Self-Duality of a Topologically Massive Born-Infeld Theory

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Abstract

We consider self-duality in a 2 + 1 dimensional gauge theory containing both the Born-Infeld and the Chern-Simons terms. We introduce a Born-Infeld inspired generalization of the Proca term and show that the corresponding model is equivalent to the Born-Infeld-Chern-Simons model.

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Many years ago Townsend et. al. studied self-duality in gauge theories in $4k-1$ dimensions [1]. In particular, in $2+1$ dimensions they considered the Proca equation for the massive gauge field:

$$\partial^\mu F_{\mu\nu} + m^2 A_\nu = 0 \ , \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. As a consequence of the antisymmetry of the field strength, it follows from above that $\partial_\mu A^\mu = 0$, and hence there are two, independent, propagating modes of equal mass. They observed that any gauge field which is proportional to the dual of it's field strength does satisfy the above equation. In particular any gauge field which satisfy

$$A_\mu = \frac{1}{2m}\epsilon_{\mu\nu\rho}F^{\nu\rho} \ , \tag{2}$$

is a solution of the second order Eq. (1). They called Eq. (2) as the self-duality equation. This equation propagates one massive mode instead of two and it can be viewed as a square root of the second order Eq. (1). The self-dual Eq. (2) can be derived from the Lagrangian

$$\mathcal{L}_P = \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{4}m\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \ . \tag{3}$$

It is straightforward to see that the above Lagrangian is not gauge invariant. However, interestingly it was soon observed [2] that the above model is equivalent to gauge invariant, topologically massive, electrodynamics characterized by the Lagrangian [3, 4]

$$\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}m\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} \ . \tag{4}$$

The corresponding field equation is

$$\partial_\mu F^{\mu\nu} + \frac{1}{2}m\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0 \ , \tag{5}$$

and following [2] it is easily shown that the field Eqs. (5) and (2) are equivalent. In fact, in [2] the authors have even shown the equivalence of the two Lagrangians $\mathcal{L}_P$ and $\mathcal{L}_M$ as given by Eqs. (3) and (4) respectively.

Long back Born and Infeld proposed [5] a nonlinear generalization to the Maxwell Lagrangian in order to cure the short distance divergence appearing in quantum electrodynamics. Recently it has attracted considerable attention both in field theory, because of it's remarkable form, as well as in string theory for it is the action which governs the gauge field dynamics of the D-branes [6]. Because of its importance in the open string theory, Gibbons and Rasheed studied various duality invariances of the Born-Infeld theory [7]. In particluar, they have shown that the $SO(2)$ electric-magnetic duality rotation, that appears as a symmetry at the level of equations of motion in the Maxwell theory in four spacetime dimensions, also holds in the Born-Infeld theory. Because of the importance of duality in understanding various non-perturbative aspects of field theroy as well as string theory, the above results have been generalized to nonlinear theories with more then one Abelian gauge field, theories with interacting scalar fields as well as to the supersymmetric theories [8]-[19] . However most of the discussion about the duality invariance has been restricted to theories in four space time dimensions or more generally to the even dimensional theories.

On the other hand, several interesting generalizations of the self-dual Chern-Simons-Proca model [1] and its equivalence [2] with the three dimensional massive electrodynamics [3, 4] has been studied in literature. Soon after the work of Deser and Jackiw, it has been realised that the self-duality can also occur in case both the Maxwell as well as the Proca term can simultaneously be incorporated in addition to the Chern-Simons term[20]. The above model has also been used in the study of bosonization in higher dimensions [21, 22]. Recently it has been shown that there exists a unified theory [23] from which the self-dual model[1], the massive electrodynamics [3, 4] as well as the Maxwell-Chern-Simons-Proca systems [20] can be recovered as special cases. However, to the best of our knowledge, the reuslts of Deser and Jackiw have not been generalized to the Born-Infeld theory. The purpose of this note is to consider the generalization of this equivalence in case the Maxwell term is replaced by the celebrated Born-Infeld Lagrangian.

Consider the Lagrangian

$$\mathcal{L}_{BI} = \beta^2 \sqrt{1 - \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu}} + \frac{1}{4} m \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} \ . \tag{6}$$

Here we have ignored an irrelevant constant factor proportional to square of the Born-Infeld parameter β which does not contribute to the equation of motion. This Lagrangian reduces to the topologically massive Lagrangian as given by Eq. (4) in the limit $\beta \to \infty$ when the constant factor is taken into account. The corresponding field equation is

$$\partial_\mu \left(\frac{F^{\mu\nu}}{\sqrt{1 - \frac{1}{2\beta^2} F_{\alpha\rho} F^{\alpha\rho}}} \right) + \frac{1}{2} m \epsilon^{\nu\alpha\rho} F_{\alpha\rho} = 0 \ . \tag{7}$$

The question one would like to ask is: what is the self-dual analogue of Eq. (2)? We will now show that the corresponding "generalized self-dual equation" is

$$A_\mu = \frac{\epsilon^{\mu\nu\rho} F_{\nu\rho}}{2m\sqrt{1 - \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu}}} \ . \tag{8}$$

Before we prove our assertion, let us note that Eq. (8) reduces to the self dual Eq. (2) in the $\beta \to \infty$ limit. We call it the "generalized self-dual equation" because of it's similarity with the self-dual equation even though in the literature the term self-dual is usually used while dealing with the linearized equations. To show that Eq. (7) follows from Eq. (8), differentiate both sides of Eq. (8) and multiply by $\epsilon^{\beta\gamma\mu}$. We get

$$\epsilon_{\beta\gamma\mu} \partial^\gamma A^\mu = \frac{1}{2m} \epsilon_{\beta\gamma\mu} \epsilon^{\mu\nu\rho} \partial^\gamma \left(\frac{F_{\nu\rho}}{\sqrt{1 - \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu}}} \right) \ , \tag{9}$$

from which Eq. (7) follows.

Following [1] it is worth enquiring if there is a corresponding Born-Infeld self-dual Lagrangian from which self-dual Eq. (8) will follow as a field equation. It is easily seen that such a Lagrangian is

$$\mathcal{L}_{BIP} = \beta^2 \sqrt{1 + \frac{1}{\beta^2} f_\mu f^\mu} - \frac{1}{2m} \epsilon^{\alpha\mu\nu} f_\alpha \partial_\mu f_\nu \ . \tag{10}$$

Here we have changed the notation from A_μ to $f_\mu = mA_\mu$ in order to avoid confusion with the system described by Eq. (6). The corresponding field equation is

$$\epsilon^{\mu\nu\rho} \partial_\nu f_\rho - \frac{m f_\mu}{\sqrt{1 + \frac{1}{\beta^2} f_\mu f^\mu}} = 0 \ . \tag{11}$$

Let us now show that the above equation is equivalent to the generalized self-dual Eq. (8). Taking the square of Eq. (11) we get

$$\frac{1}{2} f_{\mu\nu} f^{\mu\nu} = \frac{m^2 f_\mu f^\mu}{1 + \frac{1}{\beta^2} f_\mu f^\mu} \ , \tag{12}$$

which implies

$$\sqrt{1-\frac{1}{2\beta^2 m^2}f_{\mu\nu}f^{\mu\nu}}=\frac{1}{\sqrt{1+\frac{1}{\beta^2}f_\mu f^\mu}}\ . \tag{13}$$

Substituting this again in Eq. (11) we get the generalized self-dual equation which can be identified with Eq. (8) after the field redefinition. Thus both the theories described by Eqs (6) and (10) admit identical solution for the self-dual gauge field.

In fact the equivalence of the two theories is at a more basic level. In particular we show that the corresponding Hamiltonians of the two theories are equivalent after the constraints are taken into account. Let us start from the Born-Infeld Lagrangian (6). The corresponding canonical momentum is

$$\begin{aligned}\Pi^i &= \frac{\delta\mathcal{L}_{BI}}{\delta\dot{A}^i}\\ &= -\frac{E^i}{\sqrt{1-\frac{1}{2\beta^2}F_{\mu\nu}F^{\mu\nu}}}+\frac{1}{2}\epsilon^{ij}A^j\ .\end{aligned} \tag{14}$$

For convenience we define

$$R=\sqrt{1-\frac{1}{2\beta^2}F_{\mu\nu}F^{\mu\nu}}\ ,\qquad D^i=\frac{E^i}{R}\ . \tag{15}$$

After some straightforward calculation we can write the expression for R as

$$R=\sqrt{\frac{1-\frac{1}{\beta^2}B^2}{1-\frac{1}{\beta^2}\mathbf{D}^2}}\,, \tag{16}$$

where $B=F_{12}$.

Now the Hamiltonian density is

$$\begin{aligned}\mathcal{H}_{BI} &= \Pi^i\dot{A}^i-\mathcal{L}_{BI}\\ &= -\frac{E^i\dot{A}^i}{R}-\beta^2 R\\ &= -\frac{\beta^2}{R}\left(1-\frac{1}{\beta^2}B^2\right)\\ &= -\beta^2\sqrt{\left(1-\frac{1}{\beta^2}\mathbf{D}^2\right)\left(1-\frac{1}{\beta^2}B^2\right)}\ .\end{aligned} \tag{17}$$

The equal time commutation relation is

$$i\left[\Pi^i(\mathbf{r}), A^j(\mathbf{r}')\right] = \delta^{ij}\delta(\mathbf{r}-\mathbf{r}') \tag{18}$$

from which we can derive the following equal time commutators:

$$\begin{aligned} i\left[D^i(\mathbf{r}), D^j(\mathbf{r}')\right] &= m\epsilon^{ij}\delta(\mathbf{r}-\mathbf{r}') && (19)\\ i\left[D^i(\mathbf{r}), B(\mathbf{r}')\right] &= -\epsilon^{ij}\partial_j\delta(\mathbf{r}-\mathbf{r}') && (20)\\ i\left[B(\mathbf{r}), B(\mathbf{r}')\right] &= 0\ . && (21)\end{aligned}$$

These relations along with the Gauss law constraint

$$\partial_i\Pi^i + \frac{m}{2}B = 0\,, \tag{22}$$

can be solved in terms of a scalar field ϕ as

$$\begin{aligned} D^i &= -\epsilon^{ij}\hat{\partial}_j\dot{\phi} - m\hat{\partial}_i\phi && (23)\\ B &= \sqrt{-\nabla^2}\phi\ , && (24)\end{aligned}$$

where $\hat{\partial}_j = \partial_j/\sqrt{-\nabla^2}$.

Now we consider the system described by Lagrangian (10). The conjugate momentum ${\Pi^i}_f$ is given by

$${\Pi^i}_f \equiv \frac{\delta\mathcal{L}_{BIP}}{\delta\dot{f}^i} = -\frac{1}{2m}\epsilon^{ij}f^j\ , \tag{25}$$

which implies the canonical commutation relation

$$i\left[f^i(\mathbf{r}), f^j(\mathbf{r}')\right] = m\epsilon^{ij}\delta(\mathbf{r}-\mathbf{r}')\ . \tag{26}$$

This along with the Gauss law constraint

$$\frac{f_0}{\sqrt{1+\frac{1}{\beta^2}f_\mu f^\mu}} = -\frac{1}{m}\epsilon^{ij}\partial_i f_j\ , \tag{27}$$

can also be solved in terms of the field ϕ as

$$f^i = -\hat{\partial}_i\dot{\phi} + m\epsilon^{ij}\hat{\partial}_j\phi \tag{28}$$

$$\frac{f_0}{\sqrt{1+\frac{1}{\beta^2}f_\mu f^\mu}} = -\sqrt{-\nabla^2}\phi\ . \tag{29}$$

Solving Eq. (29) for f_0 we get

$$f_0 = -\sqrt{-\nabla^2}\phi \left(\sqrt{\frac{1 - \frac{1}{\beta^2}\mathbf{f}^2}{1 - \frac{1}{\beta^2}(\sqrt{-\nabla^2}\phi)^2}} \right) . \tag{30}$$

Now the Hamiltonian density is given by

$$\begin{aligned} \mathcal{H}_f &= \Pi^i{}_f \dot{f}^i - \mathcal{L}_{BIP} \\ &= -\frac{1}{2m}\epsilon^{ij}\left(f_0 \partial_i f_j + (\partial_i f_j) f_o\right) - \beta^2 \sqrt{1 + \frac{1}{\beta^2} f_\mu f^\mu} \\ &= \frac{f_0^2}{\sqrt{1 + \frac{1}{\beta^2} f_\mu f^\mu}} - \beta^2 \sqrt{1 + \frac{1}{\beta^2} f_\mu f^\mu} \end{aligned} \tag{31}$$

where we have used the Gauss law to obtain the last step. On using Eq. (30) the Hamiltonian density $\mathcal{H}_f$ takes the form

$$\mathcal{H}_f = -\beta^2 \sqrt{\left(1 - \frac{1}{\beta^2} f_i^2\right)\left(1 - \frac{1}{\beta^2}(\sqrt{-\nabla^2}\phi)^2\right)} . \tag{32}$$

which is identical to the Hamiltonian density $\mathcal{H}_{BI}$ because of Eqs. (23), (24) and (28). This gives us the following identification of the fields f_μ in terms of the field A_μ.

$$f^\mu = \frac{F^\mu}{R} , \tag{33}$$

where

$$F^\mu = \epsilon^{\mu\alpha\beta} \partial_\alpha A_\beta$$

while R is given by Eq. (15).

Before finishing this note it is worth enquiring if there is a single "generalized master Lagrangian" from which the Lagrangians (6) and (10) follow. In this context it is worth pointing out that in [2], the authors have noted the common origin of the Lagrangians (3) and (4). In particular they have shown that these Lagrangians follow from a single "master Lagrangian". Let us consider the Lagrangian

$$\mathcal{L}_{Mas} = \beta^2 \sqrt{1 + \frac{1}{\beta^2} f_\mu f^\mu} - \epsilon^{\mu\alpha\beta} f_\mu \partial_\alpha A_\beta + \frac{1}{2} m \epsilon^{\mu\alpha\beta} A_\mu \partial_\alpha A_\beta . \tag{34}$$

Here we treat f_μ and A_μ as independent variables. Varying this "generalized master Lagrangian" with respect to f_μ and after a little algebra we get Eq. (33). On eliminating f_μ from $\mathcal{L}_{Mas}$ by using Eq. (33), we get the Lagrangian $\mathcal{L}_{BI}$ as given by Eq. (6). On the other hand, on varying Eq. (34) with respect to A_μ we get

$$\epsilon^{\mu\alpha\beta}\partial_\alpha f_\beta = mF^\mu \ . \tag{35}$$

On using this to eliminate A_μ from Lagrangian (34) gives the Lagrangian $\mathcal{L}_{BIP}$ as given by Eq. (10). Note that in the limit $\beta \to \infty$ the Lagrangian (34) reduces to the "master Lagrangian" of [2].

To conclude, we have studied the generalized self-duality in the topologically massive Born-Infeld theory and shown that the equivalence of Maxwell-Chern-Simons theory [3, 4] with the Chern-Simons-Proca theory [1] also holds in the nonlinear Born-Infeld theory. Here it is worth mentioning that the $2+1$ dimensional Born-Infeld action is the world volume action for D2-brane which can appear in the type IIA superstring theory and can have Ramond-Ramond coupling via the Chern-Simons term. More generally, the action for n Dp-branes at small separation is descried by the Dirac-Born-Infeld action

$$S_{dbi} = \int d^{p+1}\sigma Tr \left(e^{\Phi}\sqrt{-det(G+B+F)} \right) \tag{36}$$

which can couple to the Ramond-Ramond background via the Chern-Simons term

$$S_{cs} = \int_{p+1} Tr \left[e^{(B+F)\wedge C} \right] \tag{37}$$

(Here we have used the stringy notation, where B is the Neveu-Schwarz 2-form pulled back on the world volume, C is the Ramond-Ramond field. Φ is the dilaton, G the pull back of the metric on the world volume of the brane and σ is the coordinate on it.) It will be remarkable, if similar results, as given in our present work, can also hold for the above more genreal D-brane action for arbitrary p in presence of the B-field. Clearly, further investigation is required to explore this point.

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