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HOW GOOD IS THE SUPERSYMMETRY-INSPIRED WKB QUANTIZATION CONDITION?

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ABSTRACT

I show that unlike the standard lowest order WKB, the supersymmetry-inspired WKB quantization condition gives correct n-dependence for all the energy eigenvalues (except ground state) of the potential $V(x) = A^2x^6 + 3Ax^2$. I also display a number of other potentials for which SWKB is then shown to give exact eigenvalues. Finally, I conjecture that for the class of models given by $V(x) = A^2x^{4d+2} + (2d+1)Ax^{2d}$ $(d = 0,1,2,\ldots)$, the exact energy eigenvalues obey the SWKB-predicted energy dependence $E_n \propto (n+1)^{2d+1/d+1}$.

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In a recent paper Comtet et al.[1] have applied semi-classical methods to supersymmetric (SUSY) quantum mechanics and have proposed a modified semi-classical quantization condition (SWKB). In particular, they have argued that for quantum mechanical models which can be written in the form

$$H_- = -\hbar^2 \frac{d^2}{dx^2} + \phi^2(x) - \hbar\phi'(x) \tag{1}$$

one can formally treat ϕ^2 term as $O(\hbar^0)$ while $\hbar\phi'$ term is explicitly an $O(\hbar)$ effect. In this way, for those models for which SUSY is unbroken [i.e., ground state energy $E_0^{(-)}=0$ and ground state wave function $\psi_0^{(-)}(x)$ is normalizable] they were led to the SWKB quantization condition

$$\int_a^b \sqrt{E - \phi^2(x)}\, dx = n\pi\hbar \,, \quad n = 0, 1, 2, \cdots \tag{2}$$

Here a and b are turning points defined by $\phi^2(a) = E = \phi^2(b)$. The attractive point of the SWKB is that it is not only exact for large-n eigenvalues, but by construction it is also exact for $E_0^{(-)}$ and $\psi_0^{(-)}(x)$. Using this modified condition, Comtet et al.[1] have shown that SWKB gives exact eigenvalues for one- and three-dimensional harmonic oscillator*), Morse, Coulomb*) and Rosen-Morse potentials. Note that the standard lowest-order WKB prescription does not give exact answers for Rosen-Morse or multidimensional Coulomb and oscillator potentials unless it is supplemented by Langer-like correction which is in general different for different potentials[2].

Encouraged by these successes, Comtet et al.[1] have then raised the question if SWKB is exact for only solvable models or if it is more generally a property of potentials for which energy eigenvalues can be expressed in terms of functions of simple powers of n.

The purpose of this note is to explore this question in some detail. In particular, using SWKB, I study a class of models characterized by

$$V(x) = A^2 x^{4d+2} + (2d+1)\hbar A x^{2d} \;;\; A>0, d=0,1,2\cdots \tag{3}$$

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*) Actually it is also exact in arbitrary number of dimensions.

and show that for d = 0 and d → ∞ the energy spectrum is almost exactly reproduced. Further, for the case of d = 1, [i.e., $v(x) = A^2x^6 + 3A\hbar x^2$], I show that even though SWKB does not reproduce the correct eigenvalues, it nevertheless correctly predicts their n-dependence (except for the ground state). Finally, I apply SWKB to the potentials

$$\text{(i)}\ V(r) = -\frac{\lambda e^{-\alpha r}}{(1-e^{-\alpha r})} + \frac{b e^{-\alpha r}}{(1-e^{-\alpha r})^2} \tag{4a}$$

$$\text{(ii)}\ V(x) = -V_0 \operatorname{sech}^2 \alpha x + \beta \tanh \alpha x + \beta \tag{4b}$$

$$\text{(iii)}\ V(x) = V_1 \operatorname{cosec}^2 \alpha x + V_2 \sec^2 \alpha x\ ;\ 0 < \alpha x < \frac{\pi}{2}\ ,\quad (\psi(0) = \psi(\pi/2\alpha) = 0) \tag{4c}$$

and show that SWKB gives exact spectrum in all three cases.

Consider the class of Hamiltonians

$$H_- = -\hbar^2 \frac{d^2}{dx^2} + A^2 x^{4d+2} - \hbar A(2d+1)x^{2d}\ ;\ A > 0 \tag{5}$$

all of which have double-well structure with two degenerate minima and hence an instanton. Nevertheless, for all these models SUSY is clearly unbroken. In fact, it is easy to see that all of them have $E_0^{(-)} = 0$ and the ground state wave function

$$\psi_0^{(-)}(x) = N \exp\left[-\frac{A}{2(d+1)}\, x^{2d+2}\right] \tag{6}$$

is clearly normalizable. On applying the SWKB quantization condition given by Eq. (2) to this class of models it follows that [note $\phi(x) = Ax^{2d+1}$]

$$E_n^{(-)} = A^{\left(\frac{1}{1+d}\right)} \left[\frac{n\hbar\sqrt{\pi}\,\Gamma\left(\frac{3d+2}{2d+1}\right)}{\Gamma\left(\frac{4d+3}{4d+2}\right)}\right]^{\frac{2d+1}{d+1}}\ ;\quad n = 0, 1, \cdots \tag{7}$$

Since SUSY relates the spectra of H_- and H_+ we therefore conclude that the energy eigenvalues of the Hamiltonian

$$H_+ = -\hbar^2 \frac{d^2}{dx^2} + A^2 x^{4d+2} + \hbar A(2d+1)x^{2d} \tag{8}$$

are given by

$$E_n^{(+)} = E_{n+1}^{(-)} = A^{\left(\frac{1}{1+d}\right)} \left[\frac{(n+1)\hbar\sqrt{\pi}\,\Gamma\left(\frac{3d+2}{2d+1}\right)}{\Gamma\left(\frac{4d+3}{4d+2}\right)} \right]^{\frac{2d+1}{d+1}} \tag{9}$$

in the SWKB approximation.

Several comments are in order at this stage

(a) As expected, in the limit $d = 0$, when H_+ is essentially the harmonic oscillator Hamiltonian, $E_n^{(+)}$ is <u>exact.</u>

(b) What is however remarkable is the fact that even for $d \to \infty$ when H_+ is that of a square well potential, $E_n^{(+)}$ is almost exact. We obtain

$$E_n^{(+)} \xrightarrow[d \to \infty]{} \frac{1}{4}(n+1)^2 \hbar^2 \pi^2 \tag{10}$$

which has the correct n-dependence for all values of n. This has to be contrasted with the conventional WKB which does not reproduce the correct n-dependence unless n is very large.

(c) The SWKB eigenvalues $E_n^{(\pm)}$ also satisfy the scaling relation satisfied by the exact eigenvalues for $H_\pm$, i.e.,

$$E_n^{(\pm)}((2d+1)\hbar A, A^2) = [(2d+1)\hbar A]^{\frac{1}{d+1}} E_n^{(\pm)}\left(1, \frac{1}{(2d+1)^2\hbar^2}\right) \tag{11}$$

In view of the fact that $E_n^{(+)}$ is exact for $d = 0$ and almost exact for $d \to \infty$

it is natural to enquire if it also gives exact eigenvalues for all d. Unfortunately none of these models ($0 < d < \infty$) are exactly solvable. However, the model characterized by

$$H = -\hbar^2 \frac{d^2}{dx^2} + x^2 + \lambda x^6 \tag{12}$$

has been extensively studied in the literature[3] and very accurate energy eigenvalues have been numerically calculated by Banerjee[3] for λ varying from 0.00001 to 40000 and for n = 0,1,..., 10, 100 and 1000. On using Eqs. (8), (11) and (12) it follows that at $\lambda = 1/9$ the Hamiltonian given by Eq. (12) and H_+ for d = 1, $A = 1/3(\hbar = 1)$ coincide and hence have the same eigenvalues. In Table 1, I have therefore compared the SWKB eigenvalues (for $d = \hbar = 1$, $A = 1/3$) as given by Eq. (9) with the exact ones of Banerjee[3] (for $\lambda = 1/9 \simeq 0.1$). From Table 1 it is clear that SWKB does not reproduce the correct eigenvalues. However, when I calculated the errors in various eigenvalues, I was surprised to find that for <u>all</u> the excited levels the SWKB value was always lower by (9±0.5)% (for the ground state it was higher by 5.5%). In other words, except for n = 0, the n-dependence of the energy eigenvalues as predicted by SWKB ($E_n \alpha (n+1)^{3/2}$) must be exact. In Table 2, I have compared the exact and SWKB expressions for E_{n+1}/E_n when n = 1, 2, ..., 9 and also E_{10}/E_1, E_{100}/E_1 and E_{1000}/E_1 and quite remarkably they agree to within 0.6%.

It may be noted here that the conventional lowest order WKB fails to predict E_{n_2}/E_{n_1} that accurately (specially for low values of n_1 and n_2). One may legitimately argue that since λ is as small as 1/9, the conventional WKB expression is in any case not expected to work. I have also compared the modified predictions of Hioe et al.[3] which are valid for any λ with the exact ones and I find that even the modified WKB fails to predict E_{n_2}/E_{n_1} accurately (specially when $n_2 >> n_1$).

One might wonder if the n-dependence as predicted by SWKB is merely valid for the SUSY value of 1/9 or is true for any value of λ for the Hamiltonian (12). Using Banerjee's estimates[3], I find that as one considers λ values which are lesser (greater) than $\lambda = 1/9$, E_{n_2}/E_{n_1} is lesser (more) than the corresponding SWKB prediction (the errors are as large as 50% when $n_2 >> n_1$). I find this rather remarkable[*)]. I am, however, puzzled by the fact that the $E_0^{(+)}$ does not obey the SWKB predicted n-dependence.

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*)This is somewhat analogous to the divergence behaviour in field theories. Only when $\lambda = \lambda_c$ (when the theory is supersymmetric) do the divergences cancel while if $\lambda > (<) \lambda_c$, then the bosonic divergence term is larger (smaller) than the corresponding fermionic one.

In view of the success of SWKB in predicting the correct n-dependence of the energy eigenvalues for d = 0,1, and ∞, I conjecture that the exact energy eigenvalues for $H_{\pm}$ as given by Eqs. (5) and (8) indeed have the SWKB predicted n-dependence (note that $E_0^{(-)}$ also satisfies this dependence)

$$E_{n+1}^{(-)} = E_n^{(+)} \propto (n+1)^{\frac{2d+1}{d+1}} \; ; \; n = 1, 2, \cdots\cdots \tag{13}$$

Before finishing this discussion about $E_n^{(\pm)}$ let me point out some of their possible applications. If one considers model field theories (in 1+1 dimensions) characterized by

$$V(\phi) = A^2 \phi^{4d+2} - (2d+1) A \phi^{2d} \tag{14}$$

then all of them have degenerate absolute minima and hence kink solution. Now it is well known[4)] that in the thermodynamic limit the classical partition function of such theories (in 1+1) is essentially given by the ground state energy of the Schrödinger-like equation for the above potential which we have calculated here. Further, such potentials in (0+1) dimension will have an instanton solution. Hence one could compute the instanton contribution to $E_1^{(-)} - E_0^{(-)}$ splitting and compare it with the exact value and study the validity of instanton approximation.

Finally, let me apply the SWKB to the three potentials as given by Eqs. (4a), (4b) and (4c) which are all exactly solvable models. Consider first the potential (4a). We treat it as a radial problem with r > 0 and calculate s-wave energy eigenvalues using SWKB. On choosing

$$\phi(r) = -A \coth\left(\frac{\alpha r}{2}\right) + \frac{\lambda}{4A} \tag{15}$$

where

$$A = \frac{\hbar\alpha}{4}\left[1 + \sqrt{1 + \frac{4b}{\hbar^2\alpha^2}}\right] \tag{16}$$

and using it in the SWKB condition as given by Eq. (2), we find that

$$E_n = -\frac{\{[(n+\frac{1}{2})+\frac{1}{2}\sqrt{1+4b/\hbar^2\alpha^2}]^2\hbar^2\alpha^2-\lambda\}^2}{4\hbar^2\alpha^2[(n+\frac{1}{2})+\frac{1}{2}\sqrt{1+4b/\hbar^2\alpha^2}]^2} \tag{17}$$

which is the exact (s-wave) energy expression[5]. Note that the conventional lowest-order WKB reproduces this expression only when a Langer-like correction term $\alpha^2/16$ cosech^2 $\alpha r/2$ is added to the potential (4a)[2].

Let me now consider the potential (4b) which occurs during the stability analysis of $\phi^2(\phi^2-a^2)^2$ kink solution and which has been extensively discussed in Ref. 6). On using

$$\phi(x) = A\tan\alpha x + \frac{\beta}{2A} \tag{18}$$

where

$$A = \frac{\hbar\alpha}{2}\left[\sqrt{1+4V_0/\hbar^2\alpha^2} - 1\right] \tag{19}$$

and applying the SWKB formula, I find that

$$E_n = -\hbar^2\alpha^2\frac{\{[\sqrt{\frac{1}{4}+V_0/\hbar^2\alpha^2}-(n+\frac{1}{2})]^2-\frac{\beta}{2}\}^2}{[\sqrt{\frac{1}{4}+V_0/\hbar^2\alpha^2}-(n+\frac{1}{2})]^2} \tag{20}$$

which is the exact expression[6]. As far as I know, it has not been noticed so far in the literature that the conventional lowest order WKB also reproduces this exact E_n provided one adds the Langer-like correction term $-\alpha^2/4$ sech$^2\alpha x$ to the potential (4b).

Finally, let me consider the Pöschl-Teller potential as given by Eq. (4c). On using

$$\phi(x) = A\tan\alpha x - B\cot\alpha x \tag{21}$$

where

$$A = \frac{\hbar\alpha}{2}\left[1+\sqrt{1+4V_2/\hbar^2\alpha^2}\,\right] ; B = \frac{\hbar\alpha}{2}\left[1+\sqrt{1+4V_1/\hbar^2\alpha^2}\,\right] \tag{22}$$

and applying the SWKB quantization condition, I find that

$$E_n = (A+B+2n\hbar\alpha)^2 \tag{23}$$

which is the exact expression[7]. In this case, I find that the conventional lowest order WKB can be made to reproduce this expression by the addition of the Langer-like correction term $\alpha^2/4$ $(\mathrm{cosec}^2\alpha x + \sec^2\alpha x)$ to the potential (4c).

Summarizing, it appears that at least for those quantum mechanical models in which SUSY is unbroken, the SWKB may prove even more useful than the conventional WKB. It would be very interesting if one could generalize the SWKB to field theories and then try to obtain the energy spectrum of, say, supersymmetric Sine-Gordon model.

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Table 1: Comparison of the exact and SWKB predictions for the energy eigenvalues of H_+ for $d = 1$ and $A = 1/3$.

n	Exact E_n	SWKB E_n
0	1.109087	1.161874
1	3.596036	3.286274
2	6.644391	6.037272
3	10.237873	9.294989
4	14.307040	12.990141
5	18.801758	17.075984
6	23.685275	21.518199
7	28.928957	26.290197
8	34.509674	31.370586
9	40.408244	36.741668
10	46.608420	42.388485
100	1293.415788	1179.345169
1000	40341.683937	36796.79457

Table 2: Comparison of the exact and SWKB predictions for the ratio of the energy eigenvalues of H_+ for d = 1 and A = 1/3

E_{n_2}/E_{n_1}	Exact value	SWKB value = $(n_2+1/n_1+1)^{3/2}$
E_2/E_1	1.847699	1.837117
E_3/E_2	1.540829	1.539600
E_4/E_3	1.397462	1.397542
E_5/E_4	1.314161	1.314534
E_6/E_5	1.259737	1.260144
E_7/E_6	1.221390	1.221765
E_8/E_7	1.192911	1.193242
E_9/E_8	1.170925	1.171214
E_{10}/E_9	1.153438	1.153689
E_{10}/E_1	12.961054	12.898643
E_{100}/E_1	359.678209	358.869927
E_{1000}/E_1	11218.375	11197.115

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