Stationary Solitons of the Fifth Order KdV-type Equations and Their Stabilization

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Abstract

Exact stationary soliton solutions of the fifth order KdV type equation

$$u_t + \alpha u^p u_x + \beta u_{3x} + \gamma u_{5x} = 0$$

are obtained for any p (> 0) in case $\alpha\beta > 0$, $D\beta > 0$, $\beta\gamma < 0$ (where D is the soliton velocity), and it is shown that these solutions are unstable with respect to small perturbations in case $p \geq 5$. Various properties of these solutions are discussed. In particular, it is shown that for any p, these solitons are lower and narrower than the corresponding

$\gamma = 0$ solitons. Finally, for p = 2 we obtain an exact stationary soliton solution even when D, α, β, γ are all > 0 and discuss its various properties.

In recent years the soliton solutions of the fifth order KdV-type highly nonlinear equations

$$\frac{\partial u}{\partial t} + \alpha u^p \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + \gamma \frac{\partial^5 u}{\partial x^5} = 0 \tag{1}$$

(with p $>$ 0) have received considerable attention in the literature [1]. In particular, attention has been focused on the role of the last term in this equation which describes higher order dispersive effects and may have important influence on the properties of the solitons. The equation with p = 1 is the fifth order KdV eqution [2] which has applications in fluid mechanics (e.g., shallow water waves with surface tension), plasma physics etc. On the other hand, for p = 2 one has fifth order MKdV equation which may also be of interest in fluid mechanics, plasma physics. Finally, above equation with $p \geq 4$ has considerable theoretical interest in connection with the general problem of collapse of nonlinear waves. Indeed, it is well known that the stationary soliton solutions of the eq.(1) for $\gamma = 0$ and $\alpha > 0, \beta > 0$ as given by

$$u_0(\xi) = \left(\frac{D}{2}(p+1)(p+2)\right)^{1/p} \mathrm{sech}^{2/p}(\frac{p}{2}\sqrt{D}\xi) \tag{2}$$

are unstable with respect to collapse-type instabilities if $p \geq 4$ [3]. Here $\xi = x - Dt$ with the velocity $D > 0$ and without any loss of generality we have chosen $\alpha = \beta = 1$ throughout this paper unless mentioned otherwise. The role of the last term in eq.(1) has been discussed by several people including Karpman [1]. Based on analytical and numerical work, it has been suggested that the fifth order term stabilizes the soliton specially for $p \leq 6$ [4]. It has also been conjectured that for large enough $\mid \gamma \mid$ even soliton solutions with $p > 6$ could also be stable. Unfortunately, no analytical soliton solution is known in literature when $\gamma \neq 0$ except when p = 1 [5] and hence so far it has not been possible to check the validity of these conjectures.

Recently Hai and Xiao [6] have obtained a soliton solution of eq.(1) with p = 1 which is valid to first order in $\mid \gamma \mid$ and have shown that the corresponding soliton is lower and narrower then the unperturbed ($\gamma = 0$) soliton as given by eq.(2). Is their conclusion also valid for large $\mid \gamma \mid$? Further, is it also true for any p?

It is thus clearly of great interest to obtain an exact analytical stationary soliton solution of eq.(1) and test the validity of these conjectures. The purpose of this note is to show that an exact solution of eq.(1) is

$$u(\xi) = \Big(\frac{D}{8}\frac{(p+1)(p+4)(3p+4)}{(p+2)}\Big)^{1/p} \mathrm{sech}^{4/p}\Big(\frac{p\xi\sqrt{D(p^2+4p+8)}}{4(p+2)}\Big) \quad (3)$$

where $\xi = x - Dt$, $\alpha = \beta = 1, D > 0,\ \gamma < 0$ and

$$\epsilon \equiv (D \mid \gamma \mid)^{1/2} = \frac{2(p+2)}{(p^2+4p+8)} < 1 \quad (4)$$

Several properties of these solutions are discussed and it is shown that contrary to the expectation based on the numerical studies [4], these soliton solutions continue to be unstable in case $p \geq 5$. We also show that for any p, the soliton as given by eq.(3) is indeed lower and narrower than the unperturbed soliton as given by (2) no matter what the value of $\mid \gamma \mid$ is.

Finally we also present an another stationary soliton solution of eq.(1) in case p = 2 for which $\alpha, \beta, \gamma, D > 0$. This is interesting because recently there have been suggestions in the literature that a stationary soliton solution to eq.(1) may not exist in case $\alpha, \beta, \gamma, D > 0$ [7].

Let us consider eq.(1) and look for stationary soliton solutions of the form u = u (ξ) where $\xi = x - Dt$ with the boundary condition that $u \to 0$ as $\xi \to \pm\infty$. On integrating eq.(1) with respect to ξ and choosing the constant of integration to be zero we have

$$-Du + \frac{u^{p+1}}{(p+1)} + \frac{d^2u}{d\xi^2} + \gamma\frac{d^4u}{d\xi^4} = 0 \quad (5)$$

We now look for a solution of the form

$$u(\xi) = A\mathrm{sech}^b(m\xi) \quad (6)$$

On using eq.(6) in eq.(5) it is easily shown that $b = 4/p, D > 0,\ \gamma < 0$ and as given by eq.(4) and

$$A = \Big(\frac{D(p+1)(p+4)(3p+4)}{8(p+2)}\Big)^{1/p},\ m = \frac{p\sqrt{D(p^2+4p+8)}}{4(p+2)} \quad (7)$$

so that the solution is as given by eq.(3). Some of the interesting properties of this solution are

(i) the solution is fairly localized and is truly a nonperturbative solution in the sense that as $\mid \gamma \mid \longrightarrow 0$ the solution diverges rather than tending to the third order KdV soliton solution as given by eq.(2).

(ii) for this solution the wave travels only to the left ($D > 0$) with amplitude $\propto (velocity)^{1/p}$ and hence taller the wave, faster it moves! Also notice from eq.(4) that the velocity $\propto 1/\mid \gamma \mid$ and hence smaller the $\mid \gamma \mid$, larger is the velocity and vice a versa.

(iii) From the solution (3) we observe that

$$\int_{-\infty}^{\infty} u^{p/2}(x)dx = \frac{2\sqrt{2}}{p}\Big(\frac{(p+1)(p+2)(p+4)(3p+4)}{(p^2+4p+8)}\Big)^{1/2} = constant. \quad (8)$$

which is independent of D and hence $\mid \gamma \mid$. Thus so far as the dependence on $\mid \gamma \mid$ (or D) is concerned, one can say that $u(x) \sim [\delta(x)]^{2/p}$.

(iv) Note that solution (3) with $\xi = x - Dt$ replaced by $x - Dt + x_0$ with x_0 being an arbitrary constant is also a solution to eq.(1).

(v) On comparing the two soliton solutions as given by eqs.(2) and (3) we find that irrespective of the value of $\mid \gamma \mid$ (and hence D), the $\mid \gamma \mid \neq 0$ soliton is lower than the $\gamma = 0$ soliton. For example, the amplitude difference of the two is given by

$$(amp)_{\gamma=0} - (amp)_{|\gamma|} = \Big(\frac{D}{2}(p+1)(p+2)\Big)^{1/p}\Big(1 - (\frac{(p+4)(3p+4)}{4(p+2)^2})^{1/p}\Big) > 0 \quad (9)$$

which is equal to $\frac{D}{12}$ for p = 1 and increases with p. Similarly, on comparing the two solutions (2) and (3) it is easily seen that the $\gamma \neq 0$ soliton is narrower than the $\gamma = 0$ soliton i.e. $u < u_0$ for any $\xi \approx 0$.

(vi) As remarked by Karpman [1], eq.(1) is a Hamiltonian system for which the energy and momentum are given by

$$E_s = \int_{-\infty}^{\infty} dx\Big(\frac{1}{2}(\frac{du}{dx})^2 - \frac{u^{p+2}}{(p+1)(p+2)} - \frac{\gamma}{2}(\frac{d^2u}{dx^2})^2\Big) \quad (10)$$

$$P_s = \frac{1}{2}\int_{-\infty}^{\infty} dx u^2 \quad (11)$$

For the solution (3) we find that

$$E_s = \frac{DA^2 2^{8/p}\Gamma^2(4/p)}{4m\Gamma(8/p)(p+2)^2(p+4)}\Big((p+2)^2(p-4) - \frac{8p^3(p+3)}{(p+8)(3p+8)}\Big) \quad (12)$$

$$P_s = \frac{A^2 2^{8/p} \Gamma^2(4/p)}{4m\Gamma(8/p)} \tag{13}$$

where A and m are as given by eq.(7). Thus we find that $E_s \propto D^{(\frac{2}{p}+\frac{1}{2})}$ while $P_s \propto D^{(\frac{2}{p}-\frac{1}{2})}$ and further $E_s < 0$ for $p \leq 4$ while $E_s > 0$ for $p \geq 5$ while $P_s > 0$ for any $p (> 0)$.

Let us now address the question of the stability of the soliton solution (3) with respect to small perturbation. Based on a conjecture which is supported by some numerical results, Karpman obtained the following sufficient condition for the soliton stability in case $\gamma < 0$ [1]

$$(\frac{\partial P_s}{\partial D})_\gamma > 0 \tag{14}$$

Based on some numerical work, Karpman then conjectured [1] the stability of the $\gamma < 0$ soliton solutions for at least $p \leq 6$ in case $0 < \epsilon < \epsilon_p$ depending on p. Does our solution support Karpman's conjecture ? On using the fact that for our solutions $P_s \propto D^{(\frac{2}{p}-\frac{1}{2})}$ it follows that

$$(\frac{\partial P_s}{\partial D})_\gamma = \frac{1}{D}(\frac{2}{p} - \frac{1}{2})P_s \tag{15}$$

so that $(\frac{\partial P_s}{\partial D})_\gamma > 0$ if and only if $p < 4$.

Karpman has also given a simple and useful necessary condition for the soliton stability given by [1]

$$R \equiv \frac{\mid \gamma \mid J_2}{2DP_s} > \frac{p(p-4)}{(p^2+4p+32)} \equiv R_{cr}(p) \tag{16}$$

where

$$J_2 = \int_{-\infty}^{\infty} dx (\frac{\partial^2 u}{\partial x^2})^2 \tag{17}$$

For the solution (3), we find that

$$J_2 = \frac{128 A^2 m^3 (p+3) 2^{8/p} \Gamma^2(4/p)}{p^2(3p+8)(p+8)\Gamma(8/p)} \tag{18}$$

where A and m are as given by eq.(7). Using eqs.(13) and (4) we then find that the fifth order stationary soliton solution is stable provided

$$\frac{4p^2(p+3)}{(p+2)^2(p+8)(3p+8)} > \frac{p(p-4)}{(p^2+4p+32)} \tag{19}$$

i.e if $3p^5 + 28p^4 - 608p^2 - 1664p - 1024 < 0$. We find from here that the $\gamma \neq 0$ soliton is stable so long as $p \leq 4.75$. Since $\gamma = 0$ soliton was only stable for $p \geq 4$ hence it is clear that the fifth order term has increased the stability range but not by as much as it had been conjectured. In particular, for $p \geq 5$ the soliton solution (3) is still unstable under small perturbations.

Finally, for the special case of p = 2, we display another stationary soliton solution. In particular, on using the ansatz u = A $\mathrm{sech}(m\xi)\tanh(m\xi)$ in eq.(1) it is easily shown that an exact stationary soliton solution is

$$u(\xi) = (\frac{360D}{11})^{1/2}\mathrm{sech}(\sqrt{\frac{100}{11}}\xi)\tanh(\sqrt{\frac{100}{11}}\xi) \tag{20}$$

provided $\gamma, D > 0$, ($\alpha = \beta = 1$ as usual) and $\xi \equiv (D\gamma)^{1/2} = \sqrt{\frac{11}{10}}$. Note that for the p = 2 soliton of the type (3), $\epsilon = 2/5$. The solution (20) is again a truly localized nonperturbative solution for which amplitude $\propto (velocity)^{1/2}$. Note that u not only vanishes as $\xi \to \pm\infty$ but also at $\xi = 0$ and further $u(\xi)$ is negative for $\xi < 0$. For this solution $P_s = 12\sqrt{\frac{100}{11}}$ while $E_s = -\frac{60}{77}\sqrt{\frac{10D^3}{11}}$. Further, it is easily seen that this soliton is also lower and narrower than the corresponding $\gamma = 0$ (and p = 2) soliton as given by eq.(2).

Before ending this note we would like to exhibit a stationary kink solution to eq.(1) in case p = 4, $D, \alpha < 0, \beta, \gamma > 0$ (or $D, \alpha > 0$ and $\beta, \gamma < 0$) and $\epsilon \equiv (\mid D \mid \gamma)^{1/2} = \sqrt{6}/10$. The solution is

$$u(\xi) = (\mid D \mid)^{1/4}\tanh(\sqrt{\frac{5 \mid D \mid}{6}}\xi) \tag{21}$$

where without any loss of generality, we have chosen $\alpha = -1, \beta = 1$.

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