Classical Angular Momentum of Light: A Paradox and its Resolution

K Vijay Kumar and N Kumar
Raman Research Institute
Bangalore 560080, India.
Also
Jawaharlal Nehru Centre for Advanced Scientific Research
Jakkur, Bangalore 560064
India.

Angular momentum carried by a classical circularly polarized electromagnetic plane wave (light) appears to be identically zero inasmuch as its linear field-momentum density is directed along wave propagation, and, therefore, the angular momentum, being the integrated moment of the linear momentum density about an axis parallel to the direction of propagation, necessarily vanishes – in detail. This, however, contradicts the established fact that circularly polarized light does carry angular momentum that remains classically non-zero. The paradox is resolved in a physically transparent manner by treating this problem as that of a transversely bounded, and hence necessarily non-transverse, electromagnetic wave propagating along a circular waveguide, in the limit as its radius tends to infinity. We get a non-zero angular momentum that bears the correct ratio to wave energy. This angular momentum derives essentially and exactly from the boundary conditions for the geometry considered. This is an interesting example of surface terms giving a volume (bulk) contribution, much as in the entirely different context of orbital diamagnetism, which was regarded as a surprise of theoretical physics by Rudolf Peierls.[1]

What is the angular momentum carried classically by a circularly polarized electromagnetic plane wave? That is the question addressed here. A straightforward calculation would, of course, suggest the simple answer that it is identically zero. This is readily seen from the following. For an electromagnetic wave, the linear wave-momentum density $\mathbf{p} = \varepsilon_o \mathbf{E} \times \mathbf{B}^* \equiv \mathbf{S}/c^2$, while the angular momentum density $\mathbf{l} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{S}$ is the Poynting vector and other symbols have their usual meaning

[2,3]. We are using here the complex representation, and the SI system of units. Thus, e.g., the physical electric field is the real part of the complex $\mathbf{E}$ field, and similarly for $\mathbf{B}$, and other quantities. Clearly, for a circularly polarized transverse electromagnetic plane wave propagating along the positive z-axis, say, the angular momentum density l_z about the z-axis is identically zero inasmuch as the linear momentum density is everywhere parallel to the (z) axis of propagation. But, of course, we must hasten to add that this cannot be the case. Non-zero angular momentum of circularly polarized light is a well established fact. Moreover, we know that the electromagnetic fields ultimately must be described correctly in terms of the quanta of radiation, the photons, and that the photon carries a spin-angular momentum of magnitude $\hbar$, directed parallel or antiparallel to the direction of propagation according as it is right- or left-circularly polarized. Now, a classical electromagnetic wave is the limiting case of the photonic state (a coherent state with a large mean photon number in the mode), and in this classical limit the angular momentum does survive. This then is the paradox! *Where is the angular momentum*? In this work we show that the resolution of this paradox lies in treating correctly the boundary conditions at the transverse infinity for the transversely unbounded plane wave in question. This we will do now in a physically transparent way by treating the problem in question as the limiting case of a well defined problem of a transversely bounded, and hence necessarily non-transverse electromagnetic wave propagating in a circular waveguide, *as its radius tends to infinity.* The non-zero angular momentum will be shown to derive essentially from the boundary conditions, and to bear the correct ratio to the wave energy. Thus, this is an interesting example of surface terms giving a volume (bulk) contribution, much as in the entirely different context of orbital diamagnetism, which was regarded as a surprise of theoretical physics by Rudolf Peierls [1].

A strong motivation for presenting this work has come from the reaction of some graduate students and physicist colleagues when confronted with this paradox and our resolution of it – it was one of surprise! Furthermore, we believe that this physically motivated and transparent treatment has a didactic value – it is certainly much simpler than the involved treatment of angular momentum of EM waves found in standard textbooks [3].

Consider a monochromatic electromagnetic wave of circular frequency ω propagating in a circular cylindrical waveguide of radius R, in the mode TE_{11}, well known from standard textbooks on classical electrodynamics [2,3]. The corresponding electric and magnetic fields are (in the complex representation)

$$\mathbf{E}(\text{lin}) = (E_\rho \hat{\rho} + E_\phi \hat{\phi} + E_z \hat{z})\exp i(\omega t - k_z z) \quad (1)$$

$$\mathbf{B}(\text{lin}) = (B_\rho \hat{\rho} + B_\phi \hat{\phi} + B_z \hat{z})\exp i(\omega t - k_z z) \quad (2)$$

with

$$E_\rho = \frac{A J_1(k\rho)\sin\phi}{\rho} \quad (3)$$

$$E_\phi = A\frac{d}{d\rho}[J_1(k\rho)]\cos\phi \quad (4)$$

$$E_z = 0, \quad (5)$$

and

$$B_\rho = -\frac{Ak_z}{\omega}\frac{d}{d\rho}[J_1(k\rho)]\cos\phi \quad (6)$$

$$B_\phi = \frac{Ak_z}{\omega}\frac{J_1(k\rho)}{\rho}\sin\phi \quad (7)$$

$$B_z = -i\frac{Ak^2}{\omega}J_1(k\rho)\cos\phi, \quad (8)$$

where $\hat{\rho}$, $\hat{\phi}$ and $\hat{z}$ are the unit vectors in the respective directions; $k = \sqrt{(\omega^2/c^2 - k_z^2)}$ with k_z the wavenumber; $i = \sqrt{-1}$ and $J_m(k\rho)$ the Bessel function of order m.

The parenthetical (lin) anticipates linear polarization of the TE_{11} mode in the limit $R \to \infty$. Now, the boundary conditions for perfectly conducting waveguide walls, namely that $E_\phi = 0 = B_\rho$ at $\rho = R$, demand

$$\frac{dJ_1(k\rho)}{d\rho}|_{\rho=R} = 0 \tag{9}$$

giving

$$k = \frac{u}{R}, \tag{10}$$

where u (= 1.841) is the first root of (9). The field configuration as in (1 - 8) corresponding to the waveguide-bound TE_{11} mode clearly goes over to that for a transversely unbounded linearly plane polarized transverse electromagnetic (TEM) wave in the limit $R \to \infty$, i.e., for all finite points on the transverse planes, z= constant, the two field configurations coincide. Next, in order to realize a circularly polarized limiting field configuration, all we have to do now is to superpose on the fields of (1-8), another field configuration which is in space-time quadrature with the former. The fields in space-time quadrature are obtained from those of (1-8) by simply replacing ϕ by $\phi + \pi/2$, and multiplying the resulting expressions by i. This gives at once

$$\mathbf{E}(\text{circ}) = E_o R \left(\frac{iJ_1(k\rho)}{\rho}\hat{\rho} + \frac{d}{d\rho}J_1(k\rho)\hat{\phi} \right) \exp i(\omega t - k_z z - \phi), \tag{11}$$

$$\mathbf{B}(\text{circ}) = \frac{E_o R}{\omega} \left(-k_z \frac{dJ_1(k\rho)}{d\rho}\hat{\rho} + ik_z \frac{J_1(k\rho)}{\rho}\hat{\phi} - ik^2 J_1(k\rho)\hat{z} \right)$$
$$\exp i(\omega t - k_z z - \phi), \tag{12}$$

where we have chosen for convenience the normalization such that the field $E_\rho = E_o$ for $\phi = \pi/2$, $R \to \infty$, giving $A = RE_o$, to within a numerical constant $(2/u)$ which is ignored. Here the (circ) in parenthesis anticipates the circular polarization in the limit $R \to \infty$.

Clearly, in the limit $R \to \infty$, the above field configuration goes over to that of a right-circularly polarized plane wave propagating along the z-axis, but now with clearly defined boundary conditions admissible at the spatial infinity in the transverse plane, z = constant (see(9)).

We are now in a position to calculate the total angular momentum (L_z) carried by the above wave fields, and see if it remains non-zero in the limit $R \to \infty$. We have

$$L_z \hat{z} = \int \varepsilon_o \rho \hat{\rho} \times (\mathbf{E} \times \mathbf{B}^*) \rho d\rho d\phi dz$$

$$= \left(\frac{\varepsilon_o E_o^2}{\omega} \right) (u^2 - 1)(J_1(u))^2 \pi R^2 \hat{z}, \qquad (13)$$

where the integration is over a unit length of the waveguide. Thus, the angular momentum per unit-cross-sectional area per unit length of the cylindrical waveguide is

$$L_z \hat{z} = \frac{\varepsilon_o E_o^2}{\omega} (u^2 - 1)(J_1(u))^2 \hat{z}. \qquad (14)$$

This is clearly non-zero. Interestingly, it is independent of R!

We also verify that the calculated angular momentum bears the correct ratio to the wave energy. The wave energy per unit cross-sectional area of the waveguide per unit length is

$$U = \left(\frac{1}{\pi R^2} \right) \frac{1}{c} \int \mathbf{S} \cdot \mathbf{ds}$$

$$= \frac{E_o^2 \sqrt{(w^2/c^2) - k^2}}{c \mu_o \omega} (u^2 - 1)(J_1(u))^2. \qquad (15)$$

This gives for the ratio

$$\frac{L_z}{U} = \left(\frac{\varepsilon_o \mu_o c}{\sqrt{(\omega^2/c^2) - k^2}} \right). \qquad (16)$$

Note that this ratio depends on the radius R of the waveguide through the relation $k = u/R$ (see (10)).

Thus in the limit of present interest, $R \to \infty$, and hence $k \to 0$, the ratio turns out to be

$$\lim_{R\to\infty} \frac{L_z}{U} = (\frac{1}{\omega}) \tag{17}$$

which is a known standard result, valid in the quantum as well as in the classical limit – recall that in the quantum case, $L_z = n\hbar$ and $U = n\hbar\omega$ with n the number of photons in the wave-guide.

Physical origin of this boundary contribution to the bulk angular momentum should now be clear. In the large R limit, the transversely bounded waveguide mode, namely the TE_{11} mode plus its counterpart in space-time quadrature, is arbitrarily close to being a circularly polarized TEM wave *except near the boundary* $\rho = R$, where the curving of the **B** field lines in the longitudinal (axial) direction gives a tangential component to the field-momentum density which, when crossed with $\rho\hat{\rho}$, generates a finite contribution to the angular momentum – *the large arm length* ρ *provides the necessary leverage.* Indeed, the circumferential whispering-gallery-mode like momentum density at the boundary is reminiscent of the boundary-skipping classical orbits of electrons in a magnetic field – the Landau diamagnetic problem [1,4]. Of course, this is not to imply that the electromagnetic angular momentum problem is being mapped on to that of the Landau orbital diamagnetism. The point to note here is that in both the cases we have an orbital bulk (volume) effect coming from the boundary! Rudolf Peierls [1] had called this a surprise of theoretical physics in the context of orbital diamagnetism. Clearly, the angular momentum of electromagnetic waves, as treated above, provides yet another example of this.

Finally, we would like to make some remarks. The first one is that we have demonstrated explicitly and exactly the non-vanishing of the angular momentum only for the case of a circular perfectly conducting boundary. We believe, however, that the effect is general – the exact shape of the boundary should not matter in the limit of $R \to \infty$. Indeed, one can have internal boundaries too (again much as in the case of Landau diamagnetism). The point really is that any transversely bounded, or confined (localized) electromagnetic wave in a simply connected region of space cannot be strictly transverse – it is the non-transversality of the field (that for the waveguide model is pronounced essentially at the boundaries only) that contributes to the angular momentum. More importantly, this classical electromagnetic angular momentum coming from the tangential whispering-gallery-mode like wave propagation at the boundary is manifestly of an *orbital nature.* It is not obvious, however, how this orbital angular momentum eventually connects up with the photonic spin-angular momentum which is a purely quantum concept. The ratio in (18) turns out to be the same for both cases, the classical as well as the quantum. This question has been discussed at somewhat formal level in advanced texts (see, e.g., [3]). The reader is encouraged to give further thought to this.

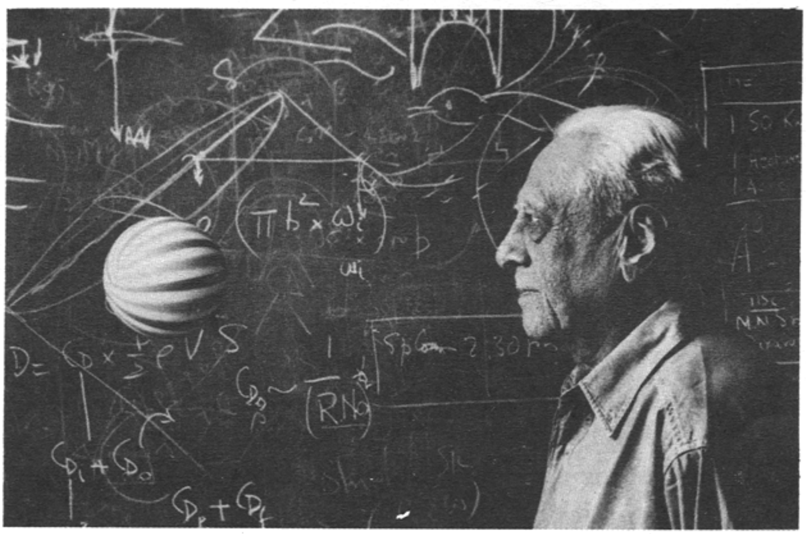
Suggested Reading

[1] R Peierls, in *Surprises in Theoretical Physics*, Princeton Univ. Press, Princeton, 1979.

[2] W C Elmore and M A Heald, *Physics of Waves*, McGraw Hill, New York, 1969.

[3] J D Jackson, *Classical Electrodynamics*, John Wiley, London, 1999.

[4] A M Jayannavar and N Kumar, *J. Phys.*, Vol. A14, p. 1399, 1984.



Satish Dhawan