PHYSICAL REVIEW B VOLUME 50, NUMBER 13 1 OCTOBER 1994-I

Localization of light in coherently amplifying random media

Prabhakar Pradhan and N. Kumar*
Department of Physics, Indian Institute of Science, Bangalore 560 012, India
(Received 25 April 1994)

We derive and analyze the statistics of reflection coefficient of light backscattered coherently from an amplifying and disordered optical medium modeled by a spatially random refractive index having a uniform imaginary part in one dimension. We find enhancement of reflected intensity owing to a synergy between wave confinement by Anderson localization and coherent amplification by the active medium. This is not the same as that due to enhanced optical path lengths expected from photon diffusion in the random active medium. Our study is relevant to the physical realizability of a mirrorless laser by photon confinement due to Anderson localization.

Light-wave propagation in a passive random medium and the associated phenomena of Anderson localization[1] and of resonance fluctuation of the reflection and the transmission coefficients are now well understood.[2] The bosonic nature of the light quanta, however, brings in some additional features, namely those of wave amplification and attenuation (absorption) that have no analog for their fermionic (electronic) counterpart. By amplification here we mean the coherent amplification, as by stimulated emission of radiation in an "active" medium, wherein the phase of the amplified wave is protected in time. And similarly for the coherent attenuation by absorption, sometimes referred to as the stochastic rather than deterministic absorption, wherein the light wave is taken to be in a coherent state—an eigenstate of the photon-annihilation operator. This persistence of phase coherence despite amplification raises the interesting possibility of obtaining synergetic enhancement of wave amplification, or laser action without mirrors, due to confinement by Anderson localization in an intentionally disordered, optically pumped, laser-active condensed matter.[3] Indeed, the recent observation[4] of multimode laser radiation from an optically pumped and strongly scattering colloidal suspension of TiO_2 (rutile) nanoparticles in methanol containing a laser active dye (rhodamine 640 perchlorate) supports the above possibility.

A proper understanding of this phenomenon, however, raises several basic questions that need to be addressed first: (a) Is the enhancement due merely to an increased sojourn time because of the slow classical diffusion of the multiply-scattered photons as in a DWS (diffusing wave spectroscopy) setup, or (b) is it due to Anderson localization, i.e., below the mobility edge anomalous diffusion of photons, and (c) how can the mode-selection possibly take place despite the non-self-averaging fluctuations of the localization length, well known from its electronic counterpart as conductance fluctuation? Motivated by its obvious relevance to photonics, and not a little by the pure physics of it, we have studied here a fundamental aspect of this phenomenon, namely, the statistics of a non-self-averaging fluctuation of the coefficient of reflection for a light-wave incident on such an active amplifying optical medium with an index-of-refraction disorder in one space dimension. More specifically, our active disordered medium stimulates an optical fiber, made active by Er doping[5] and optical pumping, say, and rendered disordered by having the real part of its dielectric constant $\epsilon'(x)$ vary randomly along the fiber length. The coherent amplification (active) aspect is modeled[6] by introducing a phenomenological nonrandom negative imaginary part ($\epsilon''<0$) to the dielectric constant, i.e., $\epsilon(x)=\epsilon'(x)+i\epsilon''(x)$ with the real part $\epsilon'(x)=\epsilon_0+\epsilon_r(x)$ and $\epsilon_r(x)$ random. Inasmuch as we are concerned here only with the aspect of enhanced coherent amplification and not the self-sustaining laser oscillations above thresold pumping, it is sufficient to confine ourselves to the linear regime where ϵ'' is independent of the wave amplitude. We will discuss briefly the nonlinear regime relevant to laser oscillations at the end. For simplicity we assume the fiber to be polarization maintaining so that a scalar wave treatment should suffice.

We take our one-dimensional sample of disordered active medium of length L to be connected to perfect leads at either end, and consider an analytic signal of circular frequency ω incident on the disordered section at one end. The complex wave amplitude $E(x)$ then obeys the Maxwell equation

$$-\frac{\partial^2 E(x)}{\partial x^2}-\frac{\omega^2}{c^2}\epsilon_r(x)E(x)=\frac{\omega^2}{c^2}(\epsilon_0+i\epsilon'')E(x) \quad \text{(1a)}$$

or

$$\frac{\partial^2 E(x)}{\partial x^2}+k_0^2[1+\eta(x)]E(x)=0, \quad \text{(1b)}$$

with $k_0^2=\omega^2\epsilon_0/c^2$ and $\eta(x)=\epsilon_r/\epsilon_0+i(\epsilon''/\epsilon_0)\equiv\eta_r(x)+i\eta_i$, where k_0 and c are, respectively, the wave vector magnitude and the speed of the light in a vacuum.

As is now well known, the Schrödinger-like wave equation (1b) can be transformed so as to give directly an equation of evolution for the emergent quantity, namely, the amplitude reflection coefficient $R(L)=\sqrt{r(L)}\exp[i\theta(L)]$ as function of the sample length:[6–8]

$$\frac{\partial R(L)}{\partial L}=2ik_0R(L)+i\frac{k_0}{2}[\eta_r(L)+i\eta_i][1+R(L)]^2. \quad (2)$$

A similar equation, though somewhat complicated, can be written down for the transmission amplitude. We are, however, interested here in the statistics of the reflection coeffi-

0163-1829/94/50(13)/9644(4)/$06.00

cient $r(L)$ and, therefore, confine our study to the reflection mode only. This stochastic equation can be solved analytically for $\epsilon_r(x)$ a Gaussian white noise, i.e., for $\langle \eta_r(L)\rangle = 0$ and $\langle \eta_r(L)\,\eta_r(L')\rangle = g\,\delta(L-L')$. With this, the "Fokker-Planck" equation for the probability distribution $P(r,l)$ associated with the stochastic equation (2) turns out to be[7,8]

$$\frac{\partial P(r,l)}{\partial l} = r(1-r)^2\frac{\partial^2 P}{\partial r^2} + [1+(-6+D)r+5r^2]\frac{\partial P(r,l)}{\partial r} + [(-2+D)+4r]P(r,l), \tag{3}$$

where we have introduced the dimensionless length $l = \frac{1}{2}gk_0^2 L \equiv L/l_c$ and $D = 4\eta_i/gk_0^3$. Also, we have assumed the phase angle θ to be uniformly distributed.[8] Here $D<0$ corresponds to coherent amplification, $D>0$ to attenuation by coherent absorption, and $D=0$ to the unitary case. To emphasize the dependence of $P(r,l)$ on D we will write explicitly $P(r,l) \equiv P^D(r,l)$ whenever necessary. It is to be noted that the assumed Gaussian white-noise randomness of a quantity (the dielectric constant) within the sample $(0 \leq x \leq L)$ translates as the equivalent Gaussian white-noise randomness of the terminal quantity at $x=L$ in the exact sense of invariant imbedding[8] where the given sample is assumed imbedded invariantly in its extension, i.e., extension of L to $L'(>L)$ maintains the statistics of randomness. Hence the dielectric constants at L and $L'(>L)$ are all statistically at par.

Some asymptotic features of $P^D(r,l)$ can be obtained directly from Eq. (3). First, let us note that our Eq. (3) reduces to the unitary case[7] ($D=0$) in the limits $\epsilon''=0$ (trivially) and $g=0$ (nontrivially). The latter case is dominated by disorder and the localization length is too short for the wave to have penetrated the amplifying medium appreciably. Next, the statistics $P^D(r,l)$ saturate to a broad limiting form as $l\to\infty$. It can be obtained by setting $\partial P(r,l)/\partial l = 0$ and solving the resulting equation analytically. We get

$$\lim_{\substack{l\to\infty\\ D<0}} P^D(r,l) \to P(r,\infty) = \begin{cases} \dfrac{|D|\exp\left(-\dfrac{|D|}{r-1}\right)}{(1-r)^2} & \text{for } r\geq 1\\[2ex] 0 & \text{for } r<1, \end{cases} \tag{4a}$$

$$\lim_{\substack{l\to\infty\\ D>0}} P^D(r,l) \to P(r,\infty) = \begin{cases} \dfrac{|D|\exp(|D|)\exp\left(-\dfrac{|D|}{1-r}\right)}{(1-r)^2} & \text{for } r\leq 1\\[2ex] 0 & \text{for } r>1, \end{cases} \tag{4b}$$

$$\lim_{\substack{l\to\infty\\ D=0}} P^D(r,l) \to P(r,\infty) = \delta(1-r). \tag{4c}$$

It is to be noted that in all cases the length scale is essentially set by the localization length l_c independently of amplification and/or absorption. It is also readily seen that for $D<0$ (amplification), the limiting form gives a weak (logarithmic) divergence for the mean $\langle r\rangle$, due to the long tail of the distribution $P^+(r,\infty)$ for large r. This is clearly due to the amplified reflections from deep within the sample for $l\to\infty$.

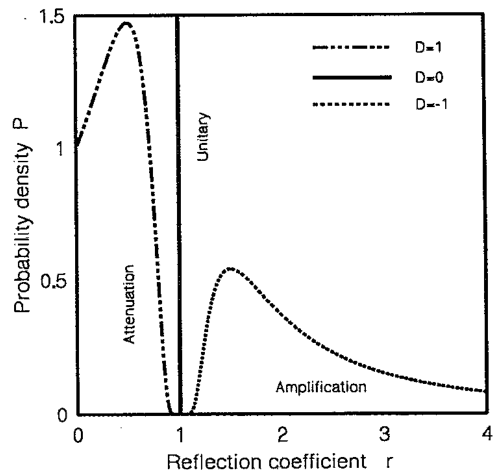

FIG. 1. Limiting probability density $P^D(r,\infty) \equiv P$ of reflection coefficient r for (a) coherent absorption ($D=1$), (b) coherent amplification ($D=-1$), and (c) unitary ($D=0$).

In Fig. 1 we have plotted $P^D(r,\infty)$ for the three limiting cases with $D=+1,0,-1$. In order to see the approach to these limiting forms (Fig. 1), we have also solved Eq. (3) numerically for finite length, and the results are plotted in Fig. 2 and Fig. 3. A technical point is to be noted here for the amplifying case. Equation (3) is essentially a diffusion equation with a diffusion coefficient $r(1-r)^2$, a function of r that vanishes as $r\to 0$ *or* 1. This is a singularity that makes the initial value problem of $P^D(r,l)$ from the initial condition, $P^D(r,l)\to\delta(r)$ as $l\to 0$, numerically difficult to follow. The plots shown in Fig. 3 have been obtained by regularizing $r(1-r)^2$ as $r(1-r)^2+\delta$ with a nonzero $\delta \ll 1$. With this it is again noticed that the probability density $P^D(r,l)$ quickly saturates to the limiting form $P^D(r,l\to\infty)$ for $l>1$, for a low value of amplification as in Fig. 3(a). For higher amplifications, however, the approach to the limiting form is relatively slow as in Fig. 3(b), where one can barely discern a shift in the peak towards the limiting position. This is readily understandable as for large amplification parameters, reflections from deep within the sample also begin to contribute despite exponential localization. The value of the reflection coefficient r_{max} at which $P^D(r,l)$ peaks, however, increases with increasing $|D|$. All of these strongly suggest that, for not too large an amplification, the reflected light is amplified mostly within a localization length of the point of incidence. This localization-enhanced reflection is quite different from what one would expect from the diffusion of photons familiar from DWS.[1] In the diffusive case the distribution of the optical path length together with the exponential growth of wave amplitude due to coherent amplification would give in one dimension $P^-(r,\infty) \sim 1/r(\ln r)^{1/2}$ for $r \gg 1$ which decays much slower than the actual $P^-(r,\infty)$ obtained in Eq. (4a). The former would represent the effect of long, diffusive return paths traversing the deep interior of the sample much more beyond the l_c than in the present case.

The above analysis based on linear amplification (i.e., the wave-amplitude independent of the imaginary part ϵ'' of the

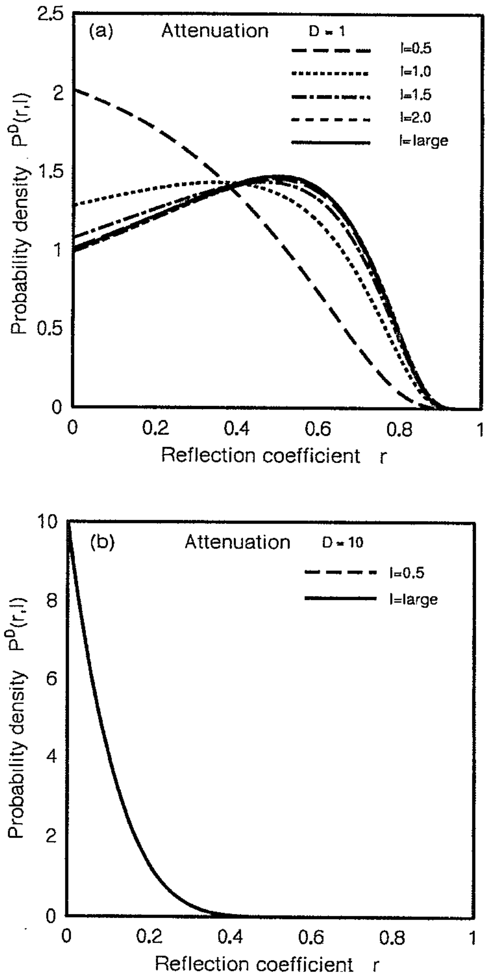

FIG. 2. Plot of $P^D(r,l)$ against the sample length l for coherent absorption parameters: (a) $D=1$ and (b) $D=10$. Solid line is the analytical result for $l \rightarrow \infty$.

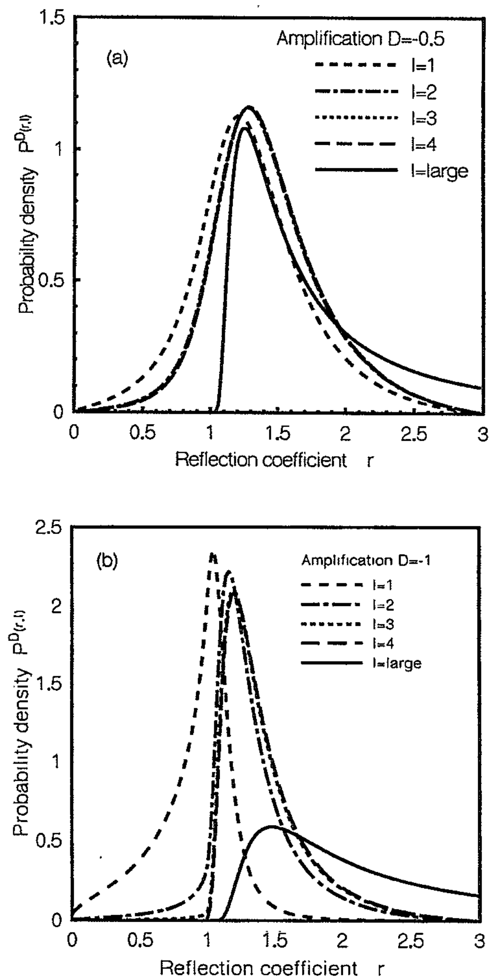

FIG. 3. Plot of $P^D(r,l)$ against the sample length l for coherent amplification parameters: (a) $D=-0.5$ and (b) $D=-1$. Solid line is the analytical result for $l \rightarrow \infty$.

dielectric constant) cannot by itself address the real problem of laser oscillations and mode selection in an optically pumped (active) random medium. For this we must make ϵ'' a function of the local wave intensity, e.g., $\epsilon \propto E^*(x)E(x)$. The resulting nonlinear Schrödinger-like equation can again be reduced quite straightforwardly to the equation for the reflection coefficient via invariant imbedding.[8,9] One expects to get a narrowing of the statistics $P^D(r,l)$ as the threshold for lasing is crossed. This nonlinear aspect, however, calls for further analysis.

In conclusion, we have analyzed the statistics of the coefficient of reflection of light from a coherently amplifying (active) medium with refractive-index randomness in one dimension. The coherent amplification has been modeled by giving an imaginary part to the refractive index. Our results strongly suggest that the enhanced reflection coefficient is due to a synergy between Anderson localization and coherent amplification. It is not simply a result of enhanced optical path length subtended by the classical diffusion of photons.

One of us (P.P.) would like to thank the Council of Scientific and Industrial Research (CSIR), India, for financial support and Mausumi Dikpati for useful discussions on numerical techniques for solving partial differential equations.

*Also at Jawaharlal Nehru Centre for Advanced Scientific Research, Indian Institute of Science Campus, Bangalore 560 012, India.

[1] P. W. Anderson, Phys. Rev. B **109**, 1492 (1958).

[2] See Sajeev John, in *Scattering and Localization of Classical Wave in Random Media*, edited by Ping Sheng (World Scientific, Singapore, 1990). Also, Sajeev John, Phys. Today **44** (5), 32 (1991).

[3] This possibility was proposed and discussed recently by one of the authors, N. Kumar, J. Indian Inst. Sci. **73**, 171 (1993). For a closely related phenomenon of modification of radiative properties of atoms and molecules by coherent scattering, see J. Mar-

torell and N. M. Lawandy, Phys. Rev. Lett. **65**, 1877 (1990); **66**, 887 (1991).

[4] N. M. Lawandy, R. M. Balachandran, A. S. L. Gomes, and E. Sauvain, Nature **368**, 436 (1994).

[5] For a review of Er-doped optical fibers, see Paul Urquhart, Proc. IEEE **135**, 385 (1988); see also Emmanuel Desurvire, Sci. Am. **266** (Jan.) 96 (1992).

[6] A. Rubio and N. Kumar, Phys. Rev. B **47**, 2420 (1993).

[7] N. Kumar, Phys. Rev. B **31**, 5513 (1985).

[8] For a detailed discussion of invariant imbedding see R. Rammal and B. Doucot, J. Phys. (Paris) **48**, 509 (1987); B. Doucot and R. Rammal, *ibid.* **48**, 527 (1987).

[9] See D. J. Pines, D. A. Weitz, G. Maret, P. E. Wolf, E. Herbolzheimer, and P. M. Chaikin, in *Scattering and Localization of Classical Wave in Random Media* (Ref. 2).