ON MASS QUANTISATION OF ELEMENTARY PARTICLES

By N. KUMAR, (MISS) M. MUTHANNA AND K. P. SINHA, F.A.Sc.

(*Departments of Physics and Applied Mathematics, Indian Institute of Science, Bangalore-12, India*)

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ABSTRACT

A semi-phenomenological theory of mass quantisation is presented, wherein different elementary particles are regarded as excited positive-energy states of a fundamental extensible object. The latter is essentially an elastic continuum which in its quiescent (classical equilibrium) state is believed to be massless and stressless. The classical Hamiltonian describing its oscillations about the equilibrium configuration is constructed by treating the mass-equivalent of the elastic potential energy as the inertial mass occurring in the denominator of the kinetic energy term. Quantisation of the resulting variable-mass oscillator is then effected by following the procedure given by Pauli and Podolsky.

The energy-mass eigenvalues (m_n) for the above Schrödinger-like equation are given by

$$\frac{m_n}{m_0} = \left[1 + \left(\frac{9}{2}\right)^{1/3} \left(\frac{\lambda\!\!\!^{-}_0}{r_c}\right)^{2/3} p_n{}^{2/3}\right],$$

where $\lambda\!\!\!^{-}_0$ is the Compton wavelength of the lowest (ground state) eigen mass m_0, r_c is the measure of the linear dimension of the object, and p_n is the nth root of the Bessel function of order 1/3. In view of their infinite lifetime we treat the electron and the proton as the ground states for the two families of particles with baryon numbers zero and unity respectively. Accordingly, for the two families, m_0 and r_c are chosen to correspond to the electron and the proton. The calculated mass values show striking agreement with the observed values for the two series.

1. INTRODUCTION

THE question of the mass spectrum of the elementary particles and of their relative stability remains one of the unsolved problems in physics. We do have, however, a symmetry-based kinematical classification of the elementary particles in agreement with the observed facts (Gell-Mann, 1962; de Broglie, 1963). The dynamical theories of mass spectrum have generally proceeded along the following lines. In the first instance we have a general relativistic model wherein a stable equilibrium configuration of charge and matter is first obtained as solution of Einstein's field equations subject to appropriate boundary conditions on the space-time metric (Penney, 1965; Dirac, 1962). Quantised oscillations of such an extensible particle-like object are then identified with different elementary particles. For example, in Dirac's (1962) theory one has an extensible electron model whose first excited state corresponds to the muon. However, the muon-electron mass ratio in his theory turns out to be about a quarter of the observed value. The second approach is quantum field theoretic and ignores the gravitational effects altogether. (Heisenberg, 1966; Katayama and Yukawa, 1968). (It should be emphasized that in these approaches a fundamental length is involved *a priori*). Thus in Heisenberg's unified field theory of elementary particles the basic non-linear field equation is uniquely obtained entirely from arguments of symmetry, relativistic causality, and the existence of a certain vacuum.

In the present paper, we discuss a semi-phenomenological approach to mass quantization of elementary particles. The basic idea behind this approach is that the various elementary particles are excited positive-energy states of a quantum-mechanical system consisting of a perfectly elastic continuum of non-zero spatial support. This object in its classical equilibrium state is assumed to be massless and stressless. The classical Hamiltonian describing its oscillations about the equilibrium configuration is constructed by treating the mass equivalent of the elastic potential energy as the inertial mass occurring in the kinetic energy term. This Hamiltonian when properly quantized yields energy-mass eigenvalues in fair agreement with results. An important single parameter of this theory is a characteristic length which represents the dimension of the primary object. It should be noted that the present treatment is based on non-relativistic Schrödinger picture; however, the essence of relativity, namely, the equivalence principle has been incorporated (Einstein, 1960).

2. Formulation of the Model Hamiltonian

As remarked above, we start with a primordial object which in its classical equilibrium state is assumed to be stressless and massless having linear dimension $\sim r_c$ (radius of the bounding spherical surface). The physics behind these assumptions will be amplified later. The classical Hamiltonian of such a 'null' system executing oscillations (radial only) about the equilibrium configuration can be written as

$$H = \frac{p^2}{2\,[V(x)/c^2]} + V(x), \tag{1}$$

where we have explicitly equated the instantaneous inertial rest-mass to the total elastic potential energy V (x) divided by the square of the speed of light. This equivalence of the inertial mass and the potential energy is the essential feature of the present treatment. In the above p is the momentum conjugate to the radial co-ordinate x. For a perfectly elastic object taken above the potential energy is quadratic in the displacement from the equilibrium configuration. Thus we have

$$V(x) = \tfrac{1}{2}kx^2 + V_0, \tag{2}$$

where k is the elastic stiffness constant and its origin lies in the details of the general relativistic restoring forces. At this stage of the phenomenology such details are not needed. Further V_0 is the equilibrium state potential energy. Inasmuch as we have assumed the system in its quiescent state to be stressless and massless, we put $V_0 = 0$. It is to be noted that because of the assumption of sphercial symmetry (radial oscillations only) the problem has been reduced to a one-dimensional situation. Thus we can parametrize the problem such that

$$-r_c \leq x \leq \propto.$$

In order to transcribe the above Hamiltonian into its quantum analogue we have to take into account the non-commutability of p and x in the kinetic energy term. For this we follow a procedure given by Podolsky (1928) and Pauli (1933). According to their prescription, for a classical Hamiltonian H written in terms of generalized co-ordinates q^i and conjugate momenta, p_i *i.e.*,

$$H = \tfrac{1}{2}\sum_{i,j} g^{ij}p_i p_j + V(q^i), \tag{3}$$

the corresponding quantum Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \tfrac{1}{2}\sum_{i,j} g^{1/4}\hat{p}_i g^{ij} g^{-1/2}\hat{p}_j g^{1/4} + V(q^i), \tag{4}$$

where

$$\hat{p}_i = - i\hbar\partial/\partial q^i$$

is the momentum operator conjugate to $q^i g^{ij}$ is the metric and g is the determinant constituted from g^{ij}. It must be emphasised that the above prescription implies that the mathematical space characterized by g^{ij} is flat, *i.e.*, the curvature tensor R_{hijk} corresponding to the metric g^{ij} is zero. This condition is trivially satisfied in the present treatment because of the one-dimensionality of the reduced problem. Then the non-vanishing metric component is

$$g^{xx} = V(x)/c^2 = \tfrac{1}{2} kx^2/c^2. \tag{5}$$

The time-independent Schrödinger-like equation

$$\hat{H}\psi = E\psi \tag{6}$$

with $\hat{H}$ given by Eqs. (4) and (5), determines the energy-mass eigenvalues. On making use of the substitution

$$\psi(x) = \phi(x)\, x^{1/2} \tag{7}$$

we get a differential equation

$$\frac{d^2\phi}{dx^2} - \frac{1}{x}\frac{d\phi}{dx} - (ax^4 - bx^2)\,\phi = 0, \tag{8}$$

where

$$a = \frac{k^2}{2\hbar^2 c^2}, \quad b = \frac{Ek}{\hbar^2 c^2}. \tag{9}$$

This equation has two linearly independent [regular $\phi_+(x)$ and irregular $\phi_-(x)$] solutions for a given E. These are

$$\phi_+(x) = (b - ax^2)^{1/2}\, J_{1/3}\left(\frac{(b - ax^2)^{3/2}}{3a}\right) \equiv \psi_+(x)/x^{1/2}, \tag{10}$$

$$\phi_-(x) = (b - ax^2)^{1/2}\, J_{-1/3}\left(\frac{(b - ax^2)^{3/2}}{3a}\right) \equiv \psi_-(x)/x^{1/2}, \tag{11}$$

where $J_\nu(x)$ is Bessel function of order ν. The general solution $\psi(x)$ is, of course, a linear combination of $\psi_+(x)$ and $\psi_-(x)$ satisfying appropriate boundary conditions. For the present case, the kinematics of the problem requires

$$\psi(x) \to 0 \quad \text{for} \quad x \to r_c$$

and, of course,

$$\psi(x) \rightarrow 0 \quad \text{for} \quad x \rightarrow \infty.$$

Further, requirements of single-valuedness, continuity of the function and its first derivative and square integrability lead to an interior solution $(-r_c \leqslant x \leqslant 0)$

$$\psi_{\text{interior}}(x) = C_1 x^{1/2} (b - ax^2)^{1/2} J_{1/3}\left(\frac{(b - ax^2)^{3/2}}{3a}\right) \tag{12}$$

and an exterior solution $(0 \leqslant x \leqslant \infty)$

$$\psi_{\text{exterior}}(x) = C_2 x^{1/2} (b - ax^2)^{1/2} H_{1/3}^{(2)}\left(\frac{(b - ax^2)^{3/2}}{3a}\right), \tag{13}$$

where $H_\nu^{(2)}(x)$ is Hankel function of the second kind of order ν. It is to be noted that the exterior solution is an appropriate combination of the two elementary solutions given by Eqs. (10) and (11). The ratio of the coefficients C_1 and C_2 is to be determined from the condition that the two solutions [cf. Eqs. (12) and (13)] must match at $x = 0$. Thus we have

$$\psi_{\text{interior}}(0) = \psi_{\text{exterior}}(0) \tag{14 a}$$

and

$$\left.\frac{d\psi_{\text{interior}}}{dx}\right|_{x=0} = \left.\frac{d\psi_{\text{exterior}}}{dx}\right|_{x=0}. \tag{14 b}$$

The condition (14 *a*) is automatically satisfied owing to the occurrence of the factor $x^{\frac{1}{2}}$ in both the solutions. The second condition [cf. Eq. (14 *b*)] gives

$$\frac{C_1}{C_2} = \frac{iH_{1/3}^{(2)}(b^{3/2}/3a)}{J_{1/3}(b^{3/2}/3a)}. \tag{15}$$

It should be emphasised that the two solutions can match with respect to the value and the first derivatives only at $x = 0$. For $X > (b/a)^{\frac{1}{2}}$, the solution $\psi_{\text{exterior}}(x)$ decays exponentially as the argument of the Hankel function of the second kind occurring in Eq. (13) becomes negaitve imaginary. The requirement of the boundary condition $\psi(x) \rightarrow 0$ for $x = -r_c$, gives us the quantisation condition

$$\frac{(b - ar_c^2)^{3/2}}{3a} = p_n, \tag{16}$$

where p_n is the n^{th} root of the Bessel function $J_{1/3}(z)$. Substituting for a and b from Eqs. (9), we get

$$\frac{m_0}{m_0} = \frac{E_n}{E_0} = \left[1 + \left(\frac{9}{2}\right)^{1/3} \left(\frac{\lambda\!\!\!^{-}}{r_c}\right)^{2/3} p_n^{\ 2/3}\right], \tag{17}$$

where

$$m_0 \equiv \frac{1}{2}\frac{kr_0^{\ 2}}{c^2} \tag{18}$$

and is to be identified with the lowest eigenmass and

$$\lambda\!\!\!^{-}_{\ 0} \equiv \hbar/m_0 c$$

is the corresponding Compton wavelength. The n^{th} excited-state mass is denoted by m_n corresponding to energy $E_n\,(= m_n c^2)$.

3. Mass Spectrum and Internal Structure

With the help of the mass formula given by Eq. (17), we are in a position to compute various eigenmasses in terms of the mass m_0 corresponding to the lowest (ground) state. A few words about the primary mass m_0 will be in order. If we examine the properties of elementary particles we find that only the electron (baryon number 0) and the proton (baryon number 1) have infinite lifetime in the free state. It appears, therefore, natural, in the framework of the present formulation, to identify the electron and the proton as representing the ground states of the two families of particles with baryon numbers 0 and 1 respectively. This may be treated as the ansatz for the present interpretation of the mass spectrum. Accordingly, for the two series of masses non-baryonic (e) and baryonic (p), m_0 is set equal to the electron mass and the proton mass respectively. The only parameter of the theory is the length r_c representing the spatial extensions of the primary object. It is expected that for the electron $r_c \sim 10^{-13}$ cm. and for the proton $r_c \sim 10^{-12}$ cm. For actual computation we have used just these values, namely, $r_c(e) = 10^{-13}$ cm. and $r_c(p) = 10^{-12}$ cm. We shall see later that the wavefunction for each is much more localised than the dimension noted above. This implies that the physical size of the particle is much smaller than what is suggested by the length r_c. A few eigenmasses for each series have been computed with the above choice of r_c and the mass formulae (17). These are labelled as $m_n(e)$ and $m_n(p)$ and are set out in Table I along with the probable assignments.

As can be seen from the table, for the non-baryonic (e) case the first excited state is close (within 15 per cent) to the muon mass. It is tempting to identify the second excited state (within 6 per cent) to the pion mass. However, we must bear in mind that the latter particles (pion, kaons, etc.) differ from leptons (electron and muon) in respect of other quantum numbers, *e.g.*, spin and isospin, etc. These are connected with other internal degrees of freedom which are outside the scope of the present work.

TABLE I

(*Mass in MeV*)

n \ m_n (e)	Calculated	Observed mass*, Probable assignment	n \ m_n (p)	Calculated	Observed mass*, Probable assignment
0	0·511	0·511, e^-	0	938·3	938·3, p
1	92·0	105·6, μ^-	1	1178	1189, Σ^+
2	148·7	139·6, π^- (?)	2	1329	1321 [illegible]$^-$
12	501·0	493·8, K^- (?)	3	1455	1470, N'^+
22	750.0	755, ρ^- (?)	4	1567	1530, [illegible]$^{*-}$
			5	1670	1672, Ω^-

* From Rosenfeld, A. H. *et al.*, *Rev. Modern Physics*, 1965, **37**, 633.

On the baryonic (p) side the first five eigenmasses seem to have the observed counterparts with a reasonably good quantitative agreement. For assignment we have displayed only one member of the isospin multiplet.

Let us now consider the relative mass-density $\rho(x)$ distribution associated with ground state of each family, *i.e.*, the electron and the proton. This is given by $\rho(x) = |\psi(x)|^2$ in the present case. A plot of $\rho(x)$ for the proton case shows that it has an internal structure (*see* Figs. 2 and 3) which can be likened to a shell structure with a dense core. The charge density will presumably follow the same distribution. The envelope (dotted portion in Figs. 2 and 3) is reminescent of the experimentally observed plot of the charge distribution (Hofstadter, 1957). It can be seen that most of the mass is confined in a region of linear dimension much smaller than r_c. For the case of the electron we do not find any detailed structure (*see* Fig. 1).

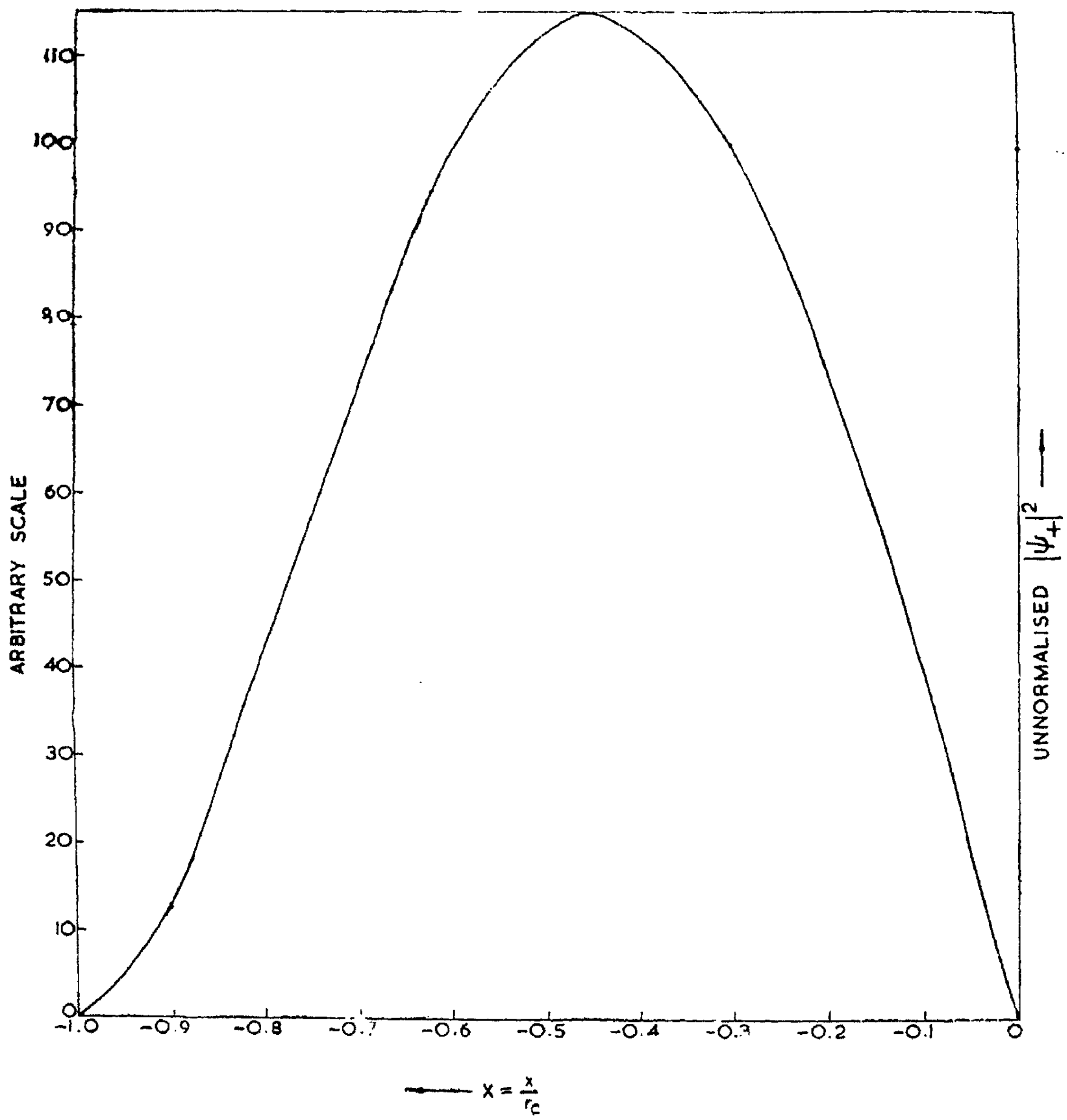

FIG. 1. A plot of the relative mass density distribution $|\psi_+|^2$ for the electron; $r_c = 10^{-13}$ cm.

4. CONCLUDING REMARKS

We shall now dwell on the physical basis of the model adopted in the present theory. The concept of a stressless and massless extensible object is justifiable within the framework of general relativity as applied to the problem of collapse of a spherically distributed charge matter. Physically, this can be understood in terms of a single principle, namely, that of the equivalence of energy (in all forms), the gravitational mass and the inertial mass. In the absence of charge, a spherical distribution of imploding matter of bare mass M (*i.e.*, mass M in the state of infinite dispersion) and of radius R has a gravitational self-energy

$$-\frac{3}{5}\frac{M^2G}{R}$$

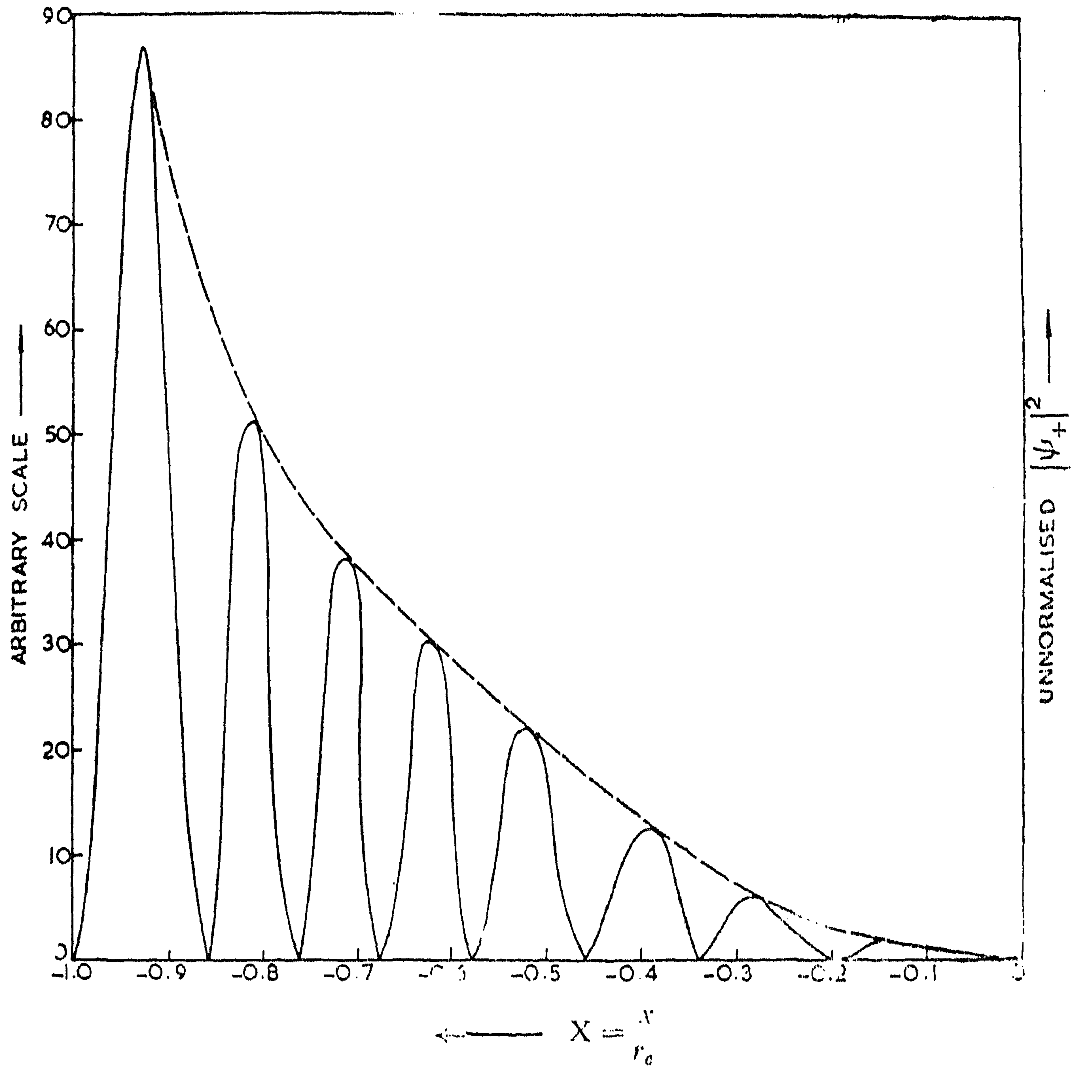

FIG. 2. A plot of the relative mass density distribution $|\psi_+|^2$ for the proton; $r_c = 10^{-12}$ cm.

in the Newtonian approximation. Thus, for

$$R = R_c = \frac{3}{5}G\frac{M}{c^2},$$

the effective energy-mass content tends to zero. This, in fact, is the physical significance of the Schwarzschild radius (R_c). Thus near this critical radius the object behaves as a massless body. If this collapse is arrested because of matter being charged, we can have an equilibrium situation where the distribution of charged matter is such that the system as a whole is essentially massless and stressless. Any radial deviation from the above situation will produce the usual elastic strain energy whose inertial mass equivalent has been used as such in equations (1) and (2) in the kinetic energy term.

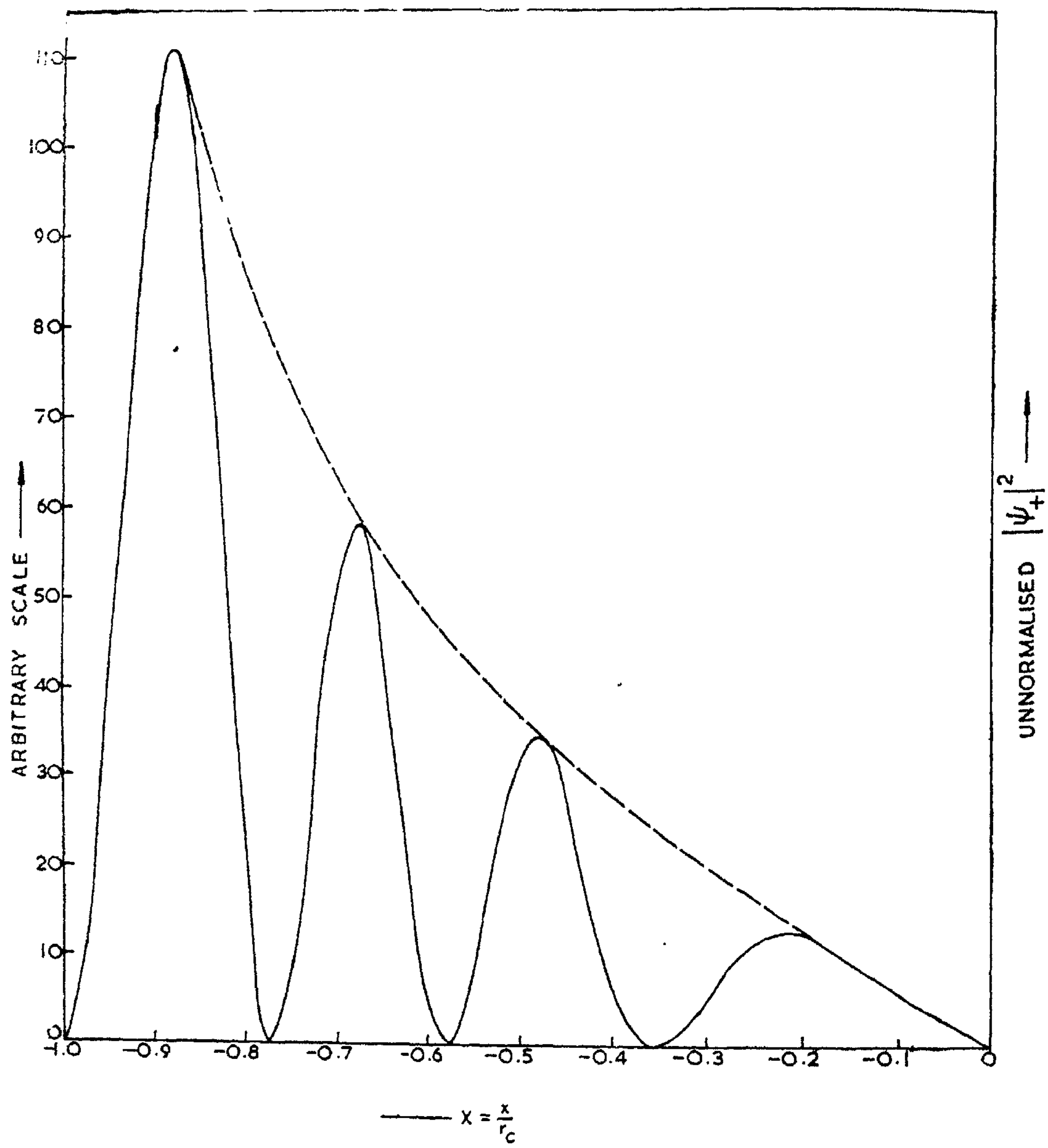

FIG. 3. A plot of the relative mass density distribution $|\psi_+|^2$ for the proton; $r_c = 5 \times 10^{-13}$ cm.

It may be noted that the assumption of purely radial mode of oscillation has effectively reduced the problem to a one-dimensional situation. The extension to non-radial oscillations is non-trivial in that the Podolsky-Pauli prescription cannot be directly applied. This is because the mathematical space described by g^{ij} (involving variable mass) will not be flat but only conformally flat.

In conclusion, it should be noted that we have suggested as plausible approach to the problem of mass quantisation. The treatment given here is, of course, far from being relativistically complete. However, the essence of relativity (equivalence principle) has been incorporated. The fact that the masses predicted are reasonably close to the observed values is suggestive, of a step in the right direction.

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