QUANTUM KINEMATICS OF FERMI-DIRAC VACUUM-*PLUS*-ONE-ELECTRON PROBLEM

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ABSTRACT

Following Weisskopf, the kinematics of quantum mechanics is shown to lead to a modified charge distribution for a test electron embedded in the Fermi-Dirac vacuum with interesting consequences.

There is a well-known argument due originally to Weisskopf (1939) wherein it has been shown that the kinematics of relativistic quantum mechanics requires that the charge distribution associated with a single Fermi particle such as an electron in absolute vacuum (as distinct from the physical vacuum) be that of a point charge. However, for an electron embedded in the physical vacuum of the Dirac theory, *i.e.*, the one in which the negative-energy states are completely filled up while the positive-energy states are completely empty, the theory predicts a charge distribution that is highly singular but still has a non-zero spatial support of linear dimensions of the order of the Compton wavelength of the particle h/mc. It may be of some interest to examine the charge distribution associated with an electron when the latter is embedded in the *Fermi-Dirac* (F-D) *Vacuum*, *i.e.*, the one in which the negative-energy states are completely filled up but the positive-energy states are filled up only to a certain maximum of the single-particle energy (Fermi-energy E_F). A situation of this kind obtains for a non-interacting electron gas at the absolute zero of temperature. The purpose of this note will be to extend the treatment given by Weisskopf to this situation. It is found that in this case, too, the charge distribution is highly singular over a region of linear dimensions of the order of the Compton wavelength h/mc of the particle. But there is, in addition to this, a long-range oscillatory tail that reflects the sharpness of the Fermi surface in a manner that is reminescent of the Friedel oscillations of the impurity problem. The two are, however, very different conceptually. It is also found that the effe tive interaction potential of two such overlapping charge distributions has regions of attraction, favouring a bound-pair state. For an electron gas at low temperature and in the low-density limit the latter is interpreted to imply an instability towards the Wigner-lattice formation. Its relevance to the purely electronic mechanism of superconductivity is also pointed out.

The present derivation closely follows the treatment given in Reference (2).

We describe the test charge distribution in general by the auto-correlation function G (ξ) given by

$$G(\xi) = \int_{\Omega} \rho\left(r + \frac{\xi}{2}\right) \rho\left(r - \frac{\xi}{2}\right) dr, \tag{1}$$

where $\rho(r)$ is the test-charge density at the point r and is given by

$$\rho(r) = e\,\{\Psi^*(r)\,\Psi(r)\} - \sigma.$$

Here $\Psi(r)$ is the four component spinor wavefunction with components Ψ_μ, ($\mu = 1, 2, 3, 4$), and the curly brackets denote the scalar product, *i.e.*,

$$\{\Psi^*(r)\,\Psi(r)\} = \sum_{\mu=1}^{4} \Psi_\mu^*(r)\,\Psi_\mu(r).$$

σ is the background charge density of the *Fermi-Dirac* (*F-D*) *Vacuum*, and Ω is the normalisation volume.

The essential point of the treatment is to take full cognizance of the fact that the test electron is indistinguishable from the electrons comprising the physical vacuum, and that electrons obey Fermi-statistics. Accordingly, we introduce second quantised waves:

$$\Psi(r) = \sum_q a_q \Phi_q(r);\quad \Psi^\dagger(r) = \sum_q a_q^\dagger\, \Phi_q^*(r),$$

where $a_q^\dagger$, a_q are Fermion creation, annihilation operators for the states $\Phi_q(r)$, and $\Phi_q(r)$ are solutions of the Dirac equation for the free electron. More explicitly

$$\Phi_q(r) = \Omega^{-\frac{1}{2}}\, u_q\, e^{i(\hbar q \cdot r - E_q t)/\hbar},$$

where u_q is the normalised four-component spinor and the other symbols have the usual meaning. The effective autocorrelation function $G_{eff}(\xi)$ for the test-charge distribution can now be written as

$$G_{eff}(\xi) = G^{(\xi)}_{F\text{-}D\ Vac+1} - G^{(\xi)}_{F\text{-}D\ Vac}. \tag{2}$$

In the present case, recalling that the negative-energy states, labelled as $q-$, are fully occupied while the positive-energy states, labelled as $q+$, are occupied only up to the Fermi-level q_F we get after some algebra

$$G_{eff}(\xi) = \Big(\sum_{q+} - \sum_{q-} - 2 \sum_{|q+| < q_F}\Big) \times \int_{\Omega} \Big\{\Phi^*_{q0F}\Big(r + \frac{\xi}{2}\Big) \Phi_q\Big(r + \frac{\xi}{2}\Big)\Big\} \times \Big\{\Phi_q^*\Big(r - \frac{\xi}{2}\Big) \Phi_{q0F}\Big(r - \frac{\xi}{2}\Big)\Big\}\, dr, \tag{3}$$

where q_F is the magnitude of Fermi wavevector and the test electron is assumed to lie just at the Fermi surface (*i.e.*, at q_{0F}).

Inserting the actual solutions of the Dirac equation in (3) we get

$$G_{eff}(\xi) = \Big(\frac{e^2\, mc^2}{8\pi^3 \hbar^3}\Big) \int_{|q| > q_F} dq\, \frac{e^{i\xi \cdot q}}{E_q},$$

and the corresponding charge distribution is given by

$$\rho(\xi) = e \int_{|q| < q_F} dq \Big(\frac{mc^2}{E_q}\Big)^{\frac{1}{2}} \frac{e^{i\xi \cdot q}}{8\pi^3 \hbar^3} \equiv \rho_0(\xi) + \rho_{osc}(\xi). \tag{4}$$

Here $\rho_0(\xi)$ is the highly singular charge density as obtained by Weisskopf and is essentially non-vanishing only over a region of linear dimensions of the order of h/mc. For all practical (low-energy) purposes the latter can well be replaced by a point charge. The second term $\rho_{osc}(\xi)$ is a long-range oscillatory charge density and for $\xi \gg h/mc$, $E_F \ll mc^2$ it can be evaluated as

$$\rho_{osc}(\xi) = \frac{e}{2\pi^2}\Big(\frac{\sin q_F\xi - q_F\xi \cos q_F\xi}{\xi^3}\Big). \tag{5}$$

Further, for two such test-charge distributions separated by a distance r the electrostatic potential energy of interaction can at once be evaluated to be

$$V(r) \cong \frac{e^2}{r}\Big(1 - \frac{2}{\pi}\, Si(q_F r)\Big), \text{ with } Si(x) = \int_0^x \frac{\sin y}{y}\, dy$$

and

$$r \gg h/mc. \quad (6)$$

One may note several interesting features of the above expressions. The oscillatory potential energy is seen to have an absolute minimum at $r = \pi/q_F$ of depth $\sim e^2 q_F/2\pi^2$ and radial width $\sim \pi/q_F$. A simple application of the Uncertainty Principle shows that a bound-pair formation is favoured for $q_F < e^2 m/h^2 \sim 5 \times 10^6 \text{ cm}^{-1}$, *i.e.*, for low-density electron gas. It is tempting to relate this to the instability towards the Wigner-lattice (Wigner, 1938) formation for a rare electron gas. The modified potential may play an important role in the purely electronic mechanisms of superconductivity as envisaged by Luttinger (1966). In conclusion, it may be noted that the above considerations do not involve any dynamical interactions such as are responsible for charge screening in the electron gas problem. The smearing out of the electron charge in the present case arises purely from the kinematic phase space considerations involving the nature of the physical vacuum in question.

REFERENCES

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