QUANTUM THEORETIC EXPLANATION OF THE SCHWARZ-HORA EFFECT

By N. Kumar and K. P. Sinha, F.A.Sc.
[*Department of Physics and Department of Applied Mathematics, Indian Institute of Science, Bangalore*–12 (*India*)]

Received July 3, 1971

Abstract

Following the path-integral approach we show that the Schwarz-Hora effect is a one-electron quantum-mechanical phenomenon in that the de Broglie wave associated with a single electron is modulated by the oscillating electric field. The treatment brings out the crucial role played by the crystal in providing a discontinuity in the longitudinal component of the electric field. The expression derived for the resulting current density shows the appropriate oscillatory behaviour in time and distance. The possibility of there being a temporal counterpart of Aharonov-Bohm effect is briefly discussed in this context.

1. Introduction

In a recent experiment Schwarz and Hora (1969) have shown that a 50 keV electron beam passing through a single-crystal film of SiO_2 or Al_2O_3 (thickness $2d \sim 1000$ Å) irradiated with a monochromatic coherent polarized intense light beam from a 10-W argon-ion laser ($\lambda = 4880$ Å) when allowed to impinge on a non-fluorescent target (polycrystalline alumina) emits radiation of the same wavelength as the exciting laser light. This result has aroused a great deal of attention and conflicting explanations have been put forward. While some authors have attributed it to a purely point-mechanical classical bunching of electrons as in a Klystron (Rubin, 1970; Oliver and Cutler, 1970), others invoke quantum-mechanical considerations involving the modulation of the one-electron wave-function (Varshalovich and D'yakonov, 1970; Oliver and Cutler, 1970; Van Zandt and Meyer, 1970; Hutson, 1970). The bunching hypothesis seems untenable owing to the fact that the electron density in the beam in question is much too small (several orders of magnitude smaller than in the case of a Klystron) (Hutson, 1970; Harris and Smith, 1970). Moreover, the depth of velocity modulation is much smaller than the velocity spread in the in coming beam itself. A quantum treatment would, therefore, seem to be more appropriate. However, the earlier workers have attempted a rather heuristic and incomplete quantum description in that certain salient features of the experimental conditions have not been explicitly accounted for, such as the role of the crystal. It is well known that free electrons cannot be scattered by photons in vacuum in a first order process for reasons of energy and momentum conservation. While this has been re-emphasized by several workers and elaborated (Van Zandt and Meyer, 1970), no precise mathematical formulation of the essential role played by the crystal has been given.

In what follows, we present a quantum-mechanical treatment of the modulation of the one-electron wave-function and derive an expression for the resulting current density which has the required oscillatory features, following a different technique.

2. Mathematical Formulation

The path integral approach as developed by Feynman (1965) is elegantly suited for the present purpose. The essential physics of the experiment is represented by the following one-dimensional Lagrangian for an electron,

$$L = \frac{m\dot{x}^2}{2} - eV(x) \sin \Omega t, \tag{1}$$

where the x-axis is along the beam direction, m is the electron mass and Ω is the circular frequency of the exciting laser light assumed to be polarized such that its electric field vector is parallel to the electron beam and $V(x)$ is the scalar potential generated by the macroscopic laser field which can be treated classically. The interaction term in the Lagrangian [*cf.* second term of equation (1)] can always be written in this form by a suitable choice of guage and neglecting the Lorentz force. The presence of the crystal will impart certain discontinuities to the normal component of the electric field associated with the laser beam. This can be represented by

$$\frac{d^2 V(x)}{dx^2} = E_0 f(\epsilon) [\delta(x-d) - \delta(x+d)] + \text{a smooth function } F(x), \tag{2}$$

where E_0 is the constant electric displacement inside the crystal, $f(\epsilon) = (\epsilon - 1)/\epsilon$, ϵ being the relative optical dielectric constant of the crystal. The δ-functions in Eqn. (2) represent discontinuities of the normal component of the electric field at the crystal boundaries. The smooth function $F(x)$ referred to above takes care of the diffuse character of the boundary of the laser beam and remains essentially constant over lengths of the order of (v/Ω), where v is mean electron velocity in the beam. It should also be noted that it does not change appreciably over lengths of the order of the de Broglie wavelength of the electron.

We now consider the scattering of an incoming free electron by the above interaction potential.

The modified final state wave-function is given by

$$\psi(x_2, t_2) = \int K_V(x_2, t_2; x_1, t_1)\, \phi(x_1, t_1)\, dx_1, \tag{3}$$

where $K(x_2, t_2; x_1, t_1)$ is the exact electron Kernal propagating from co-ordinates x_1, t_1 to x_2, t_2 and $\phi(x_1, t_1)$ is the initial state function given by

$$\phi(x_1, t_1) = e^{i/\hbar\,(p_1 x_1 - E_{p_1} t_1)}, \tag{4}$$

where $E_{p_1} = p_1^2/2m$ is the energy of the incoming electron. As usual the wave function is normalised so as to correspond to the probability density current $v\,(= p_1/m)$. The exact Kernel K is explicitly given by the following path integral (Feynman and Hibbs, 1965):

$$K_V(x_2, t_2; x_1, t_1)$$

$$= \int_{x_1}^{x_2} \exp.\left(\frac{i}{\hbar}\int_{t_1}^{t_2}\left[\frac{m\dot{x}^2}{2} - e\,V(x)\sin\Omega t\right]dt\right) Dx(t), \tag{5}$$

where $Dx(t)$ is the element of volume in the path space. In the present situation the relevant parameter in the perturbation expansion for the above Kernal is

$$\left| -\frac{i}{\hbar}\int_{t_1}^{t_2} e\,V\big(x(t)\big)\sin\Omega t\, dt \right|, \tag{6}$$

where the integral is carried out along a typical (*e.g.*, classical) path. Performing integration by parts twice and using Eqn. (2) the above reduces to

$$\frac{eE_0 f(\epsilon)\, d}{\hbar\Omega} \sim 0\cdot 1 \tag{7}$$

Accordingly, we retain terms upto first order only; thus

$$\psi(x_2, t_2)$$

$$= \phi(x_2, t_2) - \frac{i}{\hbar} \int_{-\infty}^{t_2} \int^{x_1} K_0(x_2, t_2; x_1, t_1) \sin \Omega t_1 \, \phi(x_1, t_1) \, dx_1 \, dt_1 \tag{8}$$

where (Feynman and Hibbs, 1965)

$$K_0 = \left(\frac{m}{2\pi i\hbar}\right)^{\frac{1}{2}} \frac{1}{(t_2 - t_1)^{\frac{1}{2}}} \exp. \frac{i}{\hbar} \frac{m}{2} \left[\frac{(x_2 - x_1)^2}{(t_2 - t_1)}\right]. \tag{9}$$

Making use of the identity (Appendix reference above)

$$\int_0^\infty \exp.\left(-\frac{a}{y^2} - by^2\right) dy = \sqrt{\frac{\pi}{4b}} \exp.(-2\sqrt{ab})$$

$$\text{with Re } a > 0 < \text{Re } b, \tag{10}$$

after some change of variables, the time integration can easily be performed. We get

$$\psi(x_2, t_2)$$

$$= \phi(x_2, t_2) - \frac{e}{2\hbar}\left(\frac{m}{2}\right)^{\frac{1}{2}} e^{-i/\hbar E_{p_1} t_2} \int^{x_1} dx_1 V(x_1) \, e^{i/\hbar p_1 x_1}$$

$$\times \left[\left(\frac{1}{E_{p_1} - \hbar\Omega}\right)^{\frac{1}{2}} e^{i\Omega t_2} \exp.\left(\frac{2i}{\hbar}\sqrt{\frac{m}{2}}(x_2 - x_1)^2 (E_{p_1} - \hbar\Omega)\right)\right.$$

$$\left. - \left(\frac{1}{E_{p_1} + \hbar\Omega}\right)^{\frac{1}{2}} e^{-i\Omega t_2} \exp.\left(\frac{2i}{\hbar}\sqrt{\frac{m}{2}}(x_2 - x_1)^2 (E_{p_1} + \hbar\Omega)\right)\right]. \tag{11}$$

If we note that $\hbar\Omega/E_{p_1} \sim 10^{-5}$ in the experiment of Schwarz and Hora (1969), and accordingly replace $(E_{p_1} \pm \hbar\Omega)^{\frac{1}{2}}$ by $E_{p_1}^{\frac{1}{2}}$ everywhere in Eqn. (11) except when it occurs in the exponent, integration by parts using Eqn. (2) yields

$$\psi(x_2, t_2)$$

$$= \phi(x_2, t_2)\left[1 + \frac{i}{\hbar}\,\frac{eE_0 f(\epsilon)\sin\left(\frac{\Omega d}{v}\right)}{\Omega^2}\right.$$

$$\left.\left\{\exp.\frac{i}{\hbar}\left(\hbar\Omega t_2 + x_2(p_+ - p_1)\right) + \exp.\frac{i}{\hbar}\left(\hbar\Omega t_2 - x_2(p_- - p_1)\right)\right\}\right] \tag{12}$$

with

$$p_\pm = p_1\left(1 \pm \frac{2m\hbar\Omega}{p_1^2}\right)^{\frac{1}{2}}, \tag{13}$$

where the integral involving the smooth function $F(x_1)$ has been ignored (Riemann-Lebsgue lemma). The resulting current $j(x_2, t_2)$ is given by

$$j(x_2, t_2) = \frac{ie\hbar}{2m} \times 2I_m\left(\psi^* \frac{d}{dx_2}\psi\right) \tag{14}$$

$$= ev\left[1 + \frac{2eE_0 f(\epsilon)}{mv\Omega}\sin\left(\frac{\Omega d}{v}\right)\sin\left(\Omega t_2 - \frac{x_2\Omega}{v}\right)\right.$$

$$\left. - \frac{4eE_0 f(\epsilon)\, v}{\hbar\Omega^2}\sin\left(\frac{\Omega d}{v}\right)\cos\left(\Omega t_2 - \frac{x_2\Omega}{v}\right)\sin\left(\frac{x_2\hbar\Omega^2}{mv^3}\right)\right]. \tag{15}$$

In deriving the above expression for the current we have gone up to second order terms of the expansion in Eqn (13).

3. Discussion

The above expression for the current affords an easy interpretation of the Schwarz-Hora result. The current given by Eqn. (15) consists of a constant beam current, *i.e.*, a d.c. part and a spatially modulated oscillating part as well as an unmodulated oscillating part. The d.c. part is of no consequence in the present context and the unmodulated term is relatively small in magnitude. Thus the most interesting part is the third term in the bracket of Eqn. (15). For a given distance of the non-fluorescent screen the current oscillates in time with the same frequency as the laser beam. This can excite the target material optically and lead to re-radiation through Bremsstrahlung processes. The amplitude of the exciting current is seen to vary sinusoidally with the screen distance. In this connection it may be added that in a one-dimensional treatment given here, we cannot get any aperiodic attenuation of the current amplitude. The above derivation clearly displays the role of the crystal in as much as the latter provides the discontinuity in the field [*cf.* Eqn. (2)]. In fact, in the absence of such discontinuities we will not get any optical modulation. This corresponds to the well-known fact that in first order free electrons cannot be scattered by photons in vacuum for reasons of energy momentum conservation. It seems in order at this stage to discuss the quantum *versus* the classical nature of the Schwarz-Hora effect in terms of the expression obtained above [*cf.* the third term of the right-hand side of Eqn. (15), hereafter referred to as $j(3)$]. It should be noted that in a purely formal sense for $\hbar \to 0$, $j(3)$ has a non-vanishing limiting value. This, however, is not the appropriate way of taking a classical limit in the present context. In point of fact the argument of the modulating factor sin $(x_2\hbar\Omega^2/mv^3)$ would be taken to be small in the classical limit, *i.e.*,

$$\xi = \left(\frac{\hbar\Omega}{\frac{1}{2}mv^2}\,\frac{x_2\Omega}{2v}\right) \ll 1.$$

In the experiment of Schwarz and Hora this quantity is rather large, *i.e.*, $\xi \sim 10^2$. Thus the experiment of Schwarz and Hora is in the quantum domian. It is interesting to note, however, that the ratio

$$\left|\frac{j(3)}{j(3)_{\text{classical}}}\right| = \left|\frac{\sin\xi}{\xi}\right| \leq 1, \tag{16}$$

where $j(3)_{\text{classical}}$ is the limiting value of $j(3)$ in the limit $\xi \ll 1$, *i.e.*, for very high electron energy or low photon energy (microwave region). It may be noted that in the region well beyond the effective range of the radiation field of the laser beam there will still be a time-dependent but space-independent oscillating potential V (t). While this will have no classical effect, it will introduce a time-dependent phase factor in the wave-function in the field free region. As discussed by Aharonov and Bohm (1959) in a different context, this type of potential will introduce a phase factor of the form

$$\exp.\left(-\frac{i}{\hbar}\, e \int \mathrm{V}(t)\, dt\right). \tag{17}$$

On carrying out the Bessel function expansion, *i.e.*,

$$e^{-i/\hbar\, eV_0 \int^t dt \sin \Omega t} = \sum_{n=-\infty}^{\infty} (-1)^n J_n\left(\frac{eV_0}{\hbar\Omega}\right) \exp.\,(in\Omega t), \tag{18}$$

it can readily be seen that a plane wave solution of the form

$$\exp.\left[\frac{i}{\hbar}(px - E_p t)\right]$$

gets modified to the extent that E_p is replaced by $E_p + n\hbar\Omega$ while the momentum remains unaffected. Therefore, the relationship between energy and momentum characteristic of a free particle may not be strictly retained. In the work of Varshalovich and D'yakonov, while it is not clear how the authors have derived the asymptotic wave-fnnction, the expression strictly fulfils the free-particle energy-momentum relationship for each plane wave component.

In passing it should be noted that the phase-factor occurring in Eqn. (18) will not produce any observable effect in the sence of Aharonov and Bohm owing to the fact that time is one-dimensional [*i.e.*, $V(t)\,dt$ is a perfect differential]. However, if we were to depart from the accepted concept of the co-ordinate time (t) being integrable (*i.e.*, dt is a perfect differential) and to assert that it is the proper time that is integrable (Newburgh and Phipps, 1970), there will be a temporal counterpart of the Aharonov-Bohm effect.

REFERENCES

1. Aharonov, Y. and Bohm, D. *Phys. Rev.*, 1959, **115**, 485.
2. Favro, L. D., Fradkin, D. M. and Kuo, P. K. *Phys. Rev. Lett.*, 1970, **25**, 202.
3. Feynman, R. P. and Hibbs, A. R. In *Quantum Mechanics and Path Integrals*, McGraw-Hill Book Company, New York, 1965.
4. Harris, R. L. and Smith, R. F. *Nature*, 1970, **225**, 502.
5. Hutson, A. R. .. *Appl. Phys. Letters*, 1970, **17**, 343.
6. Newburgh, R. G. and Phipps, T. E. Jr. *Nuovo Cimento*, 1970, **67**, 84.

7. Oliver, B. M. and Cutler, L. S. *Phys. Rev. Lett.*, 1970, **25**, 273.

8. Rubin, P. L. .. *ZhETF Pis. Red.*, 1970, **11**, 365; *JETP Lett.*, 1970, **11** 239.

9. Schwarz, H. and Hora, H. *Appl. Phys. Letters*, 1959, **15**, 349.

10. Van Zandt, L. L. and Meyer, J. W. *J. Appl. Phys.*, 1970, **41**, 4470.

2946-71. Printed at The Bangalore Press, Bangalore-18, by V. J. F. Jesudason, L.P.T., Superintendent. Published by B. S. Venkatachar, Editor, "Proceedings of the Indian Academy of Sciences", Bangalore