})$ where $F^{(d)} = f^{(d)}(0)$ and $\Sigma F^{(d)}(i, j) x^i y^j$ is the expansion of $F^{(d)}$ in $J[[x, y]]$; it will then suffice to take $f'(Z) = f^{(a+b+c)}(Z)$. For $d = -1$ we can take $f^{(-1)}(Z) = f(Z)$. Now let $d \geqq 0$ and suppose we have found $f^{(d-1)}(Z)$. If either $d \neq \mathrm{ord}_R F^{(d-1)}$ or $\mathrm{ord}_R F^{(d-1)} \not\equiv 0(m)$ then it is enough to take $f^{(d)}(Z) = f^{(d-1)}(Z)$. So now assume that $\mathrm{ord}_R F^{(d-1)} = d \equiv 0(m)$. Since $f^{(d-1)}(Z)$ is an R-translate of $f(Z)$ we get that $f^{(d-1)}(Z)$ is of preramified-type relative to ord_R. Therefore by Lemma 8.3 there exists $r \in R$ such that for $f^{(d)}(Z) = f^{(d-1)}(Z + r)$ we have that $f^{(d)}(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, $f^{(d)}(Z)$ is of preramified-type relative to ord_{xR}, $\mathrm{ord}_R F^{(d)} \geqq d$ where $F^{(d)} = f^{(d)}(0)$, and $F^{(d)}(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j = d$ where $\Sigma F^{(d)}(i, j) x^i y^j$ is the expansion of $F^{(d)}$ in $J[[x, y]]$. Clearly $f^{(d)}(Z)$ is an R-translate of $f(Z)$, and $F^{(d)}(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j \leqq \min(d, \mathrm{ord}_R F^{(d)})$.

Lemma 8.5. *Let* $m = p^n$ *where* n *is a positive integer, and let*

$$f(Z) = Z^m + F + \sum_{q=1}^{m-1} f_q Z^{m-q}$$

where $F, f_1, \ldots, f_{m-1}$ *are elements in* R. *Let* $\Sigma F(i, j) x^i y^j$ *be the expansion of* F *in* $J[[x, y]]$. *Assume that* $f(Z)$ *is of* $[R, x, y, J]$*-type* $(m; a, b, c)$ *where* $a \equiv 0(m)$, $f(Z)$ *is of preramified-type relative to* ord_{xR}, *and* J *is an* (x, m)*-faithful coefficient set for* R. *Let* v *be any nonnegative integer. Since* R/M *is algebraically closed, there exists a unique element* s *in* J *such that*

$$s^m + F(a, vm) \in M .$$

Let

$$r = s x^{a/m} y^v$$

and let

$$f'(Z) = f(Z + r) = Z^m + F' + \sum_{q=1}^{m-1} f'_q Z^{m-q}$$

with $F', f'_1, \ldots, f'_{m-1}$ *in* R. *Let* $\Sigma F'(i, j) x^j y^j$ *be the expansion of* F' *in* $J[[x, y]]$. *Then* $f'(Z)$ *is of* $[R, x, y, J]$*-type* $(m; a, b, c)$, $f'(Z)$ *is of preramified-type relative to* ord_{xR}, and $F'(a, vm) = 0$.

Proof. Now

1) $$F' = s^m x^a y^{vm} + F + \sum_{q=1}^{m-1} (s x^{a/m} y^v)^{m-q} f_q$$

and

2) $$f'_q = A(q, 0)(s x^{a/m} y^v)^q + \sum_{k=1}^{q} A(q, k)(s x^{a/m} y^v)^{m-q} f_k \quad \text{for } 0 < q < m$$

where $A(q,k)$ are the elements in R defined by the equations

$$(Z+1)^{m-k}=\sum_{q=k}^{m} A(q,k)\,Z^{m-q}, \qquad 0\leqq k<m,$$

in $R[Z]$. Since $m=p^n$ we get that

3) $$A(q,0)\in pR \quad \text{for} \quad 0<q<m.$$

Since $f(Z)$ is of $[R,x,y,J]$-type $(m;a,b,c)$ and $a\equiv 0(m)$ we get 4), 5), 6):

4) $$\operatorname{ord}_{yR} F\geqq b \quad \text{and} \quad \operatorname{ord}_{yR} f_q \geqq bq/m \quad \text{for} \quad 0<q<m.$$

5) $$\operatorname{ord}_{xR} F=a.$$

6) $$F(a,b+c)\neq 0 \quad \text{and} \quad b+c\not\equiv 0(m).$$

Since $f(Z)$ is of preramified-type relative to ord_{xR} we get that

7) $$p\in xR.$$

Since $f(Z)$ is of preramified-type relative to ord_{xR} and $a\equiv 0(m)$, by 5) we get that

8) $$\operatorname{ord}_{xR} f_q > aq/m \quad \text{for} \quad 0<q<m.$$

Since $s^m+F(a,vm)\in M$ and J is an (x,m)-faithful coefficient set for R we get that $s^m+F(a,vm)\in xR$ and hence

9) $$s^m x^a y^{vm}+F(a,vm)\,x^a y^{vm}\in x^{a+1}R.$$

By 2), 3), 7), 8) we get that

10) $$\operatorname{ord}_{xR} f'_q > aq/m \quad \text{for} \quad 0<q<m.$$

By 1) and 8) we get that

$$F'\equiv s^m x^a y^{vm}+F \bmod x^{a+1}R.$$

Therefore by 9) we get that

11) $$F'\equiv \sum_{(i,j)\neq(a,vm)} F(i,j)\,x^i y^j \bmod x^{a+1}R.$$

By 11) we get that

12) $$F'(a,vm)=0.$$

By 5), 6), 11) we get that $\operatorname{ord}_{xR} F'\geqq a$ and $F'(a,b+c)=F(a,b+c)\neq 0$. Therefore

13) $$\operatorname{ord}_{xR} F'=a \quad \text{and} \quad F'(a,b+c)\neq 0.$$

By 4) we get that if $vm<b$ then $F(a,vm)=0$ and hence $r=0$. Therefore our assertion is trivial when $vm<b$. Now assume that $b\leqq vm$. Then by 1), 2), 4) we get that

14) $$\operatorname{ord}_{yR} F'\geqq b \quad \text{and} \quad \operatorname{ord}_{yR} f'_q\geqq bq/m \quad \text{for} \quad 0<q<m.$$

By 6), 7), 10), 12), 13), 14) we conclude that $f'(Z)$ is of $[R,x,y,J]$-type $(m;a,b,c)$, $f'(Z)$ is of preramified-type relative to ord_{xR}, and $F'(a,vm)=0$.

Lemma 8.6. *Let $f(Z)\in R[Z]$ be of $[R,x,y,J]$-type $(m;a,b,c)$ where $[m;b,c]=0$. Assume that $f(Z)$ is of preramified-type relative to* ord_{xR}, *and J*

is an (x, m)-faithful coefficient set for R. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y]$-stable-type $(m; a, b, c')$ where $c' \leqq c$, and $f'(Z)$ is of preramified-type relative to ord_{xR}.

Proof. If $a \not\equiv 0(m)$ then upon taking $c' = \operatorname{ord}_{R/x} f(0)/(x^a y^b)$ we get that $c' \leqq c$ and hence $f(Z)$ is of $[R, x, y]$-stable-type $(m; a, b, c)$, and hence it suffices to take $f'(Z) = f(Z)$. Now assume that $a \equiv 0(m)$. Let v be the smallest integer such that $b \leqq vm$. By Lemma 8.5 there exists $r \in R$ such that for $f'(Z) = f(Z+r)$ we have that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, $f'(Z)$ is of preramified-type relative to ord_{xR}, and $F'(a, vm) = 0$ where $F' = f'(0)$ and $\Sigma F'(i,j) x^i y^j$ is the expansion of F' in $J[[x, y]]$. Let $c' = \operatorname{ord}_{R/x} F'/(x^a y^b)$. Then $F'(a, b+c') \neq 0$ and $c' \leqq c$. Since $[m; b, c] = 0$ and $c' \leqq c$ we get that $[m; b, c'] = 0$ and $c' < m$. Since $F'(b, b+c') \neq 0$ and $F(a, vm) = 0$, we get that $b + c' \neq vm$. Thus $0 \leqq c' < m$, $b + c' \neq vm$, and v is the smallest integer such that $b \leqq vm$; therefore $b + c' \not\equiv 0(m)$. Consequently F' is of $[R, x, y]$-stable-pretype $(m; a, b, c')$ and hence $f'(Z)$ is of $[R, x, y]$-stable-type $(m; a, b, c')$.

Lemma 8.7. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J]$-type $(m; a, b, c)$. Assume that $f(Z)$ is of preramified-type relative to* ord_{xR}, *and J is an (x, m)-faithful coefficient set for R. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)''$ and $f'(Z)$ is of preramified-type relative to* ord_{xR}.

Proof. If $(a, b) \not\equiv 0(m)$ then it is enough to take $f'(Z) = f(Z)$. If $(a, b) \equiv 0(m)$ then by Lemma 8.5 there exists $r \in R$ such that for $f'(Z) = f(Z+r)$ we have that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, $f'(Z)$ is of preramified-type relative to ord_{xR}, and $F'(a, b) = 0$ where $\Sigma F'(i,j) x^i y^j$ is the expansion of $f'(0)$ in $J[[x, y]]$; it follows that $f'(Z)$ is then of $[R, x, y, J]$-type $(m; a, b, c)''$.

Lemma 8.8. *Let $m = p^n$ where n is a positive integer, and let*

$$f(Z) = Z^m + F + \sum_{q=1}^{m-1} f_q Z^{m-q}$$

where $F, f_1, \ldots, f_{m-1}$ are elements in R. Let $\Sigma F(i,j) x^i y^j$ and $\Sigma f_q(i,j) x^i y^j$ be the respective expansions of F and f_q in $J[[x, y]]$. Let u and v be nonnegative integers. Since R/M is algebraically closed, there exists $s \in J$ such that

$$s^m + F(um, vm) + \sum_{q=1}^{m-1} f_q(uq, vq)\, s^{m-q} \in M. \tag{1}$$

For any such s let

$$r = s x^u y^v$$

and let

$$f'(Z) = f(Z+r) = Z^m + F' + \sum_{q=1}^{m-1} f'_q Z^{m-q}$$

with $F', f'_1, \ldots, f'_{m-1}$ in R. Let $\Sigma F'(i,j) x^i y^j$ be the expansion of F' in $J[[x, y]]$. Then we have the following.

(1) *Assume that* $\operatorname{ord}_R F \geqq (u+v)m$, *and $f(Z)$ is of prenonsplitting-type relative to* ord_R. *Then* $\operatorname{ord}_R F' \geqq (u+v)m$ *and* $F'(um, vm) = 0$.

(2) *Assume that $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$ where $a < um$, and $f(Z)$ is of prenonsplitting-type relative to* ord_{xR}. *Then $f'(Z)$ is of $[R, x, y, J]$-*

type $(m; a, b, c)$, *and* $f'(Z)$ *is of prenonsplitting-type relative to* ord_{xR}. *Moreover, if* $f(Z)$ *is of preramified-type relative to* ord_{xR} *then so is* $f'(Z)$.

Proof of (1). Now

1) $$F' = f(r) = (sx^u y^v)^m + F + \sum_{q=1}^{m-1} (sx^u y^v)^{m-q} f_q.$$

Let $e = u + v$. Since $\mathrm{ord}_R F \geqq em$ and $f(Z)$ is of prenonsplitting-type relative to ord_R, we get that $\mathrm{ord}_R f_q \geqq eq$ for $0 < q < m$. Consequently $F(i,j) = 0$ whenever $i + j < em$, and $f_q(i,j) = 0$ whenever $0 < q < m$ and $i + j < eq$. Therefore by 1) we get that

2) $$F' \equiv F_1 + F_2 \bmod M^{em+1}$$

where

$$F_1 = x^{um} y^{vm}\left[s^m + F(um, vm) + \sum_{q=1}^{m-1} f_q(uq, vq)\, s^{m-q}\right]$$

and

$$F_2 = \sum_{i+j=em,\,(i,j)\neq(um,vm)} F(i,j)\, x^i y^j + {} $$
$$+ \sum_{q=1}^{m-1} (sx^u y^v)^{m-q} \sum_{i+j=eq,\,(i,j)\neq(uq,vq)} f_q(i,j)\, x^i y^j.$$

By the definitions of s and F_1 we get that

3) $$F_1 \in M^{em+1}.$$

By the definition of F_2 we get that

$$F_2 = \sum_{i+j=em,\,(i,j)\neq(um,vm)} G(i,j)\, x^i y^j$$

with $G(i,j) \in R$. Since J is a coefficient set for R there exist unique elements $H(i,j)$ in J such that $H(i,j) - G(i,j) \in M$, and then by the above equation we get that

4) $$F_2 \equiv \sum_{i+j=em,\,(i,j)\neq(um,vm)} H(i,j)\, x^i y^j \bmod M^{em+1}.$$

By 2), 3), 4) we get that

$$F' \equiv \sum_{i+j=em,\,(i,j)\neq(um,vm)} H(i,j)\, x^i y^j \bmod M^{em+1}.$$

Since $H(i,j) \in J$ we conclude that $\mathrm{ord}_R F' \geqq em$ and $F'(um, vm) = 0$.

Proof of (2). Now

1) $$F' = (sx^u y^v)^m + F + \sum_{q=1}^{m-1} (sx^u y^v)^{m-q} f_q$$

and

2) $$f'_q = A(q,0)\,(sx^u y^v)^q + \sum_{k=1}^{q} A(q,k)\,(sx^u y^v)^{q-k} f_k \quad \text{for} \quad 0 < q < m$$

where $A(q, k)$ are the elements in R defined by the equations

$$(Z+1)^{m-k} = \sum_{q=k}^{m} A(q, k)\, Z^{m-q}, \qquad 0 \leqq k < m,$$

in $R[Z]$. Since $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$ we get 3), 4),5):

3) $\qquad \operatorname{ord}_{yR} F \geqq b \quad \text{and} \quad \operatorname{ord}_{yR} f_q \geqq bq/m \qquad \text{for } 0<q<m.$

4) $\qquad \operatorname{ord}_{xR} F = a.$

5) $\qquad (a, b+c) \not\equiv 0(m) \quad \text{and} \quad F(a, b+c) \neq 0.$

Since $f(Z)$ is of prenonsplitting-type relative to ord_{xR}, by 4) we get that

6) $\qquad \operatorname{ord}_{xR} f_q \geqq aq/m \qquad \text{for } 0<q<m.$

If $vm<b$ then by 3) we would get that $F(um, vm)=0$ and $f_q(uq, vq)=0$ for $0<q<m$, and hence by the definition of r we would get that $r=0$ and hence $f'(Z)=f(Z)$. Therefore our assertion is trivial when $vm<b$. Now assume that $b \leqq vm$. Then by 1), 2), 3) we get that

7) $\qquad \operatorname{ord}_{yR} F' \geqq b \quad \text{and} \quad \operatorname{ord}_{yR} f'_q \geqq bq/m \qquad \text{for } 0<q<m.$

Since $a<um$, by 1), 2), 4), 6) we get 8) and 9):

8) $\qquad \operatorname{ord}_{xR} F' = a \quad \text{and} \quad \operatorname{ord}_{xR} f'_q \geqq aq/m \qquad \text{for } 0<q<m.$

9) $\qquad F' \equiv F \bmod x^{a+1} R.$

By 5) and 9) we get that

10) $\qquad (a, b+c) \not\equiv 0(m) \quad \text{and} \quad F'(a, b+c) \neq 0.$

By 7), 8), 10) we conclude that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, and $f'(Z)$ is of prenonsplitting-type relative to ord_{xR}.

Note that if $f(Z)$ is of preramified-type relative to ord_{xR} then $p \in xR$. Therefore if $f(Z)$ is of preramified-type relative to ord_{xR} and $a \not\equiv 0(m)$ then by 8) we get that $f'(Z)$ is of preramified-type relative to ord_{xR}. If $f(Z)$ is of preramified-type relative to ord_{xR} and $a \equiv 0(m)$ then by 4) we get that $\operatorname{ord}_{xR} f_q > aq/m$ for $0<q<m$ and hence, because $a<um$, by 1), 2) and 4) we get that $\operatorname{ord}_{xR} F' = a$ and $\operatorname{ord}_{xR} f'_q > aq/m$ for $0<q<m$, and hence again $f'(Z)$ is of preramified-type relative to ord_{xR}.

Lemma 8.9. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J]$-type $(m; a, b, c)$ where $c<m$. Let $\Sigma F(i,j) x^i y^j$ be the expansion of $f(0)$ in $J[[x, y]]$, and let u and v be the unique integers such that $0 \leqq um-a<m$ and $0 \leqq vm-b<m$. Then the following two conditions are equivalent:*

1) *$f(Z)$ is not of $[R, x, y, J]$-type $(m; a, b, c)'$.*
2) *$\operatorname{ord}_R f(0) = um+vm$ and $F(um, vm) \neq 0$.*

Proof. Straightforward.

Lemma 8.10. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J]$-type $[m; a, b, c]$ where $c<m$. Assume that $f(Z)$ is of prenonsplitting-type relative to ord_R, $f(Z)$ is of preramified-type relative to ord_{xR}, and J is an (x, m)-faithful coefficient set for R. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$, and $f'(Z)$ is of preramified-type relative to ord_{xR}.*

Proof. If $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$ then we can take $f'(Z) = f(Z)$. Now assume that $f(Z)$ is not of $[R, x, y, J]$-type $(m; a, b, c)'$. Then by Lemma 8.9 we get that $\operatorname{ord}_R f(0) = um + vm$ where u and v are the unique integers such that $0 \leqq um - a < m$ and $0 \leqq vm - b < m$. If $um = a$ then by Lemma 8.5 and if $um \neq a$ then by Lemma 8.8 there exists $r \in R$ such that for $f'(Z) = f(Z + r)$ we have that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, $f'(Z)$ is of preramified-type relative to ord_{xR}, and $F'(um, vm) = 0$ where $\Sigma F'(i,j)\, x^i y^j$ is the expansion of $f'(0)$ in $J[[x, y]]$. By Lemma 8.9 is follows that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$.

Lemma 8.11. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J]$-type $(m; a, b, c)^*$ where $c < m$. Assume that $f(Z)$ is of prenonsplitting-type relative to ord_R, and $f(Z)$ is of prenonsplitting-type relative to ord_{xR}. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$, and $f'(Z)$ is of prenonsplitting-type relative to ord_{xR}.*

Proof. If $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$ then we can take $f'(Z) = f(Z)$. Now assume that $f'(Z)$ is not of $[R, x, y, J]$-type $(m; a, b, c)'$. Then by Lemma 8.9 we get that $\operatorname{ord}_R f(0) = um + vm$ and $F(um, vm) \neq 0$ where u and v are the unique integers such that $0 \leqq um - a < m$ and $0 \leqq vm - b < m$ and where $\Sigma F(i,j) x^i y^j$ is the expansion of $f(0)$ in $J[[x, y]]$. By assumption $F(i,j) = 0$ whenever $(i,j) \equiv 0(m)$ and $i \leqq a$. Therefore $um > a$ and hence by Lemma 8.8 there exists $r \in R$ such that for $f'(Z) = f(Z + r)$ we have that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, $f'(Z)$ is of prenonsplitting-type relative to ord_{xR}, and $F'(um, vm) = 0$ where $\Sigma F'(i,j)\, x^i y^j$ is the expansion of $f'(0)$ in $J[[x, y]]$. By Lemma 8.9 it follows that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$.

Lemma 8.12. *Let $f(Z) \in R[Z]$ be of $[R, x, y]$-stable-type $(m; a, b, c)$. Assume that $f(Z)$ is of prenonsplitting-type relative to ord_R, and $f(Z)$ is of prenonsplitting-type relative to ord_{xR}. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$, and $f'(Z)$ is of prenonsplitting-type relative to ord_{xR}.*

Proof. If $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$ then we can take $f'(Z) = f(Z)$. Now assume that $f(Z)$ is not of $[R, x, y, J]$-type $(m; a, b, c)'$. By definition $[m; b, c] = 0$ and hence $c < m$. Therefore by Lemma 8.9 we get that $\operatorname{ord}_R f(0) = um + vm$ and $F(um, vm) \neq 0$ where u and v are the unique integers such that $0 \leqq um - a < m$ and $0 \leqq vm - b < m$ and where $\Sigma F(i,j)\, x^i y^j$ is the expansion of $f(0)$ in $J[[x, y]]$. Suppose if possible that $a = um$; since $(a, b + c) \not\equiv 0(m)$ we then get that $b + c \not\equiv 0(m)$; since $F(a, vm) \neq 0$ and $F(a, j) = 0$ whenever $j < b + c$ we then must have $b + c < vm$ and hence $a + b + c < um + vm = \operatorname{ord}_R f(0)$; this is a contradiction because $F(a, b + c) \neq 0$. Therefore $a \neq um$ and hence $a < um$. Therefore by Lemma 8.8 there exists $r \in R$ such that for $f'(Z) = f(Z + r)$ we have that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, $f'(Z)$ is of prenonsplitting-type relative to ord_{xR}, and $F'(um, vm) = 0$ where $\Sigma F'(i,j)\, x^i y^i$ is the expansion of $f'(0)$ in $J[[x, y]]$. By Lemma 8.9 it follows that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$.

Lemma 8.13. *Let $m = p^n$ where n is a positive integer and let $f(Z)$ be a monic polynomial of degree m in Z with coefficients in R. Assume that $f(Z)$ is of* *ramified-type relative to* ord_R, *R is of characteristic p, and* $f(Z) \neq Z^m + f(0)$. *Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(0) \neq 0$ and $F'(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \operatorname{ord}_R f'(0)$ where $\Sigma F'(i,j)\, x^i y^j$ is the expansion of $f'(0)$ in $J[[x,y]]$.*

Proof. Let f_q be the coefficient of Z^{m-q} in $f(Z)$. Since $f(Z) \neq Z^m + f(0)$, there exists an integer k such that $0<k<m$, $f_k \neq 0$, and $f_q = 0$ whenever $0<q<k$. By induction on d we shall show that if d is any integer such that $d \geqq -1$ then there exists an R-translate $f^{(d)}(Z)$ of $f(Z)$ such that $F^{(d)}(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \min(d, \operatorname{ord}_R F^{(d)})$ where $F^{(d)} = f^{(d)}(0)$ and $\Sigma F^{(d)}(i,j)\, x^i y^j$ is the expansion of $F^{(d)}$ in $J[[x,y]]$. For $d=-1$ it suffices to take $f^{(-1)}(Z) = f(Z)$. Now let $d \geqq 0$ and suppose we have found $f^{(d-1)}(Z)$. If either $d \neq \operatorname{ord}_R F^{(d-1)}$ or $\operatorname{ord}_R F^{(d-1)} \not\equiv 0(m)$ then it is enough to take $f^{(d)}(Z) = f^{(d-1)}(Z)$. So now assume that $\operatorname{ord}_R F^{(d-1)} = d \equiv 0(m)$. Since $f^{(d-1)}(Z)$ is an R-translate of $f(Z)$ we get that $f^{(d-1)}(Z)$ is of preramified-type relative to ord_R. Therefore by Lemma 8.3 (1) there exists $r \in R$ such that for $f^{(d)}(Z) = f^{(d-1)}(Z+r)$ we have that $F^{(d)}(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq d$ where $F^{(d)} = f^{(d)}(0)$ and $\Sigma F^{(d)}(i,j)\, x^i y^j$ is the expansion of $F^{(d)}$ in $J[[x,y]]$; clearly then $F^{(d)}(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \min(d, \operatorname{ord}_R F^{(d)})$. This completes the induction on d. Since $f_k \neq 0$, there exists a positive integer h such that $h \geqq (m/k) \operatorname{ord}_R f_k$. Let $f'(Z) = f^{(h)}(Z)$, let $F' = f'(0)$, and let $\Sigma F'(i,j) x^i y^j$ be the expansion of F' in $J[[x,y]]$. Then $F'(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \min(h, \operatorname{ord}_R F')$. Let f'_q be the coefficient of Z^{m-q} in $f'(Z)$. Since R is of characteristic p we get that $f'_k = f_k$ and hence $h \geqq (m/k) \operatorname{ord}_R f'_k$. Since $f'(Z)$ is an R-translate of $f(Z)$ we get that $f'(Z)$ is of preramified-type relative to ord_R and hence $\operatorname{ord}_R F' \leqq (m/k) \operatorname{ord}_R f'_k$. Therefore $\operatorname{ord}_R F' \leqq h$ and hence $F'(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \operatorname{ord}_R F'$.

Lemma 8.14. *Let $m = p^n$ where n is a positive integer and let $f(Z)$ be a monic polynomial of degree m in Z with coefficients in R. Assume that $f(Z)$ is of ramified-type relative to ord_R, and $f(z) \neq 0$ for all $z \in R^*$ where R^* is the completion of R. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(0) \neq 0$ and $F'(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \operatorname{ord}_R f'(0)$ where $\Sigma F'(i,j)\, x^i y^j$ is the expansion of $f'(0)$ in $J[[x,y]]$.*

Proof. Let $F = f(0)$ and let $\Sigma F(i,j)\, x^i y^j$ be the expansion of F in $J[[x,y]]$. Let s_0 be the unique element in J such that $s_0^m + F(0,0) \in M$. Define $s_e \in R$ for all $e>0$ by the following recurrence equation:

$$s_e = \begin{cases} s_{e-1} + \sum\limits_{u+v=e} r(e,u,v)\, x^u y^v & \text{if } \operatorname{ord}_R f(s_{e-1}) \geqq em \\ \\ s_{e-1} & \text{if } \operatorname{ord}_R f(s_{e-1}) < em \end{cases}$$

where $r(e,u,v)$ is the unique element in J such that

$$r(e,u,v)^m + F^{(e-1)}(um, vm) \in M$$

where $\Sigma F^{(e-1)}(i,j)\, x^i y^j$ is the expansion of $f(s_{e-1})$ in $J[[x,y]]$. Note that then $s_e - s_{e-1} \in M^e$ for all $e>0$. Let $f^{(e)}(Z) = f(Z+s_e)$ and $F^{(e)} = f^{(e)}(0)$ for all $e \geqq 0$. Then $F^{(e)} = f(s_e)$. Since $f^{(e)}(Z)$ is an R-translate of $f(Z)$ we get that $f^{(e)}(Z)$ is of preramified-type relative to ord_R for all $e \geqq 0$. Therefore, in view of Lemma 8.3 (1), by induction one e it follows that for all $e \geqq 0$ we have: $F^{(e)}(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \min(em, \operatorname{ord}_R F^{(e)})$. Let M^* be the maximal ideal in R^*. Since $s_e - s_{e-1} \in M^e$ for all $e>0$, there exists $z \in R^*$ such that $z - s_e \in M^{*a}$ whenever $a \geqq 0$ and $e \geqq b(a)$ where $b(a)$ is a nonnegative integer depending on a. It follows that $z - s_e \in M^{*e}$ for all $e \geqq 0$. Clearly $f(z) - f(s_e) \in (z - s_e) R^*$ and hence $f(z) - F^{(e)} \in M^{*e}$ for all $e \geqq 0$. By assumption $f(z) \neq 0$ and hence there exists a positive integer k such that $f(z) \notin M^{*k}$. It follows that $F^{(k)} \notin M^{*k}$ and hence $\operatorname{ord}_R F^{(k)} < km$. Therefore $F^{(k)}(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \operatorname{ord}_R F^{(k)}$. Thus it suffices to take $f'(Z) = f^{(k)}(Z)$.

Lemma 8.15. *Let $f(Z)$ be a monic polynomial of degree $m>1$ in Z with coefficients in R. Assume that R is a spot over a pseudogeometric domain, $f(Z)$ is irreduzible in $K[Z]$, and the integral closure of $h(R)$ in $h(K[Z])$ is quasilocal where h is the canonical epimorphism of $K[Z]$ onto $K[Z]/f(Z)K[Z]$. Then $f(Z)$ is irreduzible in $R^*[Z]$ where R^* is the completion of R, and hence in particular $f(z) \neq 0$ for all $z \in R^*$.*

Proof. We can take an element t in an overfield of K such that $f(t)=0$, and then the last assumption is equivalent to saying that S is quasilocal where S is the integral closure of R in $K(t)$. Since R is a spot over a pseudogeometric domain, by [10: (36.5)] we get that R is pseudogeometric and hence S is a finite R-module. Therefore S is a local domain and S is pseudogeometric. Since S is a finite R-module, by [9: Proposition 7 on page 699] we get that R is subspace of S and hence we can regard the completion S^* of S to be an overring of R^* and then any finite number of elements in S which are linearly independent over R remain so over R^*. Clearly $0, t, \ldots, t^{m-1}$ are elements in S which are linearly independent over R and hence they are linearly independent over R^*; therefore if $g(Z)$ is any nonzero polynomial of degree $<m$ in Z with coefficients in R^* then $g(t) \neq 0$. Since R is regular we get that R^* is regular and hence R^* is a normal domain. Since R^* is a normal domain, S is a pseudogeometric normal local domain, and S is a finite R-module, by [10: (37.8)] we get that S^* is a domain. Suppose if possible that $f(Z) = f'(Z) f''(Z)$ where $f'(Z)$ and $f''(Z)$ are nonzero polynomials of positive degrees in Z with coefficients in R^*; since $f(t)=0$ and S^* is a domain we must then have either $f'(t)=0$ or $f''(t)=0$; this would be contradiction because the degree of $f'(Z)$ in Z is less than m and the degree of $f''(Z)$ in Z less than m. Therefore $f(Z)$ is irreducible in $R^*[Z]$ and hence in particular $f(z) \neq 0$ for all $z \in R^*$.

Lemma 8.16. *Let $m = p^n$ where n is a positive integer and let $f(Z)$ be a monic polynomial of degree m in Z with coefficients in R. Assume that $f(Z)$ is of ramified-type relative to* ord_R, *R is a spot over a pseudogeometric domain, $f(Z)$ is irreducible in $K[Z]$, and the integral closure of $h(R)$ in $h(K[Z])$ is quasilocal where h is the canonical epimorphism of $K[Z]$ onto $K[Z]/f(Z)K[Z]$. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(0) \neq 0$ and $F'(i,j)=0$ whenever $(i,j) \equiv 0(m)$ and $i+j \leqq \operatorname{ord}_R f'(0)$ where $\Sigma F'(i,j) x^i y^j$ is the expansion of $f'(0)$ in $J[[x,y]]$.*

Proof. Follows from Lemmas 8.14 and 8.15.

§ 9. Effect of a sequence of quadratic transformations on a permissible polynomial

Let R be a two dimensional regular local domain with maximal ideal M such that R/M is an algebraically closed field of characteristic $p \neq 0$. Let (x, y) be a basis of M and let J be a coefficient set for R. Let w be a valuation of the quotient field K of R such that w dominates R and w is residually algebraic over R. Let $X \subset M$.

Definition 9.1. Let $f(Z) \in K[Z]$.

$f(Z)$ is said to be of $[R, x, y, J, X, w]$-*type* $(m; a, b, c)$ if $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)$, J is an (x, m)-faithful coefficient set for R, $X \subset \mathrm{rad}_R y^b R$, and $f(Z)$ is $[R, x, y, J, w]$-permissible.

$f(Z)$ is said to be of $[R, x, y, J, X, w]$-*type* $(m; a, b, c)'$ if $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$, $X \subset \mathrm{rad}_R y^b R$, and $f(Z)$ is $[R, x, y, J, w]$-permissible.

$f(Z)$ is said to be of $[R, x, y, J, X, w]$-*type* $(m; a, b, c)''$ if $f(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)''$, $X \subset \mathrm{rad}_R y^b R$, and $f(Z)$ is $[R, x, y, J, w]$-permissible.

$f(Z)$ is said to be of $[R, x, y, J, X, w]$-*antitype* $(m; b, a, c)$ if $f(Z)$ is of $[R, y, x, J]$-type $(m; b, a, c)^*$, $X \subset \mathrm{rad}\, y^b R$, $f(Z)$ is $[R, x, y, J, w]$-permissible, and $f(Z)$ is of nonsplitting-type relative to ord_{yR}.

$f(Z)$ is said to be of $[R, x, y, J, X, w]$-*antitype* $(m; b, a, c)'$ if $f(Z)$ is of $[R, y, x, J]$-type $(m; b, a, c)'$, $X \subset \mathrm{rad}\, y^b R$, $f(Z)$ is $[R, x, y, J, w]$-permissible, and $f(Z)$ is of nonsplitting-type relative to ord_{yR}.

Note that the follwoing two conditions are equivalent: (1) $f(Z)$ is of $[R, x, y, J, X, w]$-stable-type $(m; a, b, c)$; (2) $f(Z)$ is of $[R, x, y]$-stable-type $(m; a, b, c)$, $X \subset \mathrm{rad}_R y^b R$, and $f(Z)$ is $[R, x, y, J, w]$-permissible.

Lemma 9.2. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$–type $(m; a, b, c)$. Assume that $w(y) \geqq w(x)$. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-type $(m; a, b, c)'$.*

Proof. Let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w. Then $f(Z)$ is of ramified-type relativ to ord_{xR} and $f(Z)$ is of ramified-type relative to $\mathrm{ord}_{x'R'}$. Since $w(y) \geqq w(x)$ we get that $\mathrm{ord}_{x'R'} = \mathrm{ord}_R$ and hence $f(Z)$ is of ramified-type relative to ord_R. Therefore by Lemma 8.4 there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J]$-type $(m; a, b, c)'$. It follows that $f'(Z)$ is of $[R, x, y, J, X, w]$-type $(m; a, b, c)'$.

Lemma 9.3. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-type $(m; a, b, c)''$.*

Proof. Follows from Lemma 8.7.

Lemma 9.4. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $c < m$. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-type $(m; a, b, c)'$.*

Proof. Follows from Lemma 8.10.

Lemma 9.5. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-antitype $(m; b, a, c)$ where $c < m$. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-antitype $(m; b, a, c)'$.*

Proof. Follows from Lemma 8.11.

Lemma 9.6. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $[m; b, c] = 0$. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-stable-type $(m; a, b, c')$ where $c' \leqq c$.*

Proof. Follows from Lemma 8.6.

Lemma 9.7. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-stable-type $(m; a, b, c)$. Assume that $w(y) \geqq w(x)$ and let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w. Let $b' = b$ if $w(y) > w(x)$, and $b' = 0$ if $w(y) = w(x)$. Then there exists an R'-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-stable-type $(m; a', b', c')$.*

Proof. By Lemma 8.12 there exists an R-translate $f^*(Z)$ of $f(Z)$ such that $f^*(Z)$ is of $[R, x, y, J, X, w]$-type $(m; a, b, c)'$. Now $X \subset \mathrm{rad}_R y^b R$ and $\mathrm{ord}_{yR} f_q^* \geqq bq/m$ for $0 < q < m$ where f_q^* is the coefficient of Z^{m-q} in $f^*(Z)$. If $w(y) = w(x)$ then $b' = 0$ and hence $X \subset \mathrm{rad}_{R'} y'^{b'} R'$ and $\mathrm{ord}_{y'R'} f_q^* \geqq b'q/m$ for $0 < q < m$; and if $w(y) > w(x)$ then $b' = b$ and $\mathrm{ord}_{y'R'} = \mathrm{ord}_{yR}$ and hence again $X \subset \mathrm{rad}_{R'} y'^{b'} R'$ and $\mathrm{ord}_{y'R'} f_q^* \geqq b'q/m$ for $0 < q < m$. Since $f^*(0)$ is of $[R, x, y, J]$-pretype $(m; a, b, c)'$ and $[m; b, c] = 0$, by Lemma 7.6 we get that $f^*(0)$ is of $[R', x', y', J]$-pretype $(m; a', b', c^*)$ where $a' = \mathrm{ord}_R f^*(0)$ and $[m; b', c^*] = 0$. Since $w(y) \geqq w(x)$ we also get that J is an x'-faithful coefficient set for R'. Therefore $f^*(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c^*)$. Since $[m; b', c^*] = 0$, by Lemma 9.6 there exists an R'-translate $f'(Z)$ of $f^*(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-stable-type $(m; a', b', c')$ where $c' \leqq c^*$.

Lemma 9.8. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-stable-type $(m; a, b, c)$. Assume that $w(y) < w(x)$ and let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w. Let $b' = \mathrm{ord}_R f(0)$ and $c' = a + b + c - b'$. Then $f(Z)$ is of $[R', x', y', J, X, w]$-stable-type $(m; a, b', c')$.*

Proof. Now $\mathrm{ord}_{y'R'} = \mathrm{ord}_R$. Since $X \subset M$ we get that $X \subset \mathrm{rad}_{R'} y'^{b'} R'$. Since $f(Z)$ is of nonsplitting-type relative to ord_R and $\mathrm{ord}_R f(0) = b'$ we get that $\mathrm{ord}_R f_q \geqq b'q/m$ for $0 < q < m$ where f_q is the coefficient of Z^{m-q} in $f(Z)$, and hence $\mathrm{ord}_{y'R'} f_q \geqq b'q/m$ for $0 < q < m$. Since $f(0)$ is of $[R, x, y]$-stable-pretype $(m; a, b, c)$, by Lemma 7.19 we get that $f(0)$ is of $[R', x', y']$-stable-pretype $(m; a, b', c')$. Therefore $f(Z)$ is of $[R', x', y', J, X, w]$-stable-type $(m; a, b', c')$.

Lemma 9.9. *Let $f^{(0)}(Z) \in K[Z]$ be of $[R, x, y, J, X, w]$-stable-type $(m; a_0, b_0, c_0)$. Let (R_j, x_j, y_j) be the canonical j^{th} quadratic transform of (R, x, y, J) along w. Then for each $j > 0$ there exists an R_j-translate $f^{(j)}(Z)$ of $f^{(0)}(Z)$ such that $f^{(j)}(Z)$ is of $[R_j, x_j, y_j, J, X, w]$-stable-type $(m; a_j, b_j, c_j)$ where for all $j \geqq 0$ we have the following: (1) if $w(y_j) < w(x_j)$ then $f^{(j+1)}(Z) = f^{(j)}(Z)$, $a_{j+1} = a_j$, $b_{j+1} = \mathrm{ord}_{R_j} f^{(j)}(0)$, and $c_{j+1} = a_j + b_j + c_j - b_{j+1}$; (2) if $w(y_j) = w(x_j)$ then $b_{j+1} = 0$; (3) if $w(y_j) > w(x_j)$ then $b_{j+1} = b_j$. In connection with (2) note that if w is rational then there exist infinitely many distinct nonnegative integers j for which $w(y_j) = w(x_j)$.*

Proof. Follows from Lemma 9.7 and 9.8.

Lemma 9.10. *Let $m = p^n$ where n is a positive integer, and let $f(Z)$ be a monic polynomial of degree m in Z with coefficients in R. Assume that $f(0) \neq 0$ and $F(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j \leqq \mathrm{ord}_R f(0)$ where $\Sigma F(i, j) x^i y^j$ is the expansion of $f(0)$ in $J[[x, y]]$. Assume that $f(Z)$ is $[R, x, y, J, w]$-permissible.*

Assume that $w(y) \geqq w(x)$ and let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w. Then $f(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', 0, c')$ where $a' = \operatorname{ord}_R f(0)$ and $c' \leqq a'$.

Proof. By Lemma 7.4 we get that $f(0)$ is of $[R', x', y', J]$-pretype $(m; a', 0, c')$ where $a' = \operatorname{ord}_R f(0)$ and $c' \leqq a'$. Since $w(y) \geqq w(x)$ we also get that J is an x'-faithful coefficient set for R'. Therefore $f(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', 0, c')$.

Lemma 9.11. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $b \equiv 0(m)$. Assume that $w(y) \geqq w(x)$ and let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w. Let $b' = b$ if $w(y) > w(x)$, and $b' = 0$ if $w(y) = w(x)$. Then $b' \equiv 0(m)$ and there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $c' \leqq c$.*

Proof. Clearly $b' \equiv 0(m)$. By Lemma 9.2 there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-type $(m; a, b, c)'$. Now $X \subset \operatorname{rad}_R y^b R$ and $\operatorname{ord}_{yR} f'_q \geqq bq/m$ for $0 < q < m$ where f'_q is the coefficient of Z^{m-q} in $f'(Z)$. If $w(y) = w(x)$ then $b' = 0$ and hence $X \subset \operatorname{rad}_{R'} y'^{b'} R'$ and $\operatorname{ord}_{y'R'} f'_q \geqq b'q/m$ for $0 < q < m$; and if $w(y) > w(x)$ then $b' = b$ and $\operatorname{ord}_{y'R'} = \operatorname{ord}_{yR}$ and hence again $X \subset \operatorname{rad}_{R'} y'^{b'} R'$ and $\operatorname{ord}_{y'R'} f'_q \geqq b'q/m$ for $0 < q < m$. Since $w(y) \geqq w(x)$ we get that J is an x'-faithful coefficient set for R'. Since $f'(0)$ is of $[R, x, y, J]$-pretype $(m; a, b, c)'$, by Lemma 7.5 we get that $f'(0)$ is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $a' = \operatorname{ord}_R f'(0)$ and $c' \leqq c$. It follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$.

Lemma 9.12. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $[m; b, c] < m$. Assume that $w(y) \geqq w(x)$ and let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w. Let $b' = b$ if $w(y) > w(x)$, and $b' = 0$ if $w(y) = w(x)$. Then there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $[m; b', c'] \leqq [m; b, c]$.*

Proof. By Lemma 9.4 there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-type $(m; a, b, c)'$. Now $X \subset \operatorname{rad}_R y^b R$ and $\operatorname{ord}_{yR} f'_q \geqq$ $\geqq bq/m$ for $0 < q < m$ where f'_q is the coefficient of Z^{m-q} in $f'(Z)$. If $w(y) = w(x)$ then $b' = 0$ and hence $X \subset \operatorname{rad}_{R'} y'^{b'} R'$ and $\operatorname{ord}_{y'R'} f'_q \geqq b'q/m$ for $0 < q < m$; and if $w(y) > w(x)$ then $b' = b$ and $\operatorname{ord}_{y'R'} = \operatorname{ord}_{yR}$ and hence again $X \subset \operatorname{rad}_{R'} y'^{b'} R'$ and $\operatorname{ord}_{y'R'} f'_q \geqq b'q/m$ for $0 < q < m$. Since $w(y) \geqq w(x)$ we get that J is an x'-faithful coefficient set for R'. Since $f'(0)$ is of $[R, x, y, J]$-pretype $(m; a, b, c)'$, by Lemma 7.6 we get that $f'(0)$ is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $a' = \operatorname{ord}_R f'(0)$ and $[m; b', c'] \leqq [m; b, c]$. It follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$.

Lemma 9.13. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-antitype $(m; b, a, c)$ where $a \equiv 0(m)$ and $c < m$. Assume that $w(y) \geqq w(x)$ and let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w. Then we have the following.*

(1) *Either there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a', c')$ where $a' \equiv 0(m)$ and $c' < c$, or there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $b' \equiv 0(m)$ and $c' < m$.*

(2) *If* $c=0$ *then there exists an* R*-translate* $f'(Z)$ *of* $f(Z)$ *such that* $f'(Z)$ *is of* $[R', x', y', J, X, w]$*-type* $(m; a', b', c')$ *where* $b' \equiv 0(m)$ *and* $c' < m$.

Proof. By Lemma 9.5 there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-antitype $(m; b, a, c)'$. Now $f'(0)$ is of $[R, y, x, J]$-pretype $(m; b, a, c)'$ and $X \subset \mathrm{rad}_R y^b R$. Since $\mathrm{ord}_{yR} f'(0) = b$ and $f'(Z)$ is of nonsplitting-type relative to ord_{yR} we get that $\mathrm{ord}_{yR} f'_q \geqq bq/m$ for $0<q<m$ where f'_q is the coefficient of Z^{m-q} in $f'(Z)$. Since $w(y) \geqq w(x)$ we also get that J is an x'-faithful coefficient set for R'. Let $d = \mathrm{ord}_R f'(0)$. If $w(y) = w(x)$ then by Lemma 7.17 we get that $f'(0)$ is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $a' = d$, $b' = 0$, and $c' < m$; since $b' = 0$ it follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$. Now assume that $w(y) > w(x)$. Then $\mathrm{ord}_{y'R'} = \mathrm{ord}_{yR}$ and $\mathrm{ord}_{x'R'} = \mathrm{ord}_R$. Let b^* be the greatest integer such that $b^* \equiv 0(m)$ and $b^* \leqq b$. If $d - b^* - a < m$ then by Lemma 7.18 (2) we get that $f'(0)$ is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $b' = b^*$, $a' = d$, and $c' < m$; since $b' \leqq b$ and $\mathrm{ord}_{y'R'} = \mathrm{ord}_{yR}$ we get that $X \subset \mathrm{rad}_{R'} y'^{b'} R'$ and $\mathrm{ord}_{y'R'} f'_q \geqq b'q/m$ for $0<q<m$; it follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$. If $d - b^* - a \geqq m$ then by Lemma 7.18 (1) we get that $f'(0)$ is of $[R', y', x', J]$-pretype $(m; b', a', c')^*$ where $b' = b$, a' is the greatest integer such that $a' \equiv 0(m)$ and $a' \leqq d$, and $c' = b + a + c - a' < c$; since $\mathrm{ord}_{y'R'} = \mathrm{ord}_{yR}$ and $b' = b$ we get that $X \subset \mathrm{rad}_{R'} y'^{b'} R'$ and $f'(Z)$ is of nonsplitting-type relative to $\mathrm{ord}_{y'R'}$; since $\mathrm{ord}_{x'R'} = \mathrm{ord}_R$ and $f'(Z)$ is of nonsplitting-type to ord_R we also get that $\mathrm{ord}_{x'R'} f'_q \geqq a'q/m$ for $0<q<m$; it follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a', c')$. It only remains to note that if $c=0$ then clearly $d = a+b$ and hence $d - b^* - a < m$.

Lemma 9.14. *Let* $f(Z) \in R[Z]$ *be of* $[R, x, y, J, X, w]$*-antitype* $(m; b, a, c)$ *where* $c \leqq m/p$ *and either* $c < m/p$ *or* $a + b + m/p \not\equiv 0(m)$. *Assume that* $w(y) \geqq w(x)$ *and let* (R', x', y') *be the canonical first quadratic transform of* (R, x, y, J) *along* w. *Let* $b' = b$ *if* $w(y) > w(x)$, *and* $b' = 0$ *if* $w(y) = w(x)$. *Then there exists an* R*-translate* $f'(Z)$ *of* $f(Z)$ *such that* $f'(Z)$ *is of* $[R', x', y', J, X, w]$*-type* $(m; a', b', c')$ *where* $[m; b', c'] = 0$.

Proof. By Lemma 9.5 there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-antitype $(m; b, a, c)'$. Now $f'(0)$ is of $[R, y, x, J]$-pretype $(m; b, a, c)'$ and $X \subset \mathrm{rad}_R y^b R$. Since $\mathrm{ord}_{yR} f'(0) = b$ and $f'(Z)$ is of nonsplitting-type relative to ord_{yR} we get that $\mathrm{ord}_{yR} f'_q \geqq bq/m$ for $0<q<m$ where f'_q is the coefficient of Z^{m-q} in $f'(Z)$. If $w(y) = w(x)$ then $b' = 0$ and hence $X \subset \mathrm{rad}_{R'} y'^{b'} R'$ and $\mathrm{ord}_{y'R'} f'_q \geqq b'q/m$ for $0<q<m$; and if $w(y) > w(x)$ then $b' = b$ and $\mathrm{ord}_{y'R'} = \mathrm{ord}_{yR}$ and hence again $X \subset \mathrm{rad}_{R'} y'^{b'} R'$ and $\mathrm{ord}_{y'R'} f'_q \geqq b'q/m$ for $0<q<m$. Since $w(y) \geqq w(x)$ we get that J is an x'-faithful coefficient set for R'. Since $c \leqq m/p$ and either $c < m/p$ or $a + b + m/p \not\equiv 0(m)$, by Lemmas 7.8 and 7.9 we get that $f'(0)$ is of $[R', x', y', J]$-pretype $(m; a', b', c')$ where $a' = \mathrm{ord}_R f'(0)$ and $[m; b', c'] = 0$. It follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$.

Lemma 9.15. *Let* $f(Z) \in R[Z]$ *be of* $[R, x, y, J, X, w]$*-antitype* $(m; b, a, c)$ *where* $c < m$. *Assume that* $w(y) < w(x)$ *and let* (R', x', y') *be the canonical first quadratic transform of* (R, x, y, J) *along* w. *Then there exists an* R*-translate*

$f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a, c')$ where $c' \leqq c$.

Proof. By Lemma 9.5 there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-antitype $(m; b, a, c)'$. Now $\mathrm{ord}_{x'R'} = \mathrm{ord}_{xR}$ and $\mathrm{ord}_{xR} f'_q \geqq aq/m$ for $0<q<m$ where f'_q is the coefficient of Z^{m-q} in $f'(Z)$; therefore $\mathrm{ord}_{x'R'} f'_q \geqq aq/m$ for $0<q<m$. Let $b' = \mathrm{ord}_R f'(0)$. Now $X \subset M$, $\mathrm{ord}_{y'R'} = \mathrm{ord}_R$, and $f'(Z)$ is of nonsplitting-type relative to ord_R; therefore $M \subset \mathrm{rad}_{R'} y'^{b'} R'$ and $f'(Z)$ is of nonsplitting-type relative to $\mathrm{ord}_{y'R'}$. Since $f'(0)$ is of $[R, y, x, J]$-pretype $(m; b, a, c)'$, by Lemma 7.11 we get that $f'(0)$ is of $[R', y', x', J]$-pretype $(m; b', a, c')^*$ where $c' \leqq c$. It follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a, c')$.

Lemma 9.16. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $b \equiv 0(m)$. Assume that $w(y) < w(x)$ and let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w. Then either there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a, b', c')$ where $b' \equiv 0(m)$ and $c' < c$, or there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a', c')$ where $a' \equiv 0(m)$ and $c' < m$.*

Proof. By Lemma 9.3 there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-type $(m; a, b, c)''$. Now $f'(0)$ is of $[R, x, y, J]$-pretype $(m; a, b, c)''$. Let $d = \mathrm{ord}_R f'(0)$ and let a' be the greatest integer such that $a' \equiv 0(m)$ and $a' \leqq a$. Now $\mathrm{ord}_{y'R'} = \mathrm{ord}_R$, $X \subset M$, and $f'(Z)$ is of nonsplitting-type relative to ord_R; therefore $f'(Z)$ is of nonsplitting-type relative to $\mathrm{ord}_{y'R'}$, $X \subset \mathrm{rad}_{R'} y'^{d} R'$, and $\mathrm{ord}_{y'R'} f'_q \geqq dq/m$ for $0<q<m$ where f'_q is the coefficient of Z^{m-q} in $f'(Z)$. Also $\mathrm{ord}_{x'R'} = \mathrm{ord}_{xR}$, $a' \leqq a = \mathrm{ord}_{xR} f'(0)$, and $f'(Z)$ is of ramified-type relative to ord_{xR}; therefore $\mathrm{ord}_{x'R'} f'_q \geqq a'q/m$ for $0<q<m$. Since J is an (x, m)-faithful coefficient set for R and $xR \subset x'R'$, we also get that J is an (x', m)-faithful coefficient set for R'. If $d - a' - b \geqq m$ then by Lemma 7.14 (1) we get that $f'(0)$ is of $[R', x', y', J]$-pretype $(m; a, b', c')$ where b' is the greatest integer such that $b' \equiv 0(m)$ and $b' \leqq d$, and where $c' = a + b + c - b' < c$; since $b' \leqq d$ we get that $X \subset \mathrm{rad}_{R'} y'^{b'} R'$ and $\mathrm{ord}_{y'R'} f'_q \geqq b'q/m$ for $0<q<m$; it follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a, b', c')$. If $d - a' - b < m$ then by Lemma 7.14 (2) we get that $f'(Z)$ is of $[R', y', x', J]$-pretype $(m; b', a', c')^*$ where $b' = d$ and $c' < m$; it follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a', c')$.

Lemma 9.17. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $[m; b, c] < m$. Assume that $w(y) < w(x)$ and let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w. Then either there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a, b', c')$ where $[m; b', c'] \leqq [m; b, c]$ and $c' \leqq c - m/p$, or there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a, c')$ where $c' < m/p$.*

Proof. By Lemma 9.4 there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-type $(m; a, b, c)'$. Now $f'(0)$ is of $[R, x, y, J]$-pretype $(m; a, b, c)'$. Let $b' = \mathrm{ord}_R f'(0)$. Now $\mathrm{ord}_{y'R'} = \mathrm{ord}_R$, $X \subset M$, and $f'(Z)$ is of nonsplitting-type relative to ord_R; therefore $X \subset \mathrm{rad}_{R'} y'^{b'} R'$, $f'(Z)$ is of nonsplitting-type relative to $\mathrm{ord}_{y'R'}$, and $\mathrm{ord}_{y'R'} f'_q \geqq b'q/m$ for $0<q<m$ where f'_q is the

coefficient of Z^{m-q} in $f'(Z)$. Also $\mathrm{ord}_{x'R'} = \mathrm{ord}_{xR}$, $\mathrm{ord}_{xR} f'(0) = a$, and $f'(Z)$ is of ramified-type relative to ord_{xR}; therefore $\mathrm{ord}_{x'R'} f'_q \geqq aq/m$ for $0 < q < m$. Since J is an (x, m)-faithful coefficient set for R and $xR \subset x'R'$ we also get that J is an (x', m)-faithful coefficient set for R'. If $b' - a - b \geqq m/p$ then by Lemma 7.15 (1) we get that $f'(0)$ is of $[R', x', y', J]$-pretype $(m; a, b', c')$ where $c' \leqq c - m/p$ and $[m; b', c'] \leqq [m; b, c]$; it follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a, b', c')$. If $b' - a - b < m/p$ then by Lemma 7.15 (2) we get that $f'(0)$ is of $[R', y', x', J]$-pretype $(m; b', a, c')^*$ where $c' < m/p$; it follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a, c')$.

Lemma 9.18. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $[m; b, c] < m$. Assume that $w(y) < w(x)$ and let (R', x', y') be the canonical first quadratic transform of (R, x, y, J) along w. Then either there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a, b', c')$ where $[m; b', c'] \leqq \max(0, [m; b, c] - 1)$, or there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a, c')$ where $c' \leqq m/p$ and either $c' < m/p$ or $a + b' + m/p \not\equiv 0(m)$.*

Proof. By Lemma 9.4 there exists an R-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R, x, y, J, X, w]$-type $(m; a, b, c)'$. Now $f'(0)$ is of $[R, x, y, J]$-pretype $(m; a, b, c)'$. Let $d = \mathrm{ord}_R f'(0)$. Now $\mathrm{ord}_{y'R'} = \mathrm{ord}_R$, $X \subset M$, and $f'(Z)$ is of nonsplitting-type relation to ord_R; therefore $f'(Z)$ is of nonsplitting-type relative to $\mathrm{ord}_{y'R'}$, $X \subset \mathrm{rad}_{R'} y'^d R'$, and $\mathrm{ord}_{y'R'} f'_q \geqq dq/m$ for $0 < q < m$ where f'_q is the coefficient of Z^{m-q} in $f'(Z)$. Also $\mathrm{ord}_{x'R'} = \mathrm{ord}_{xR}$, $\mathrm{ord}_{xR} f'(0) = a$, and $f'(Z)$ is of ramified-type relative to ord_{xR}; therefore $\mathrm{ord}_{x'R'} f'_q \geqq aq/m$ for $0 < q < m$. Since J is an (x, m)-faithful coefficient set for R and $xR \subset x'R'$ we also get that J is an (x', m)-faithful coefficient set for R. Let a' be the greatest integer such that $a' \equiv 0(m)$ and $a' \leqq a$. Clearly one of the following six conditions hold: 1) $d - a - b > m/p$; 2) $d - a - b = m/p$ and $b \not\equiv 0(m)$; 3) $d - a - b = m/p$ and $p = 2$; 4) $b \equiv 0(m)$ and $d - a' - b \geqq m$; 5) $d - a - b < m/p$; 6) $d - a - b = m/p$, $b \equiv 0(m)$, $p \neq 2$, and $d - a' - b < m$. If one of the conditions 1) to 4) holds then by Lemma 7.16 (2) we get that $f'(0)$ is of $[R', x', y', J]$-pretype $(m; a, b', c')$ where $b' \leqq d$ and $[m; b', c'] \leqq \max(0, [m; b, c] - 1)$; since $b' \leqq d$ we get that $X \subset \mathrm{rad}_{R'} y'^{b'} R'$ and $\mathrm{ord}_{y'R'} f'_q \geqq b'q/m$ for $0 < q < m$; it follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a, b', c')$. If condition 5) holds then by Lemma 7.16 (1) we get that $f'(0)$ is of $[R', y', x', J]$-pretype $(m; b', a, c')^*$ where $b' = d$ and $c' < m/p$; it follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a, c')$. If conditions 6) holds then by Lemma 7.16 (3) we get that $f'(0)$ is of $[R', y', x', J]$-pretype $(m; b', a, c')^*$ where $b' = d$, $c' \leqq m/p$, and $a + b' + m/p \not\equiv 0(m)$; it follows that $f'(Z)$ is of $[R', x', y', J, X, w]$-antitype $(m; b', a, c')$.

Lemma 9.19. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $[m; b, c] < m$. Assume that $w(y) < w(x)$ and let (R'', x'', y'') be the canonical first quadratic transform of (R, x, y, J) along w. Assume that $w(y'') \geqq w(x'')$ and let (R', x', y') be the canonical first quadratic transform of (R'', x'', y'', J) along w. Then there exists an R'-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $[m; b', c'] \leqq \max(0, [m; b, c] - 1)$.*

Proof. By Lemma 9.18 we get that either 1) there exists an R-translate $f''(Z)$ of $f(Z)$ such that $f''(Z)$ is of $[R'', x'', y'', J, X, w]$-type $(m; a'', b'', c'')$ where $[m; b'', c''] \leqq \max(0, [m; b, c]-1)$, or 2) there exists an R-translate $f''(Z)$ of $f(Z)$ such that $f''(Z)$ is of $[R'', x'', y'', J, X, w]$-antitype $(m; b'', a'', c'')$ where $c'' \leqq m/p$ and either $c'' < m/p$ or $a''+b''+m/p \not\equiv 0(m)$. In case 1), by Lemma 9.12 there exists an R''-translate $f'(Z)$ of $f''(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $[m; b', c'] \leqq [m; b'', c'']$ and hence $[m; b', c'] \leqq \max(0, [m; b, c]-1)$. In case 2), by Lemma 9.14 there exists an R''-translate $f'(Z)$ of $f''(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $[m; b', c'] = 0$.

Lemma 9.20. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $[m; b, c] < m$. Assume that w is real and $w(y) < w(x)$. Then there exists a canonical quadratic transform (R', x', y') of (R, x, y, J) along w and an R'-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $[m; b', c'] \leqq \leqq \max(0, [m; b, c]-1)$.*

Proof. Let (R_i, x_i, y_i) be the canonical i^{th} quadratic transform of (R, x, y, J) along w. Since w is real and $w(y) < w(x)$, there exists a positive integer j such that $w(y_i) < w(x_i)$ for $0 \leqq i < j$ and $w(y_j) \geqq w(x_j)$. If $j=1$ then upon taking $(R', x', y') = (R_2, x_2, y_2)$ by Lemma 9.19 we get that there exists an R'-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $[m; b', c'] \leqq \max(0, [m; b, c]-1)$. So now assume that $j \geqq 2$. By induction on i we shall show that if i is any integer such that $0 \leqq i \leqq j$ then either: (1_i) there exists an R_i-translate $f_i(Z)$ of $f(Z)$ such that $f_i(Z)$ is of $[R_i, x_i, y_i, J, X, w]$-type $(m; a, b_i, c_i)$ where $c_i \leqq c - im/p$ and $[m; b_i, c_i] < m$, or: $(1'_i)$ there exists an R_i-translate $f'_i(Z)$ of $f(Z)$ such that $f'_i(Z)$ is of $[R_i, x_i, y_i, J, X, w]$-antitype $(m; b'_i, a, c'_i)$ where $c'_i < m/p$. For $i=0$ it suffices to take $f_0(Z) = f(Z)$, $b_0 = b$, $c_0 = c$. Now let $i > 0$ and assume that the assertion is true for all values of i smaller than the given one. If case (1_{i-1}) prevails then by Lemma 9.17 either there exists an R_i-translate $f_i(Z)$ of $f_{i-1}(Z)$ such that $f_i(Z)$ is of $[R_i, x_i, y_i, J, X, w]$-type $(m; a, b_i, c_i)$ where $c_i \leqq c_{i-1} - m/p$ and $[m; b_i, c_i] \leqq [m; b_{i-1}, c_{i-1}]$ and hence $c_i \leqq c - im/p$ and $[m; b_i, c_i] < m$, or there exists an R_i-translate $f'_i(Z)$ of $f_{i-1}(Z)$ such that $f'_i(Z)$ is of $[R_i, x_i, y_i, J, X, w]$-antitype $(m; b'_i, a, c'_i)$ where $c'_i < m/p$. If case $(1'_{i-1})$ prevails then by Lemma 9.15 there exists an R_{i-1}-translate $f'_i(Z)$ of $f'_{i-1}(Z)$ such that $f'_i(Z)$ is of $[R_i, x_i, y_i, J, X, w]$-antitype $(m; b'_i, a, c'_i)$ where $c'_i \leqq c'_{i-1}$ and hence $c'_i < m/p$. This completes the induction on i. If case (1_j) prevails then $[m; b_j, c_j] < [m; b, c]$ because $j \geqq 2$ and hence it suffices to take $R' = R_j$, $x'_j = x_j$, $y' = y_j$, $a' = a$, $b' = b_j$, $c' = c_j$, $f'(Z) = f_j(Z)$. If case $(1'_j)$ prevails then upon taking $(R', x', y') = (R_{j+1}, x_{j+1}, y_{j+1})$ by Lemma 9.14 we get that there exists an R'-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $[m; b', c'] = 0$.

Lemma 9.21. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $[m; b, c] < m$. Assume that w is real nondiscrete. Then there exists a canonical quadratic transform (R', x', y') of (R, x, y, J) along w and an R'-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is $[R', x', y', J, X, w]$-stable.*

Proof. We shall make induction on $[m;\, b,\, c]$. If $[m;\, b,\, c]=0$ then upon taking $(R', x', y')=(R, x, y)$ by Lemma 9.6 we get that there exists an R'-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is $[R', x', y', J, X, w]$-stable. Now let $[m; b, c]>0$ and assume that the assertion is true for all values of $[m;\, b,\, c]$ smaller than the given one. Let (R_i, x_i, y_i) be the canonical i^{th} quadratic transform of (R, x, y, J) along w. Since w is real nondiscrete, by Lemma 3.13 there exists a nonnegative integer j such that $w(y_i)\geqq w(x_i)$ for $0\leqq i<j$ and $w(y_j)<w(x_j)$. Let $f_0(Z)=f(Z)$, $a_0=a$, $b_0=b$, $c_0=c$. Upon applying Lemma 9.12 successively j times we find an R_i-translate $f_i(Z)$ of $f_{i-1}(Z)$ such that $f_i(Z)$ is of $[R_i, x_i, y_i, J, X, w]$-type $(m;\, a_i,\, b_i,\, c_i)$ where $[m;\, b_i,\, c_i]\leqq[m;\, b_{i-1},\, c_{i-1}]$ for $0<i\leqq j$. In particular then $f_j(Z)$ is an R_j-translate of $f(Z)$ such that $f_j(Z)$ is of $[R_j, x_j, y_j, J, X, w]$-type $(m;\, a_j,\, b_j,\, c_j)$ where $[m;\, b_j,\, c_j]\leqq[m;\, b,\, c]$. Since $w(y_j)<w(x_j)$, by Lemma 9.20 there exists a canonical quadratic transform (R^*, x^*, y^*) of (R_j, x_j, y_j, J) along w and an R^*-translate $f^*(Z)$ of $f_j(Z)$ such that $f^*(Z)$ is of $[R^*, x^*, y^*, J, X, w]$-type $(m;\, a^*,\, b^*,\, c^*)$ where $[m;\, b^*,\, c^*]\leqq\max(0, [m;\, b_j,\, c_j]-1)$. Since $[m;\, b_j,\, c_j]\leqq$ $\leqq[m;\, b,\, c]>0$, it follows that $[m;\, b^*,\, c^*]<[m;\, b,\, c]$. Therefore by the induction hypothesis there exists a canonical quadratic transform (R', x', y') of (R^*, x^*, y^*, J) along w and an R^*-translate $f'(Z)$ of $f^*(Z)$ such that $f'(Z)$ is $[R', x', y', J, X, w]$-stable.

Lemma 9.22. *Let $f(Z)\in R\{Z\}$ be of $[R, x, y, J, X, w]$-antitype $(m;\, b,\, a,\, c)$ where $a\equiv 0(m)$ and $c<m$. Assume that w is real. Then there exists a canonical quadratic transform (R', x', y') of (R, x, y, J) along w and an R'-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m;\, a',\, b',\, c')$ where $b'\equiv 0(m)$ and $c'<m$.*

Proof. We shall make induction on c. Let (R_i, x_i, y_i) be the canonical i^{th} quadratic transform of (R, x, y, J) along w. Since w is real there exists a nonnegative integer j such that $w(y_i)<w(x_i)$ for $0\leqq i<j$ and $w(y_j)\geqq w(x_j)$. Let $f_0(Z)=f(Z)$, $b_0=b$, $c_0=c$. Upon applying Lemma 9.15 successively j times we find an R_i-translate $f_i(Z)$ of $f_{i-1}(Z)$ such that $f_i(Z)$ is of $[R_i, x_i, y_i, J, X, w]$-antitype $(m;\, b_i,\, a,\, c_i)$ where $c_i\leqq c_{i-1}$ for $0<i\leqq j$. In particular then $f_j(Z)$ is an R_j-translate of $f(Z)$ such that $f_j(Z)$ is of $[R_j, x_j, y_j, J, X, w]$-antitype $(m;\, b_j,\, a,\, c_j)$ where $c_j\leqq c$. If $c=0$ then $c_j=0$ and hence upon taking $(R', x', y')=(R_{j+1}, x_{j+1}, y_{j+1})$ by Lemma 9.13 (2) we get that there exists an R'-translate $f'(Z)$ of $f_j(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m;\, a',\, b',\, c')$ where $b'\equiv 0(m)$ and $c'<m$. Now let $c>0$ and assume that the assertion is true for all values of c smaller than the given one. By Lemma 9.13 (1) either (1) there exists an R_{j+1}-translate $f'(Z)$ of $f_j(Z)$ such that $f'(Z)$ is of $[R_{j+1}, x_{j+1}, y_{j+1}, J, X, w]$-type $(m;\, a',\, b',\, c')$ where $b'\equiv 0(m)$ and $c'<m$, or (2) there exists an R_{j+1}-translate $f^*(Z)$ of $f_j(Z)$ such that $f^*(Z)$ is of $[R_{j+1}, x_{j+1}, y_{j+1}, J, X, w]$-antitype $(m;\, b^*,\, a^*,\, c^*)$ where $a^*\equiv 0(m)$ and $c^*<c_j$. If case (1) prevails then it is enough to take $(R', x', y')=(R_{j+1}, x_{j+1}, y_{j+1})$. If case (2) prevails then $c^*<c$ and hence by the induction hypothesis there exists a canonical quadratic transform (R', x', y') of $(R_{j+1}, x_{j+1}, y_{j+1}, J)$ along w and an R'-translate $f'(Z)$ of $f^*(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m;\, a',\, b',\, c')$ where $b'\equiv 0(m)$ and $c'<m$.

Lemma 9.23. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $b \equiv 0(m)$. Assume that w is real nondiscrete. Then there exists a canonical quadratic transform (R', x', y') of (R, x, y, J) along w and an R'-translate $f'(Z)$ of $f(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $b' \equiv 0(m)$ and $c' < m$.*

Proof. We shall make induction on c. If $c < m$ then it suffices to take $R' = R$, $x' = x$, $y' = y$, $a' = a$, $b' = b$, $c' = c$, $f'(Z) = f(Z)$. Now let $c \geqq m$ and assume that the assertion is true for all values of c smaller than the given one. Let (R_i, x_i, y_i) be the canonical i^{th} quadratic transform of (R, x, y, J) along w. Since w is real nondiscrete, by Lemma 3.13 there exists a nonnegative integer j such that $w(y_i) \geqq w(x_i)$ for $0 \leqq i < j$ and $w(y_j) < w(x_j)$. Let $f_0(Z) = f(Z)$, $a_0 = a$, $b_0 = b$, $c_0 = c$. Upon applying Lemma 9.11 successively j times we find an R_i-translate $f_i(Z)$ of $f_{i-1}(Z)$ such that $f_i(Z)$ is of $[R_i, x_i, y_i, J, X, w]$-type $(m; a_i, b_i, c_i)$ where $b_i \equiv 0(m)$ and $c_i \leqq c_{i-1}$ for $0 < i \leqq j$. In particular then $f_j(Z)$ is an R_j-translate of $f(Z)$ such that $f_j(Z)$ is of $[R_j, x_j, y_j, J, X, w]$-type $(m; a_j, b_j, c_j)$ where $b_j \equiv 0(m)$ and $c_j \leqq c$. Since $w(y_j) < w(x_j)$, by Lemma 9.16 we get that either (1) there exists an R_{j+1}-translate $f^*(Z)$ of $f_j(Z)$ such that $f^*(Z)$ is of $[R_{j+1}, x_{j+1}, y_{j+1}, J, X, w]$-antitype $(m; b^*, a^*, c^*)$ where $a^* \equiv 0(m)$ and $c^* < m$, or (2) there exists an R_{j+1}-translate $f^*(Z)$ of $f(Z)$ such that $f^*(Z)$ is of $[R_{j+1}, x_{j+1}, y_{j+1}, J, X, w]$-type $(m; a^*, b^*, c^*)$ where $b^* \equiv 0(m)$ and $c^* < c_j$. If case (1) prevails then by Lemma 9.22 there exists a canonical quadratic transform (R', x', y') of $(R_{j+1}, x_{j+1}, y_{j+1}, J)$ along w and an R'-translate $f'(Z)$ of $f^*(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $b' \equiv 0(m)$ and $c' < m$. If case (2) prevails then $c^* < c$ and hence by the induction hypothesis there exists a canonical quadratic transform (R', x', y') of $(R_{j+1}, x_{j+1}, y_{j+1}, J)$ along w and an R'-translate $f'(Z)$ of $f^*(Z)$ such that $f'(Z)$ is of $[R', x', y', J, X, w]$-type $(m; a', b', c')$ where $b' \equiv 0(m)$ and $c' < m$.

Lemma 9.24. *Let $f(Z) \in R[Z]$ be of $[R, x, y, J, X, w]$-type $(m; a, b, c)$ where $b \equiv 0(m)$. Assume that w is real nondiscrete and let (R_j, x_j, y_j) be the canonical j^{th} quadratic transform of (R, x, y, J) along w. Then there exists a nonnegative integer e and for each $j \geqq e$ an R_j-translate $f^{(j)}(Z)$ of $f(Z)$ such that $f^{(j)}(Z)$ is of $[R_j, x_j, y_j, J, X, w]$-stable-type $(m; a_j, b_j, c_j)$ where for all $j \geqq e$ we have the following:* (1) *if $w(y_j) < w(x_j)$ then $f^{(j+1)}(Z) = f^{(j)}(Z)$, $a_{j+1} = a_j$, $b_{j+1} = \operatorname{ord}_{R_j} \times \times f^{(j)}(0)$, and $c_{j+1} = a_j + b_j + c_j - b_{j+1}$;* (2) *if $w(y_j) = w(x_j)$ then $b_{j+1} = 0$;* (3) *if $w(y_j) > w(x_j)$ then $b_{j+1} = b_j$. In connection with* (2) *note that if w is rational then there exist infinitely many distinct nonnegative integers j for which $w(y_j) = w(x_j)$.*

Proof. In view of Lemma 9.9, our assertion follows by first applying Lemma 9.23 and then applying Lemma 9.21.

Lemma 9.25. *Let $m = p^n$ where n is a positive integer and let $f(Z)$ be a monic polynomial of degree m in Z with coefficients in R. Assume that w is real nondiscrete and $f(Z)$ is $[R, x, y, J, w]$-permissible. Also assume that either:* 1) *R is of characteristic p and $f(Z) \neq Z^m + f(0)$; or:* 2) *R is a spot over a pseudogeometric domain, $f(Z)$ is irreducible in $K[Z]$, and $h(R_w)$ does not split in $h(K[Z])$ where h is the canonical epimorphism of $K[Z]$ onto $K[Z]/f(Z)K[Z]$. Let (R_j, x_j, y_j)*

be the canonical j^{th} quadratic transform of (R, x, y, J) along w. Then there exists a nonnegative integer e and for each $j \geqq e$ an R_j-translate $f^{(j)}(Z)$ of $f(Z)$ such that $f^{(j)}(Z)$ is of $[R_j, x_j, y_j, J, X, w]$-stable-type $(m; a_j, b_j, c_j)$ where for all $j \geqq e$ we have the following: (1) if $w(y_j) < w(x_j)$ then $f^{(j+1)}(Z) = f^{(j)}(Z)$, $a_{j+1} = a_j$, $b_{j+1} = \operatorname{ord}_{R_j} f^{(j)}(0)$, and $c_{j+1} = a_j + b_j + c_j - b_{j+1}$; (2) if $w(y_j) = w(x_j)$ then $b_{j+1} = 0$; (3) if $w(y_j) > w(x_j)$ then $b_{j+1} = b_j$, In connection with (2) note that if w is rational then there exist infinitely many distinct nonnegative integers j for which $w(y_j) = w(x_j)$.

Proof. Since w is real, there exists a canonical quadratic transform (R', x', y') of (R, x, y, J) along w such that $w(y') \geqq w(x')$. Let (R'', x'', y'') be the canonical first quadratic transform of (R', x', y', J) along w. Since $f(Z)$ is $[R, x, y, J, w]$-permissible we get that $f(Z)$ is $[R', x', y', J, w]$-permissible and $f(Z)$ is $[R'', x'', y'', J, w]$-permissible and hence in particular $f(Z)$ is of ramified-type relative to $\operatorname{ord}_{x''R''}$. Now $\operatorname{ord}_{R'} = \operatorname{ord}_{x''R''}$ and hence $f(Z)$ is of ramified-type relative to $\operatorname{ord}_{R'}$. Also note that if condition 2) holds then R' is a spot over a pseudogeometric domain and the integral closure of $h(R')$ in $h(K[Z])$ is quasilocal. Therefore, if condition 1) holds then by Lemma 8.13 and if condition 2) holds then by Lemma 8.16, there exists an R'-translate $f'(Z)$ of $f(Z)$ such that $f'(0) \neq 0$ and $F'(i, j) = 0$ whenever $(i, j) \equiv 0(m)$ and $i + j \leqq \operatorname{ord}_{R'} f'(0)$ where $\Sigma F'(i, j) x'^i y'^j$ is the expansion of $f'(0)$ in $J[[x', y']]$. Our assertion now follows by first applying Lemma 9.10 and then applying Lemma 9.24.

Bibliography

[1] ABHYANKAR, S.: On the ramification of algebraic functions. Am. J. Math. **77**, 575—592 (1955).
[2] — On the valuations centered in a local domain. Am. J. Math. **78**, 321—348 (1956).
[3] — Ramification theoretic methods in algebraic geometry. Princeton: Princeton University Press 1959.
[4] — Reduction to multiplicity less than p in a p-cyclic extension of a two dimensional regular ring (p = characteristic of the residue field). Math. Ann. **154**, 28—55 (1964).
[5] — Uniformization of Jungian local domains. Math. Ann. **159**, 1—43 (1965). Correction, Math. Ann. **160**, 319—320 (1965).
[6] — Uniformization in a p-cyclic extension of a two dimensional regular local domain of residue field characteristic p. Festschrift zur Gedächtnisfeier für KARL WEIERSTRASS, 1815—1965, Wissenschaftliche Abhandlungen des Landes Nordrhein-Westfalen, vol. **33**, pp. 243—317. Köln und Opladen: Westdeutscher Verlag 1966.
[7] — Resolution of singularities of arithmetical surfaces. Arithmetical Algebraic Geometry, pp. 111—152, Proceedings of a Conference held at Purdue University. New York: Harper and Row 1965.
[8] ALBERT, A. A.: Modern Higher Algebra. Chicago: University of Chicago Press 1937.
[9] CHEVALLEY, C.: On the theory of local rings. Ann. Math. **44**, 690—708 (1943).
[10] NAGATA, M.: Local Rings. New York: Interscience Publishers 1962.
[11] ZARISKI, O., and P. SAMUEL: Commutative Algebra, vol. I. Princeton: Van Nostrand 1958.
[12] ZASSENHAUS, H.: The Theory of Groups. New York: Chelsea Publishing Company 1949.

Professor SHREERAM S. ABHYANKAR
Division of Mathematical Sciences
Purdue University
Lafayette, Indiana

(Received May 21, 1965)