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ASTRONOMY AND ASTROPHYSICS

Modelling the variance of dispersion measures of radio pulsars

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Abstract. On the basis of a highly simplified model for small scale structure in the electron density distribution in the Galaxy, we argue that the spread of expectation values for the dispersion measure at given distance is proportional to the square root of the dispersion measure as found from a smooth model for the electron distribution. We show that this simple method leads to appreciable improvement in the description of dispersion measures in a full population synthesis.

Key words: pulsars: general – ISM: structure

1. Introduction

One method to study the evolution of the properties of radio pulsars is radio pulsar population synthesis. In such a synthesis, neutron stars are given initial properties, such as position, velocity, rotation period and magnetic field, and these properties are allowed to evolve according to given prescriptions. Observations of the resulting population are simulated, and the simulated detected pulsars are compared with the real detected pulsars.

A recent example of such a study is the synthesis by Hartman et al. (1997). Whereas the overall results of the synthesis compare well with observations, a detailed comparison of the simulated dispersion measures DM with the observed values for real pulsars shows systematic differences. It is suggested by Hartman et al. (1997) that this is due to small scale structure in the electron distribution in the Galaxy. In the model for this distribution by Taylor & Cordes (1993) several components are present (a thin layer, a thick layer, spiral arms, the Gum nebula), but each of these components is modelled with a smooth distribution. In this paper we investigate a highly simplified model for small scale variations in the electron density, in which all electrons are in uniform clouds. Based on this model, we propose a method to describe such fluctuations in population synthesis (Sect. 2). From the observed distribution of high values of $DM \sin b$ (where b is the galactic latitude) we derive the characteristic dispersion measure of one cloud (Sect. 3). We use this to implement the method in the population synthesis (Sect. 4). A discussion of our results is given in Sect. 5.

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2. A simple model

To study the effect of small scale structure in the electron distribution in the Galaxy, we investigate a very simple model, in which we compare a smooth, homogeneous distribution of constant electron density $n_{\rm e}$ with one in which all electrons are in spherical clouds with radius R_c. In the clouds, the electron density is enhanced with respect to the density in the homogeneous model by $1/f$, where f is the filling factor. Denoting the number density of the clouds with $N_{\rm c}$, we have

$$f = N_{\rm c}\frac{4\pi}{3}{R_{\rm c}}^3. \quad (1)$$

A line of sight that passes at distance $r_{\rm c} \leq R_{\rm c}$ from the center of one cloud has a dispersion measure $dm_1(r_{\rm c}) = (2n_{\rm e}/f)\sqrt{{R_{\rm c}}^2 - {r_{\rm c}}^2}$. The average dispersion measure for a collection of lines of sight passing through one cloud is given by

$$\langle dm_1\rangle = \int_0^{R_{\rm c}} \frac{2n_{\rm e}}{f}\sqrt{{R_{\rm c}}^2 - {r_{\rm c}}^2}\,\frac{2\pi r_{\rm c} dr_{\rm c}}{\pi {R_{\rm c}}^2} = \frac{4n_{\rm e}R_{\rm c}}{3f} \quad (2)$$

and the variance on this value is given by

$${\sigma_1}^2 = \int_0^{R_{\rm c}} \left(dm_1(r_{\rm c}) - \langle dm_1\rangle\right)^2 \frac{2\pi r_{\rm c} dr_{\rm c}}{\pi {R_{\rm c}}^2} = \frac{\langle dm_1\rangle^2}{8}. \quad (3)$$

We now divide the free path length L of photons travelling between the clouds in small subdivisions $\delta L > R_{\rm c}$, so that the probability that a subdivision δL encounters a cloud is given by $p = \delta L/L \ll 1$. The number of subdivisions required to reach a pulsar at distance d is given by $K = d/\delta L \gg 1$. Thus, the probability $P(k)$ of encountering k clouds on the way to a pulsar is given by a Poisson probability for K trials with individual probabilities p, and with $Kp = d/L$:

$$P(k) = \frac{(d/L)^k}{k!}e^{-d/L} \quad (4)$$

independent of δL, as it should be.

The distribution of the dispersion measures after encountering k clouds is given by an k-fold convolution of the distribution of the dispersion measures for an individual cloud. We now use the central limit theorem to state that the resulting distribution is a Gaussian $G_k(DM)$ with average $\langle dm_k \rangle$ and variance ${\sigma_k}^2$ given by

$$G_k(DM) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{(DM-\langle dm_k \rangle)^2/2{\sigma_k}^2}$$

$$\text{where} \quad \langle dm_k \rangle = k \langle dm_1 \rangle \quad \text{and} \quad {\sigma_k}^2 = k{\sigma_1}^2 \tag{5}$$

By comparing the exact distributions with the ones given by the central limit theorem for the first few convolutions, it can be verified that the approximate distribution already gives a fairly accurate description for values of k as small as 2.

The dispersion measure to a radio pulsar at distance d has a finite probability of being caused by passage through 0, 1, 2, 3, etc. clouds, and to calculate the average value $\langle DM \rangle$ and the variance $\sigma_{\rm DM}$ of its distribution we must combine all these possibilities. The maximum number of clouds that fit between the pulsar and Earth is given by $k_{\rm max} = d/R_{\rm c}$. It is shown in the appendix that for sufficiently large $k_{\rm max}$

$$\langle DM \rangle = DM_{\rm h} \quad \text{and} \quad {\sigma_{\rm DM}}^2 = DM_{\rm h} \frac{9}{8} \langle dm_1 \rangle \tag{6}$$

where $DM_{\rm h}$ is the dispersion measure from the homogeneous model.

The above results suggest the following method to include fluctuations in the electron distribution in the Galaxy in a pulsar population synthesis. A smooth model is used to compute a dispersion $DM_{\rm h}$ to the simulated pulsar at distance d; its dispersion measure DM is chosen from a Gaussian distribution with average and variance given by Eq. 6.

3. Confrontation with observations

Before going to the population synthesis, we confront our suggestion with observations, and in doing so determine the parameter $\langle dm_1 \rangle$. According to the smooth model by Taylor & Cordes (1993) for the electron distribution in the galaxy, all pulsars above or below the thick electron layer, i.e. with $|z| \gtrsim 1.75$ kpc, have a maximum DM given roughly by $DM_{\rm max} \sin b \simeq 16.5\,{\rm pc\,cm^{-3}}$. However in the sample of currently known radio pulsars we do not see a sharp cut–off in the $DM \sin b$ distribution. We can interpret this as a distribution of pulsars with $DM_{\rm h} \sin b = DM_{\rm max} \sin b$ but a real DM around $DM_{\rm h}$ as in our simple model. Some of these pulsars may in fact have an even smaller $DM_{\rm h}$, but we argue that this is only a relatively small fraction since they come from a thin layer whereas the pulsars with $DM_{\rm h} = DM_{\rm max}$ can come from all the volume above the layer. A test with the full population synthesis confirms this.

So we assume all the pulsars with $DM \sin b > DM_{\rm max} \sin b$ to have $DM_{\rm h} = DM_{\rm max}$. For each of these pulsars we can determine the difference between the real DM and $DM_{\rm h}$. According to our simple model we expect Δ, defined as

$$\Delta \equiv \frac{(DM - DM_{\rm h})}{\sqrt{DM_{\rm h}}} \tag{7}$$

to follow a Gaussian distribution with fixed width of $\sqrt{9/8\langle dm_1 \rangle}$ (see Eq. 6).

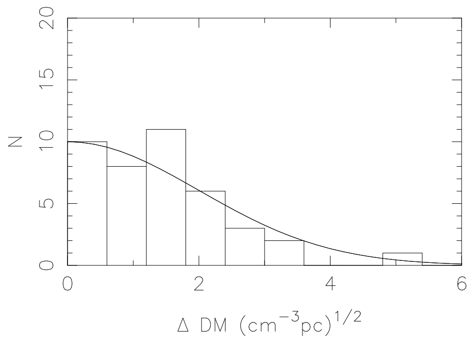

Fig. 1. Histogram of Δ (see Eq. 7) for pulsars above the electron layer. The solid line is a Gaussian with $\sigma_\Delta = 2.0\,({\rm pc\,cm^{-3}})^{1/2}$. Data from the Princeton Pulsar Catalogue, updated by Camilo on May 3, 1995 (see Taylor et al. 1993).

In Fig. 1 we plotted the histogram of Δ with (the right hand side of) a Gaussian with $\sigma_\Delta = 2.0\,({\rm pc\,cm^{-3}})^{1/2}$. From this figure we conclude that the data are consistent with our simple model and that $\langle dm_1 \rangle \simeq 3.6\,{\rm pc\,cm^{-3}}$.

4. Inclusion in population synthesis

In the population synthesis as computed by Hartman et al. (1997) the dispersion measure is assumed to be an exact measure of the distance and therefore the derived and actual distance of a radio pulsar are the same. We implement our simple model in this synthesis as follows. The synthesis gives the actual distance of a simulated radio pulsar, and from this distance a flux measured at Earth is derived. From the actual distance, we calculate the smooth dispersion measure $DM_{\rm h}$ according to the model by Taylor & Cordes (1993), and then randomly choose the simulated dispersion measure DM from a Gaussian distribution centered on $DM_{\rm h}$ and with width $\sigma_{\rm DM}$ given by Eq. 6, where we use $\langle dm_1 \rangle = 3.6\,{\rm pc\,cm^{-3}}$. The value of DM is also used to compute the scatter broadening of the pulse profile. From DM and the Taylor & Cordes model we find a derived distance, and a derived luminosity. These values are used for the pulsar in the remainder of the simulation, and in particular its derived distance is used to determine whether the pulsar is within the volume selected for the comparison with observation.

In Fig. 2 we compare the results relating to the simulated and observed distributions of the dispersion measures for the synthesis model B, with decay time $\tau = 100$ Myr, with and without inclusion of small scale structure in the electron distribution. It is seen that our simple model leads to a significantly better description of the distributions of the dispersion measure DM,

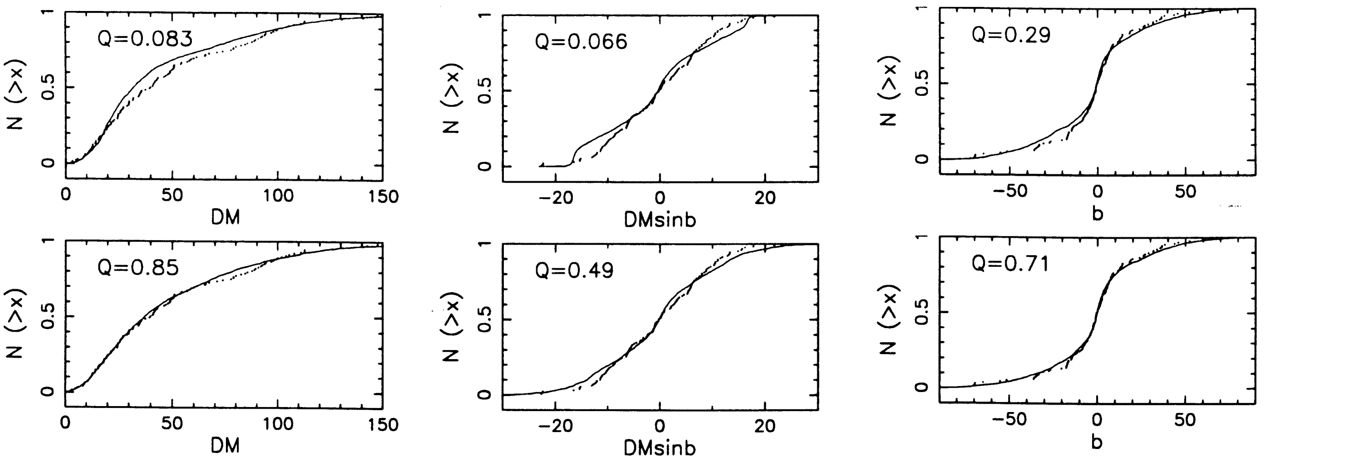

Fig. 2. Comparison between cumulative distributions of the dispersion measure DM, galactic latitude b and the product DM sin b of real pulsars (dots) and 2000 simulated pulsars (solid line) for the population synthesis according to model B from Hartman et al. (1997), without (upper row) and with (bottom row) the model for the variance in the dispersion measure. The Kolmogorov-Smirnov probabilities Q that the real and simulated distributions are drawn from the same population are indicated in the frames. (The values of Q vary somewhat between runs with different random number initializations; the improvement shown in this figure is more dramatic than for most other initializations.)

of the vertical component of the dispersion measure DM sin b, and of the galactic latitude distribution b.

5. Discussion

5.1. Distance distribution

The inclusion of a spread in the DM in the population synthesis has two effects on the results of the simulation. The first is that the DM distribution of the simulated population changes (see Fig. 2). The second effect is a change of the sample of simulated pulsars that is retained for comparison with the real pulsars, because these pulsars are selected on the basis of the derived distance instead of the actual distance. In the new simulation both real and simulated pulsars with $DM > DM_{\rm max}$ are placed at a derived $|z| = 1.75$ kpc. Pulsars with an actual distance projected on the Galactic Plane $d0_{\rm proj} > 4$ kpc thus can have a projected derived distance < 4 kpc (see Fig. 3). In fact, at $|z| > 2$ kpc, almost half of the pulsars in the simulated comparison sample has $d0_{\rm proj} > 4$ kpc. More importantly, the derived luminosity is based on the derived distance, and is lower than the real luminosity for pulsars above the electron layer. Thus, the luminosity distribution derived from the fluxes in the simulation shifts towards lower values; to compensate for this, a higher intrinsic luminosity distribution of the pulsars is required. (In terms of Eq. 3 of Hartman et al. (1997) for the luminosity distribution, the best value of a changes from 1.5 in their model B to 0.9 in our model.)

5.2. Cloud size

Because some of the parameters we use can be derived independently, we can determine the actual cloud size given by our model. From EM and DM measurements Reynolds (1991) derived a filling factor $f \simeq 0.2$ (see also Anantharamaiah & Bhattacharya 1986). For an average electron density $0.025\,{\rm cm}^{-3}$ (e.g. Weisberg et al. 1979) together with the obtained value of $\langle dm_1 \rangle$ and Eq. 2 we find

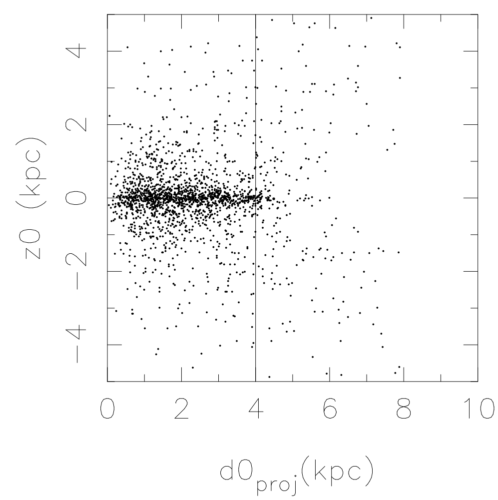

Fig. 3. Actual distances projected on the Galactic Plane $d0_{\rm proj}$, as function of actual distance to the Galactic Plane $z0$ of the pulsars in the sample obtained with Model B of Hartman et al. (1997) with the variations in the DM. The high z pulsars cover a large fraction of the pulsars with large actual distances

$$R_{\rm c} = \frac{3}{4}\frac{f}{n_{\rm e}}\langle dm_1 \rangle \simeq 21\ {\rm pc} \tag{8}$$

Remarkably, this is similar to the sizes of clouds containing both neutral and ionized hydrogen that have been found by Reynolds et al. (1995).

From $R_{\rm c}$ we can check the assumption made in the appendix, that we can replace $k_{\rm max}$ with infinity in the summation over the Poisson probabilities. This is strictly only possible if $k_{\rm max} \gg$

d/L, i.e. $L \gg R_c$. From the filling factor and the equation for L (see Appendix) we find $L/R_c = 4/(3f) \simeq 6.7$. However since the Poisson distribution drops off rapidly, the error in replacing k_{max} with ∞ in the summations in the Appendix is smaller than 1 %.

For small distances, and therefore small DM, we make an error in applying this model since the individual inhomogeneities become important. However, the sizes of the clouds are relatively small compared to the scales involved in the simulation ($\sim$ kpc), and the number of pulsars in our simulations at distances less than the free path length L is negligible.

5.3. DM variations in other pulsar simulations

Lorimer et al (1993) model the spread in the dispersion measures expected at a given distance, by assuming that the logarithm of the ratio DM/DM_h has a Gaussian distribution with width log 2. (In a model of constant electron density this is identical to the assumption by Gunn & Ostriker (1970) that the logarithm of the ratio of real to derived distance of radio pulsars has a Gaussian distribution.) In the description by Lorimer et al. (1993) the spread in the DM is roughly proportional to DM itself. In principle, the relation between DM and σ_{DM} can be derived from the deviation of the directly measured distances (i.e. by HI absorption, association with an object of known distance, or parallax) from the distances derived from the dispersion measure, but in practice the number of accurate distance measurements is too small.

Because of the wide applicability of the central limit theorem, the simple model discussed in Sect. 2 suggests that $\sigma_{DM} \propto \sqrt{DM}$ for a wide variety of models for small scale structure in the electron density distribution. Our simulations show that such a variance adequately describes the currently available observations.

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Appendix A

For a line-of-sight passing through k clouds, the distribution of dispersion measures is given by a Gaussian $G_k(DM)$ (Eq. 5). The average value of the distribution of dispersion measures to a distance d is found by averaging the distributions for all k-values, weighted with the Poisson probability of getting k clouds:

$$\begin{aligned}
\langle DM\rangle &= \int_0^\infty \sum_{k=0}^{\infty} P(k) G_k(DM) DM dDM \\
&= \sum_{k=0}^{\infty} P(k) \int_{-\infty}^{\infty} G_k(DM) DM dDM \\
&= \sum_{k=0}^{\infty} P(k) k \langle dm_1\rangle = \frac{d}{L}\langle dm_1\rangle \qquad \text{(A1)}
\end{aligned}$$

where we use the fact that the integration over the dispersion measure is independent of the summation over k and we can extend the integration to $-\infty$ because the mean is sufficiently displaced from 0. We now substitute $1/L = N_c \pi R_c^2$ and $\langle dm_1\rangle$ from Eq. 2 to find

$$\langle DM\rangle = n_e d = \int_0^d n_e ds \equiv DM_h \qquad \text{(A2)}$$

In a similar fashion we may estimate the variance of the dispersion measure distribution for distance d:

$$\begin{aligned}
{\sigma_{DM}}^2 &= \int_0^\infty \sum_{k=0}^{\infty} P(k) G_k(DM) \left(DM - \langle DM\rangle\right)^2 dDM \\
&= \sum_{k=0}^{\infty} P(k) \int_{-\infty}^{\infty} G_k(DM) DM^2 dDM - \langle DM\rangle^2 \\
&= \sum_{k=0}^{\infty} P(k) \left({\sigma_k}^2 + \langle dm_k\rangle^2\right) - \langle DM\rangle^2 \\
&= \sum_{k=0}^{\infty} P(k) \left(k{\sigma_1}^2 + k^2\langle dm_1\rangle^2\right) - \langle DM\rangle^2 \\
&= \frac{9d}{8L}\langle dm_1\rangle^2 = \frac{9}{8}\langle DM\rangle\langle dm_1\rangle \qquad \text{(A3)}
\end{aligned}$$

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