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On the Coleman-Hill Theorem

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Abstract

The Coleman-Hill theorem prohibits the appearance of radiative corrections to the topological mass (more precisely, to the parity-odd part of the vacuum polarization tensor at zero momentum) in a wide class of abelian gauge theories in 2+1 dimensions. We re-express the theorem in terms of the effective action rather than in terms of the vacuum polarization tensor. The theorem so restated becomes somewhat stronger: a known exception to the theorem, spontaneously broken scalar Chern-Simons electrodynamics, obeys the new non-renormalization theorem. Whereas the vacuum polarization *does* receive a one-loop, parity-odd correction, this does not translate to a radiative contribution to the Chern-Simons term in the effective action. We also point out a new situation, involving scalar fields and parity-odd couplings, which was overlooked in the original analysis, where the conditions of the theorem are satisfied and where the topological mass does, in fact, get a radiative correction.

The existence of the Chern-Simons (CS) term in 2+1 dimensional gauge theories [1] has fueled a large body of research over the last several years, in fields varying from condensed matter physics to pure mathematics. The term leads to fractional-statistics excitations (relevant to the fractional quantum Hall effect) [2], while its topological nature in the nonabelian case has yielded information on the classification of lower-dimensional manifolds and knot invariants [3]. It is odd under parity, and provides for a gauge-invariant mass for the relevant vector bosons.

The coefficient of the non-abelian CS term must be quantized for the theory to be consistent [4–6]. This quantization must be respected by radiative corrections, and, indeed, in pure $SU(N)$ gauge theory, it has been found that the coefficient (appropriately normalized so that the quantization is to integer values) receives a one-loop correction which changes its value by the integer N [7].

If the gauge field is coupled to matter fields which spontaneously break the symmetry, the situation is much more delicate in the non-abelian case. With complete breaking of the symmetry, the topological mass itself receives a correction which is a complicated function of the parameters of the theory, and certainly no quantization condition is satisfied, in general [8]. However, the quantization of the coefficient of the CS term itself might be salvaged, since there exist other terms which are not of a topological nature and therefore whose coefficients need not be quantized, yet which contribute to the topological mass. The non-quantization of the radiative correction to the topological mass might thus be a combination of a quantized correction to the CS term along the lines of [7] along with a non-quantized correction to the other terms, as was suggested [8].

More alarming is the case of a non-abelian theory spontaneously broken to a non-abelian subgroup. There, the topological mass is found *not to be* quantized (similar to the situation in [8]), yet there are no terms other than the CS term which make a contribution to the topological mass [9]. It would appear, therefore, that in such theories the CS term does receive a non-quantized radiative correction, and thus that they are not consistent at a quantum level, following the reasoning of [4–6].

A parallel but quite different situation arises in the abelian case, where no quantization condition is required. There is thus no *a prioro* restriction on radiative corrections, yet such corrections are in fact few and far between. Coupling the photon to a fermion yields a correction to the linear term in a momentum expansion of the parity-odd part of the photon vacuum polarization tensor $\pi^{\rm odd}_{\mu\nu}$ (whether or not the CS term is there initially) at one loop [1,4,10], but not at two loops [11]. Inspired by this unexpected result, Coleman and Hill [12] devised a proof that, under very general conditions, the only correction to the linear term in a momentum expansion of $\pi^{\rm odd}_{\mu\nu}$ comes from fermions at one loop. In particular, they have emphasized that the result is valid even for nonrenormalizable interactions in the presence of gauge- and Lorentz-invariant regularization.

Situations exist where the conditions given by Coleman and Hill are satisfied, yet where radiative corrections to the topological mass do nonetheless arise. Namely, this can occur if there are new parity-violating interaction terms in the initial Lagrangian, a possibility which was overlooked in [12]. For instance, if the photon is coupled to massive vector particles which themselves violate parity (a possibility in 2+1 dimensions) there is a correction to the topological mass [13].

Furthermore, even scalar fields can have such parity-violating interactions. Indeed, the following interaction Lagrangian

$$\mathcal{L}_{int} = j_\mu \left(ieA^\mu + \frac{\alpha}{e}\epsilon^{\mu\nu\lambda}F_{\nu\lambda}\right) \tag{1}$$

yields, after a straightforward evaluation of the vacuum polarization,

$$\pi^{\rm odd}_{\mu\nu}\,\frac{\epsilon^{\mu\nu\lambda}p_\lambda}{2p^2}\bigg|_{p^2=0} \equiv \pi^{\rm odd}(0) = \frac{\alpha m}{\pi}, \tag{2}$$

where m is the mass of the scalar and j^μ is the usual particle current.

In addition to these actual counter-examples to the Coleman-Hill theorem, a number of situations have been found where the initial assumptions of the theorem are not satisfied, and where the vacuum polarization tensor does get further radiative corrections. One such situation is if there are massless particles present, in which case infrared divergences spoil the proof of the theorem [14–16]. Another is if Lorentz or gauge invariance is not manifest, a situation found in the nonabelian case (where, as outlined above, radiative corrections indeed exist).

A third such case is that of spontaneously broken scalar electrodynamics [17,18], where the interaction term explicitly violates one of Coleman and Hill's initial assumptions. (In this case, in their words, there are parts of the gauge boson kinetic terms "lurking" about in the interaction Lagrangian.) Here, it has been found that, with even an infinitesimal CS term at the tree level, a macroscopic parity-odd part of the vacuum polarization is induced to one loop. Since the tree level Lagrangian respects parity (in the absence of a CS term), this system exhibits spontaneous parity violation, very reminiscent of the case of massless spinor electrodynamics [10]. This induced parity violation is not found in the phase where the symmetry is respected; indeed, the limit of letting the bare CS term go to zero does not commute with that of letting the expectation value of the scalar field go to zero.

In the case of spontaneous symmetry breaking, particularly, the relation between the parity-odd part of the vacuum polarization and the renormalization of the CS term is rather indirect. This is best seen within the framework of the effective action. The vacuum polarization tensor, in a phase with scalar field expectation value ϕ_0, is the second derivative of the effective action with respect to the photon, evaluated at zero photon field and at $\phi = \phi_0$:

$$\pi_{\mu\nu}(x,y) = \left\{ \frac{\delta^2 \Gamma_{\text{eff}}}{\delta A^\mu(x) \delta A^\nu(y)} \right\} \Bigg|_{A=0,\phi=\phi_0} . \tag{3}$$

We are interested in the parity-odd part of this for small momenta. By Lorentz invariance, this will be proportional to $\epsilon_{\mu\nu\rho} k^\rho$; let us call the coefficient $\pi^{\text{odd}}(0)$.

$\pi^{\text{odd}}(0)$ will certainly receive contributions from the tree-level CS term as well as from radiative corrections to it. But, as we will show, other terms in the effective action which reduce to the CS term if one sets $\phi \to \phi_0$ (which we refer to as "would-be CS terms") will also contribute to $\pi^{\text{odd}}(0)$, making the extraction of the radiative correction to the CS term itself more complicated. In fact, if we rephrase the statement of the Coleman-Hill theorem in terms of the non-renormalization of the coefficient of the CS term in the effective action, then, as will be shown below, at least in the case of minimally-coupled scalar electrodynamics, the newly-stated theorem remains valid even in the presence of spontaneous symmetry breaking. More precisely, in spontaneously-broken, minimally-coupled scalar electrodynamics, the coefficient of the CS term in the effective action *does not receive* a radiative correction at one loop.

To be specific, the model we consider consists of a real scalar doublet with gauged SO(2) symmetry and with bare CS term, described by the Lagrangian

$$\begin{aligned} \mathcal{L} = & -\tfrac{1}{4}{F_{\mu\nu}}^2 + \tfrac{\mu}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - \tfrac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ & +\tfrac{1}{2}{(D_\mu\phi)_a}^2 - \tfrac{m^2}{2}\phi^2 - \tfrac{\lambda}{4!}\phi^4 - \tfrac{\tau}{6!}\phi^6, \end{aligned} \tag{4}$$

where $\phi^2 = (\phi_a)^2$ and $(D_\mu\phi)_a = (\partial_\mu\delta_{ab} - eA_\mu\epsilon_{ab})\phi_b$. The third term is for gauge fixing; we will eventually work in the Landau gauge $\xi \to 0$.

In what follows, we will compute the terms of interest in the effective action $\Gamma[\hat{\phi}, \hat{A}]$ to one loop, the ultimate goal being to compute the one-loop correction to the CS term, and to show that it is indeed zero. Naively, the CS term is calculated by computing the term in Γ which is parity-odd, bilinear in $\hat{A}$, and which contains one derivative. In the phase of unbroken symmetry, this is perfectly unambiguous and correct; however, in the presence of spontaneous symmetry breaking there are would-be CS terms which have the identical structure if we let $\phi \to \phi_0$, and which necessitate additional work in order to separate them from the true CS term.

To see this, consider the low-momentum terms in the parity-odd part of Γ. Such terms must be expressed in terms of $\epsilon_{\mu\nu\lambda}$; one finds

$$\begin{aligned} \Gamma^{odd}[\hat{\phi}(x), \hat{A}(x)] = & \textstyle\int d^3x\, \epsilon_{\mu\nu\lambda}(c_1\hat{A}_\mu\partial_\nu\hat{A}_\lambda + c_2(\hat{\phi}^2)\hat{\phi}_a D_\mu\hat{\phi}_a\partial_\nu\hat{A}_\lambda \\ & +c_3(\hat{\phi}^2)\epsilon_{ab}\hat{\phi}_a D_\mu\hat{\phi}_b\partial_\nu\hat{A}_\lambda + \text{higher order terms}). \end{aligned} \tag{5}$$

We can, in fact, put the coefficient c_2 to zero without loss of generality, since $c_2(\hat{\phi}^2)\hat{\phi}_a D_\mu\hat{\phi}_a$ is a total derivative, and integration by parts demonstrates that the term itself is, in fact, also a total derivative.

If we were to proceed according to the naive approach outlined above, setting $\hat{\phi} \to \phi_0$, the third term would make an unwanted contribution to the effective action:

$$\Gamma^{odd}[\phi_0, \hat{A}(x)] = \int d^3x\, \epsilon_{\mu\nu\lambda}\left(c_1 + e\phi_0^2 c_3(\phi_0^2)\right)\hat{A}_\mu\partial_\nu\hat{A}_\lambda + \cdots. \tag{6}$$

We must therefore perform a supplementary calculation in order to separate the undesired c_3 part from the genuine CS term.

This can be most easily done by considering a field configuration whose scalar part has a space-dependent piece: $\hat{\phi} = \phi_0 + \phi_1(x)$. If we now evaluate Γ^{odd} to linear order in ϕ_1 and in $\hat{A}$, the CS term makes no contribution, and we find

$$\Gamma^{odd}[\phi_0 + \phi_1(x), \hat{A}(x)] = \int d^3x\, \epsilon_{\mu\nu\lambda} c_3 \epsilon_{ab} \phi_{0a} \partial_\mu \phi_{1b} \partial_\nu \hat{A}_\lambda + \cdots . \tag{7}$$

This second calculation then gives c_3, which then enables us to extract c_1.

To zero loops, the effective action is the ordinary action: $\Gamma_0[\hat{\phi}, \hat{A}] = S[\hat{\phi}, \hat{A}] = \int d^3x\, \mathcal{L}[\hat{\phi}, \hat{A}]$. The one-loop contribution to the effective action is obtained according to the following prescription [19]. One expands the ordinary action about the desired field values, $S[\hat{\phi} + \phi, \hat{A} + A]$. It is the part quadratic in A and ϕ which is relevant to one loop; this is

$$\begin{aligned} S^q = \int d^3x \Big\{ & \tfrac{1}{2} A_\mu \left(g_{\mu\nu}\partial^2 - \left(1 - \tfrac{1}{\xi}\right)\partial_\mu\partial_\nu + \mu\epsilon_{\mu\alpha\nu}\partial_\alpha + e^2\hat{\phi}^2 g_{\mu\nu}\right) A_\nu \\ & + A_\mu \left(e\epsilon_{ac} \left(\hat{\phi}_c \partial_\mu - (\partial_\mu \hat{\phi}_c)\right) + 2e^2 \hat{A}_\mu \hat{\phi}_a \right) \phi_a \\ & + \tfrac{1}{2}\phi_a \Big(-\left(\partial_\mu \delta_{ab} - e\hat{A}_\mu \epsilon_{ab}\right)\left(\partial_\mu \delta_{bc} - e\hat{A}_\mu \epsilon_{bc}\right) \\ & \quad - \left(m^2 + \tfrac{\lambda}{6}\hat{\phi}^2 + \tfrac{\tau}{120}\hat{\phi}^4\right)\delta_{ac} - \left(\tfrac{\lambda}{3} + \tfrac{\tau}{30}\hat{\phi}^2\right)\hat{\phi}_a\hat{\phi}_c \Big)\phi_c \Big\} \\ \equiv \int d^3x \Big\{ & \tfrac{1}{2} A_\mu U_{\mu\nu}(\hat{\phi}) A_\nu + A_\mu V_{\mu a}(\hat{\phi}, \hat{A})\phi_a \\ & + \phi_a \tfrac{1}{2} W_{ab}(\hat{\phi}, \hat{A})\phi_b \Big\} \end{aligned} \tag{8}$$

The one-loop contribution to the effective action Γ_1 is then obtained by functional integration; the result is

$$\Gamma_1[\hat{\phi}, \hat{A}] = \frac{i}{2}\mathrm{Tr}\log W + \frac{i}{2}\mathrm{Tr}\log(U - VW^{-1}V). \tag{9}$$

Were we interested in the effective potential, we could take constant field values and evaluate the traces exactly. However, for terms in the effective action involving derivatives,we must allow $\hat{\phi}$ and $\hat{A}$ to depend on x, and some sort of approximation must be employed. Fortunately, we are interested only in the CS and would-be CS terms, so an expansion in derivatives of the fields and in powers of the fields themselves will suffice.

Let us first outline the calculation of the combined genuine and would-be CS terms, letting $\hat{\phi} = \phi_0$. We must compare (6), on the one hand, with an expansion of (9) on the other.[1] The first term in (9) can be ignored since it makes no contribution to Γ^{odd}. In the second term, the argument of the log can be written

$$X \equiv U - VW^{-1}V = U^{(0)} - (V^{(0)} + V^{(1)})(W^{(0)} + W^{(1)} + W^{(2)})^{-1}(V^{(0)} + V^{(1)}), \qquad (10)$$

where the superscripts indicate powers of $\hat{A}$. We can expand X in powers of $\hat{A}$, $X = X^{(0)} + X^{(1)} + X^{(2)} + \cdots$; higher terms are unnecessary since we are only interested in terms quadratic in $\hat{A}$. The relevant terms in the one-loop, parity-odd part of the effective action are

$$\Gamma_1^{odd}[\phi_0, \hat{A}] = \frac{i}{2}\mathrm{Tr}[((X^{(0)})^{-1})^{odd} X^{(2)}] - \frac{i}{2}\mathrm{Tr}[((X^{(0)})^{-1})^{odd} X^{(1)}((X^{(0)})^{-1})^{even} X^{(1)}] + \cdots. \qquad (11)$$

There are rather a large number of terms; however, after some work, only one survives; re-expressing it in terms of U, V and W,

$$\Gamma_1^{odd}[\phi_0, \hat{A}] = -\frac{i}{2}\mathrm{Tr}\left[\left(U^{(0)} - V^{(0)}W^{(0)^{-1}}V^{(0)}\right)^{-1\,odd}_{\mu\nu} V^{(1)}_{\nu a} W^{(0)}{}^{-1}_{ab} V^{(1)}_{\mu b}\right] + \cdots. \qquad (12)$$

The relevant expressions can be read more or less directly off (8). The trace is not yet calculable, since it involves both derivatives (in the zeroeth-order parts) and space dependance (in the first-order parts). A separation of these can be achieved in a derivative expansion [20], which suffices for our purposes; the trace can then be performed. After some work, one finds (after Wick rotation)

$$\Gamma_1^{odd}[\phi_0, \hat{A}(x)] = \frac{4\mu e^4}{3}\int d^3x\, \epsilon_{\mu\nu\lambda}\phi_0^2 \hat{A}_\mu \partial_\nu \hat{A}_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{((p^2 + e^2\phi_0^2)^2 + \mu^2 p^2)\,(p^2 + m_L^2)^2} + \cdots, \qquad (13)$$

where $m_L^2 = m^2 + \lambda\phi_0^2/5 + \tau\phi_0^4/24$ and the integral is in momentum space; comparing with (6), the coefficients of the CS and would-be CS terms satisfy the following relation:

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[1] For simplicity, we redefine c_1 so that it excludes the original (zero-loop) coefficient of the CS term, $\mu/2$.

$$c_1 + e\phi_0^2 c_3(\phi_0^2) = \frac{4\mu e^4 \phi_0^2}{3} I, \tag{14}$$

where I is the integral in (13); it is easy to evaluate, although not particularly transparent.

To calculate c_3, we proceed in a similar fashion, this time giving a space-dependent piece to the scalar field: $\hat{\phi} = \phi_0 + \phi_1(x)$. The term linear in ϕ_1 and in $\hat{A}$ is, after some work,

$$\Gamma_1^{odd}[\phi_0 + \phi_1(x), \hat{A}(x)] = \frac{4\mu e^3}{3} \int d^3x\, \epsilon_{\mu\nu\lambda} \phi_0 \epsilon_{ab} \partial_\mu \phi_{1b} \partial_\nu \hat{A}_\lambda\, I + \cdots. \tag{15}$$

Combining (7), (14) and (15), we find our main result: the radiative correction to the coefficient of the CS term in the effective action is

$$c_1 = 0. \tag{16}$$

In summary, we have shown that in the Higgs phase of 2+1 dimensional scalar electrodynamics, all one-loop contributions to the topological mass arise from manifestly gauge invariant terms in the effective action which reduce to the CS term once the scalar field is set equal to its vacuum expectation value, rather than being attributable to the CS term itself. This suggests a more general theorem: namely, that the Coleman-Hill theorem restated as a non-renormalization theorem for the coefficient of the CS term in the effective action would be valid. We have also shown, in passing, that one-loop corrections to the topological mass arise simply from parity-violating interactions, which exist even for scalar fields, in addition to the known cases of fermions [4] and vector bosons [7,13].

These ideas fit in nicely with the suggestion of Khlebnikov and Schaposhnikov [8] for the case of a completely spontaneously broken non-abelian gauge symmetry, where an apparent violation of the quantization condition on the topological mass was postulated to be attributable to the existence of other would-be CS terms. There, however, the analysis is much more involved and has yet to be done. The case of partial breaking to a non-abelian subgroup remains a puzzle [9]: the violation of the quantization condition on the topological mass has thus far eluded a similar explanation, since no would-be CS terms can be constructed in an analogous fashion.

As this manuscript was being finalized, a paper has appeared which discusses the renormalization of the CS term in self-dual and supersymmetric versions of the Abelian Higgs model [21].

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