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On some one-parameter families of three-body problems in one dimension: Exchange operator formalism in polar coordinates and scattering properties

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Abstract

We apply the exchange operator formalism in polar coordinates to a one-parameter family of three-body problems in one dimension and prove the integrability of the model both with and without the oscillator potential. We also present exact scattering solution of a new family of three-body problems in one dimension.

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In recent years, the Calogero-Sutherland type [1, 2] N-body problems in one dimension have attracted a lot of attention [3]. Some time ago, Brink et al. [4] and Polychronakos [5] independently introduced an exchange operator formalism, leading to covariant derivatives, known in the mathematical literature as Dunkl operators [6], and an S_N-extended Heisenberg algebra [7, 8]. In terms of this formalism, the N-body quantum Calogero model has been shown to be equivalent to a set of free modified oscillators and hence integrable.

In the last few years, there has been renewed interest in three-body Calogero-Marchioro-Wolfes (CMW) problem [9, 10], and some other related three-body problems [11, 12], all of which include a three-body potential. It is then natural to enquire if the exchange operator formalism [4, 5] can also be extended to these problems and further if using it, one can show the integrability of these models. Recently, one of us (CQ) took the first step in that direction when she showed the integrability of the quantum CMW problem by using the exchange operator formalism [13].

The purpose of this letter is to extend the exchange operator formalism to the class of three-body problems, with and without the oscillator potential, discovered recently by Sukhatme and one of the author (AK) [14], and hence to prove the integrability of the quantum model. It is worth pointing out that in this model there is an interesting relationship between the incoming and outgoing momenta of the three particles. In particular, we show that introducing an exchange operator formalism in polar coordinates is very useful in analyzing this model (both with and without the oscillator potential) and proving its integrability. Finally, we discuss the scattering solution for a new one-parameter family of three-body problems and show that even for these problems there is a very simple relationship between the incoming and outgoing momenta of the particles.

The exchange operator formalism in polar coordinates has not been discussed so far in the literature, hence it may be worthwhile to first discuss the CMW three-body problem (a known integrable model [13]) using this formalism. We shall see that the generalization of the formalism to the class of three-body problems to be discussed below [14] is then straightforward.

The three-particle Hamiltonian for the CMW problem is given by [9, 10]

$$H = \sum_{j=1}^{3} \left(-\partial_j^2 + \omega^2 x_j^2 \right) + g \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{1}{(x_i - x_j)^2} + 3f \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} \frac{1}{(x_i + x_j - 2x_k)^2}, \tag{1}$$

where x_i ($i = 1, 2, 3$) denotes the particle coordinates, $\partial_i \equiv \partial/\partial x_i$, and the inequalities $g, f > -1/4$ are assumed to prevent collapse. Let $x_{ij} \equiv x_i - x_j$ and $y_{ij} \equiv x_i + x_j - 2x_k$ ($i \neq j \neq k \neq i$), where in the latter we suppress index k since it is entirely determined by i and j.

Let us introduce the Jacobi coordinates

$$R = \frac{x_1 + x_2 + x_3}{3}, \qquad x \equiv \frac{x_{12}}{\sqrt{2}} = r\sin\phi, \qquad y \equiv \frac{y_{12}}{\sqrt{6}} = r\cos\phi. \tag{2}$$

It is then easily verified that $r^2 = \frac{1}{3}\sum_{i<j}^{3}(x_i - x_j)^2$, and

$$x_{ij} = \sqrt{2}\, r \sin\left(\phi + \frac{2\pi k}{3}\right), \qquad y_{ij} = \sqrt{6}\, r \cos\left(\phi + \frac{2\pi k}{3}\right), \tag{3}$$

where $(ijk) = (123)$. One can show that the differential operators ∂_R, ∂_r and ∂_ϕ are given in terms of ∂_i $(i = 1, 2, 3)$ by

$$\partial_R = \sum_j \partial_j, \qquad \partial_r = -\sqrt{\frac{2}{3}} \sum_j \cos\left(\phi + \frac{2\pi}{3} j\right) \partial_j, \qquad \partial_\phi = \sqrt{\frac{2}{3}}\, r \sum_j \sin\left(\phi + \frac{2\pi}{3} j\right) \partial_j. \tag{4}$$

As a result, in polar coordinates the CMW Hamiltonian (1) takes the form $H = H_R + H_r$, where

$$H_R = -\frac{1}{3}\partial_R^2 + 3\omega^2 R^2, \tag{5}$$

$$H_r = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\phi^2 + \omega^2 r^2 + \frac{9}{r^2}\left(\frac{g}{\sin^2 3\phi} + \frac{f}{\cos^2 3\phi}\right). \tag{6}$$

In Ref. [13], the CMW problem was analyzed in terms of some exchange operators belonging to a D_6-group. The latter is generated by the particle permutation operators K_{ij}, and the inversion operator I_r in relative coordinate space. Let us now consider the effect of K_{ij} and I_r on the polar coordinates R, r, ϕ. Using the fact that

$$K_{ij} x_j = x_i K_{ij}, \qquad K_{ij} x_k = x_k K_{ij}, \qquad I_r x_i = (2R - x_i) I_r, \tag{7}$$

it is easy to show that

$$\begin{aligned} K_{ij} R &= R K_{ij}, \qquad K_{ij} r = r K_{ij}, \qquad K_{ij}\phi = \left(-\phi + \frac{2\pi}{3}k\right) K_{ij}, \\ I_r R &= R I_r, \qquad I_r r = r I_r, \qquad I_r \phi = (\phi + \pi) I_r. \end{aligned} \tag{8}$$

Hence it follows that

$$L_{ij} R = R L_{ij}, \qquad L_{ij} r = r L_{ij}, \qquad L_{ij}\phi = \left[-\phi + (2k+3)\frac{\pi}{3}\right] L_{ij}, \tag{9}$$

where $L_{ij} \equiv K_{ij} I_r$. We thus see that K_{ij}, I_r, and hence all the operators of the D_6-group act only on ϕ (and not on r and R). Furthermore, the operations may be written in terms of the two operators $\mathcal{R}$ and $\mathcal{I}$, defined by

$$\mathcal{R} = \exp\left(\frac{\pi}{3}\partial_\phi\right), \qquad \mathcal{I} = \exp(\mathrm{i}\pi\phi\partial_\phi), \tag{10}$$

where $\mathcal{R}$ and $\mathcal{I}$ denote the rotation operator by angle $\pi/3$ and the inversion operator, respectively, i.e.,

$$\mathcal{R}\psi(\phi) = \psi(\phi + \pi/3)\mathcal{R}, \qquad \mathcal{I}\psi(\phi) = \psi(-\phi)\mathcal{I}. \tag{11}$$

In particular, in terms of $\mathcal{R}$ and $\mathcal{I}$ the 12 generators of the D_6-group are given by

$$\begin{aligned} &I, \qquad K_{ij} = \mathcal{I}\mathcal{R}^{2k}, \qquad K_{123} = \mathcal{R}^2, \qquad K_{132} = \mathcal{R}^4, \\ &I_r = \mathcal{R}^3, \qquad L_{ij} = \mathcal{I}\mathcal{R}^{2k+3}, \qquad L_{123} = \mathcal{R}^5, \qquad L_{132} = \mathcal{R}, \end{aligned} \tag{12}$$

where $K_{123} \equiv K_{12}K_{23}$, $K_{132} \equiv K_{23}K_{12}$, $L_{123} \equiv K_{123}I_r$, and $L_{132} \equiv K_{132}I_r$.

Following Eq. (4), it is natural to define the covariant derivatives D_R, D_r, and D_ϕ in polar coordinates by

$$D_R = \sum_j D_j, \qquad D_r = -\sqrt{\frac{2}{3}}\sum_j \cos\left(\phi + \frac{2\pi}{3}j\right) D_j, \qquad D_\phi = \sqrt{\frac{2}{3}}\, r \sum_j \sin\left(\phi + \frac{2\pi}{3}j\right) D_j, \tag{13}$$

where D_i ($i = 1, 2, 3$) are those in cartesian coordinates.

On using the fact that in CMW model the generalized derivatives are defined by [13]

$$D_i = \partial_i - \kappa\left(\frac{1}{x_{ij}}K_{ij} - \frac{1}{x_{ki}}K_{ki}\right) - \lambda\left(\frac{1}{y_{ij}}L_{ij} + \frac{1}{y_{ki}}L_{ki} - \frac{2}{y_{jk}}L_{jk}\right), \tag{14}$$

where $g = \kappa(\kappa - 1)$, and $f = \lambda(\lambda - 1)$, D_R, D_r, and D_ϕ can be shown to be given by

$$\begin{aligned} D_R &= \partial_R, \qquad D_r = \partial_r - \frac{\kappa}{r}\left(\sum_j \mathcal{R}^{2j}\right)\mathcal{I} - \frac{\lambda}{r}\left(\sum_j \mathcal{R}^{2j+1}\right)\mathcal{I}, \\ D_\phi &= \partial_\phi - \kappa\sum_j \cot\left[\phi + (3-j)\frac{2\pi}{3}\right]\mathcal{R}^{2j}\mathcal{I} \\ &\quad + \lambda\sum_j \tan\left[\phi + (4-j)\frac{2\pi}{3}\right]\mathcal{R}^{2j+1}\mathcal{I}. \end{aligned} \tag{15}$$

It is easy to show that the covariant derivatives for the pure three-body case ($\kappa = 0$) can be obtained from those for the pure two-body one ($\lambda = 0$) by making the rotation $\phi \to \phi' = \phi + \pi/6$. In the proof, one uses the fact that the operators $\mathcal{R}'$ and $\mathcal{I}'$, defined in terms of ϕ' in the same way as $\mathcal{R}$ and $\mathcal{I}$ in terms of ϕ (see Eq. (10)), can be expressed in terms of the latter as $\mathcal{R}' = \mathcal{R}$, and $\mathcal{I}' = \mathcal{R}\mathcal{I}$, respectively.

Let us now consider the exchange operator formalism for the one-parameter family of three-body problems [14] characterized by

$$H = \sum_{j=1}^{3}(-\partial_j^2 + \omega^2 x_j^2) + g\sum_{\substack{i,j=1\\ i\neq j}}^{3}\frac{1}{x_{ij}'^2} + 3f\sum_{\substack{i,j=1\\ i\neq j}}^{3}\frac{1}{y_{ij}'^2}, \tag{16}$$

where

$$x'_{ij} \equiv x_{ij} \cos\delta + \frac{y_{ij}}{\sqrt{3}} \sin\delta, \qquad y'_{ij} = -\sqrt{3}\, x_{ij} \sin\delta + y_{ij} \cos\delta, \tag{17}$$

and $0 \le \delta \le \pi/6$. On following Eqs. (2) to (5), the Hamiltonian (16) takes the form $H = H_R + H'_r$, where

$$H'_r = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\phi^2 + \omega^2 r^2 + \frac{9}{r^2}\left[\frac{g}{\sin^2(3\phi + 3\delta)} + \frac{f}{\cos^2(3\phi + 3\delta)}\right], \tag{18}$$

while H_R is again as given by Eq. (5).

Now notice that H for this problem can be obtained from the CMW Hamiltonian (Eqs. (5), (6)) by making the change of variables

$$R \to R' = R, \qquad r \to r' = r, \qquad \phi \to \phi' = \phi + \delta. \tag{19}$$

Hence the exchange operator formalism developed above for the CMW case remains valid for even this case, provided we replace all coordinates and operators by the corresponding primed ones. For example, by using Eq. (15), the generalized derivatives in the primed polar coordinates are given by

$$\begin{aligned} D'_R &= D_R = \partial_R, \qquad D'_r = \partial_r - \frac{\kappa}{r}\left(\sum_j \mathcal{R}'^{2j}\right)\mathcal{I}' - \frac{\lambda}{r}\left(\sum_j \mathcal{R}'^{2j+1}\right)\mathcal{I}', \\ D'_\phi &= \partial_\phi - \kappa \sum_j \cot\left[\phi + \delta + (3-j)\frac{2\pi}{3}\right]\mathcal{R}'^{2j}\mathcal{I}' \\ &\quad + \lambda \sum_j \tan\left[\phi + \delta + (4-j)\frac{2\pi}{3}\right]\mathcal{R}'^{2j+1}\mathcal{I}', \end{aligned} \tag{20}$$

where $\mathcal{R}'$, $\mathcal{I}'$ are again defined in terms of ϕ' as $\mathcal{R}$, $\mathcal{I}$ in terms of ϕ. From Eq. (13), it follows that D'_R, D'_r, D'_ϕ can be expressed in terms of generalized derivatives D'_i in some primed coordinates x'_i by

$$\begin{aligned} D'_R &= \sum_j D'_j, \qquad D'_r = -\sqrt{\frac{2}{3}} \sum_j \cos\left(\phi + \delta + \frac{2\pi}{3} j\right) D'_j, \\ D'_\phi &= \sqrt{\frac{2}{3}}\, r \sum_j \sin\left(\phi + \delta + \frac{2\pi}{3} j\right) D'_j. \end{aligned} \tag{21}$$

Note that x'_i is defined in terms of $R'(= R)$, $r'(= r)$, ϕ' in the same way as x_i in terms of R, r, ϕ.

Further, following Eq. (14), D'_i can also be written as

$$D'_i = \partial'_i - \kappa\left(\frac{1}{x'_{ij}} K'_{ij} - \frac{1}{x'_{ki}} K'_{ki}\right) - \lambda\left(\frac{1}{y'_{ij}} L'_{ij} + \frac{1}{y'_{ki}} L'_{ki} - \frac{2}{y'_{jk}} L'_{jk}\right), \tag{22}$$

in terms of the primed coordinates x'_i, and some primed operators K'_{ij}, L'_{ij}. The latter have the same action on x'_k as K_{ij}, L_{ij} on x_k, respectively, and belong to a tranformed D_6-group. The operators D'_i are covariant under this transformed D_6-group, and commute with one another (hence they may be called Dunkl operators [6]).

We now need to determine the relation between x'_i and x_i to study the action of the transformed D_6-group on x_i. To this end we start from the relation

$$x'_k = -\frac{1}{3}y'_{ij} + R = -\frac{1}{\sqrt{3}}\left(-x_{ij}\sin\delta + \frac{y_{ij}}{\sqrt{3}}\cos\delta\right) + R. \tag{23}$$

On using Eqs. (2) and (3), we then have

$$x'_k = s_2 x_i + s_1 x_j + s_3 x_k, \qquad (ijk) = (123), \tag{24}$$

where

$$s_k \equiv s_k(\delta) = \frac{1}{3}\left[1 + 2\cos\left(\delta + \frac{2\pi}{3}k\right)\right]. \tag{25}$$

In a compact matrix form, we can write $\boldsymbol{x}' = \boldsymbol{x}\boldsymbol{S}$, where $\boldsymbol{x} = (x_1 x_2 x_3)$, $\boldsymbol{x}' = (x'_1 x'_2 x'_3)$, and the matrix $\boldsymbol{S}$ is given by

$$\boldsymbol{S} = \boldsymbol{S}(\delta) = \begin{pmatrix} s_3 & s_1 & s_2 \\ s_2 & s_3 & s_1 \\ s_1 & s_2 & s_3 \end{pmatrix}. \tag{26}$$

It is easily checked that $\boldsymbol{S}$ is orthogonal, i.e., $\boldsymbol{S}\tilde{\boldsymbol{S}} = \tilde{\boldsymbol{S}}\boldsymbol{S} = \boldsymbol{I}$, hence we also have

$$\boldsymbol{x} = \boldsymbol{x}'\tilde{\boldsymbol{S}}, \qquad \boldsymbol{\partial}' = \boldsymbol{\partial}\boldsymbol{S}, \qquad \boldsymbol{\partial} = \boldsymbol{\partial}'\tilde{\boldsymbol{S}}, \tag{27}$$

where $\boldsymbol{\partial} = (\partial_1\partial_2\partial_3)$, $\boldsymbol{\partial}' = (\partial'_1\partial'_2\partial'_3)$. Thus, by definition $\boldsymbol{S}$ is the matrix representing the operator $\mathcal{S} = \exp(\delta\partial_\phi)$ of rotation through an angle δ.

Under the transformation $\phi \to \phi' = \phi + \delta$, the functions transform as

$$\psi(\phi) = \psi(\phi' - \delta) = \psi'(\phi'), \tag{28}$$

hence it is obvious that $\mathcal{R}' = \mathcal{R}$, while $\mathcal{I}' \equiv \exp(\mathrm{i}\pi\phi'\partial'_\phi)$ is such that

$$\mathcal{I}'\psi(\phi) = \mathcal{I}'\psi'(\phi') = \psi'(-\phi') = \psi(-\phi' - \delta) = \psi(-\phi - 2\delta) = \mathcal{S}^2\mathcal{I}\psi(\phi). \tag{29}$$

In other words, $\mathcal{I}' = \mathcal{S}^2\mathcal{I}$, where $\mathcal{S}^2$ is the operator of rotation through an angle 2δ. Using these results, the operators D'_R, D'_r, and D'_ϕ of Eq. (20) may be expressed in terms of unprimed variables and operators. On the other hand, from Eq. (12) and its primed counterpart, it follows that the operators of the transformed D_6-group may be written as

$$\begin{aligned} &I, \qquad K'_{ij} = \mathcal{S}^2 K_{ij}, \qquad K'_{123} = K_{123}, \qquad K'_{132} = K_{132}, \\ &I'_r = I_r, \qquad L'_{ij} = \mathcal{S}^2 L_{ij}, \qquad L'_{123} = L_{123}, \qquad L'_{132} = L_{132}. \end{aligned} \tag{30}$$

From these relations, their action on x_k can be easily determined, but will not be given in detail here.

Finally, we may introduce a Hamiltonian with exchange terms

$$H_{exch} = \sum_{j=1}^{3} \left(-\partial_j^2 + \omega^2 x_j^2\right) + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{1}{x_{ij}^{\prime 2}} \kappa(\kappa - K'_{ij}) + 3 \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{1}{y_{ij}^{\prime 2}} \lambda(\lambda - L'_{ij}). \tag{31}$$

In case $\omega = 0$, since $\mathcal{S}$ is an orthogonal matrix, it is easily shown that

$$H_{exch} = -\sum_{j=1}^{3} D_j^{\prime 2}. \tag{32}$$

It is worth noting that H_{exch} may also be written as $H_{exch} = -\sum_{j=1}^{3} D_j^2$ in terms of Dunkl operators in unprimed coordinates, defined as in Eq. (27) by $\boldsymbol{D} = \boldsymbol{D}'\tilde{\boldsymbol{S}}$. Further, the operators $I_n = \sum_{i=1}^{3} \Pi_i^{\prime 2n}$ $(n = 1,\, 2,\, 3)$, where $\Pi'_i = -\mathrm{i} D'_i$ are generalized momenta, commute with one another, and are left invariant under the transformed D_6-group. Hence, their projection in the subspaces of Hilbert space characterized by $(K'_{ij}, L'_{ij}) = (1,1), (1,-1), (-1,1)$ or $(-1,-1)$ also commute. In these subspaces, $I_1 = H_{exch}$ is nothing but the one-parameter family of Hamiltonians as given by Eq. (16) with $\omega = 0$, corresponding to the parameter values $(\kappa, \lambda), (\kappa, \lambda + 1), (\kappa + 1, \lambda)$ or $(\kappa + 1, \lambda + 1)$, respectively, while I_2 and I_3 become the integrals of motion of such Hamiltonians.

In case $\omega \neq 0$, i.e., when the oscillator potential is present, the Hamiltonian (31) can be written as

$$H_{exch} = \sum_{i=1}^{3} (-D_i^{\prime 2} + \omega^2 x_i^{\prime 2}) = \omega \sum_{i=1}^{3} \{a_i^{\prime +}, a'_i\}, \tag{33}$$

where

$$a'_i = \frac{1}{\sqrt{2\omega}}(\omega x'_i + D'_i), \qquad a_i^{\prime +} = \frac{1}{\sqrt{2\omega}}(\omega x'_i - D'_i). \tag{34}$$

One can now show that a'_i, $a_i^{\prime +}$ $(i = 1,\, 2,\, 3)$, and the operators of the transformed D_6-group, as given by Eq. (30), generate a D_6-extended oscillator algebra, whose commutation relations are entirely similar to Eq. (4.2) of Ref. [13]. It may also be noted that since $\boldsymbol{S}$ is an orthogonal matrix, H_{exch} as given by Eq. (31) can also be written in terms of D_i, x_i, or a_i, a_i^+ defined exactly as in Eqs. (33) and (34). Thus we see that Hamiltonian (16) corresponding to the parameter values (κ, λ), $(\kappa, \lambda + 1)$, $(\kappa + 1, \lambda)$ or $(\kappa + 1, \lambda + 1)$ can be regarded as a free modified boson Hamiltonian. The corresponding conserved quantities are

$$I_n = \sum_{i=1}^{3} h_i^{\prime n}, \qquad h'_i = \frac{1}{2}\{a_i^{\prime +}, a'_i\}, \qquad n = 1, 2, 3, \tag{35}$$

and following Ref. [13], it is easily shown that I_1, I_2, I_3 are mutually commuting operators. Note that I_n, $n = 1,\, 2,\, 3$, are invariant under the transformed D_6-group, and hence their projections in the subspaces characterized by K'_{ij} and L'_{ij} equal to $+1$ or -1 still commute with one another.

Before ending this note, we would like to present a new one-parameter family of three-body problems in one dimension, and show that there is an interesting, simple relation between the incoming and outgoing momenta of the three particles. Since the philosophy is similar to that of Ref. [14], we avoid giving all the details here, but merely point out the steps that are different in the two cases. As in Ref. [14], we work in the centre-of-mass frame and consider the following relative Hamiltonian in polar coordinates ($\hbar = 2m = 1$)

$$H = -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{B^2}{r^2}, \tag{36}$$

where

$$B^2 = -\frac{\partial^2}{\partial \phi^2} + \frac{9d^2 g}{\sin^2(3d\phi)}, \tag{37}$$

with $0 \le \phi \le \pi/3d$, and $d = 1,\ 2,\ 3$. Note that for $d = 1$, we obtain the famous Calogero problem, while for other d values, we get some new three-body problems. For $d = 2$, for instance, the potential in Eq. (36) is $V = 12gr^2 \sum_{i<j}^3 x_{ij}^{-2} y_{ij}^{-2}$. The angular Schrödinger equation is easily solved by following Calogero [1], yielding

$$B_l = 3d(l + \kappa), \qquad g = \kappa(\kappa - 1). \tag{38}$$

If d is even, then on running through the derivation in Refs. [1, 14], it is easily shown that if p_i and p'_i ($i = 1,\ 2,\ 3$) are the incoming and outgoing momenta, then

$$p'_i = -p_i, \qquad i = 1, 2, 3. \tag{39}$$

On the other hand, if d is odd then one has to introduce a symmetry operation T such that

$$Tr = r, \qquad T\phi = \frac{\pi}{3d} - \phi, \tag{40}$$

so that T transforms $0 \le \phi \le \pi/3d$ into itself. One can easily show that this T operator, when applied to the angular eigenfunctions Φ_l of the problem (see Eq. (2.17c) of Ref. [10]) yields

$$T\Phi_l = (-1)^l \Phi_l. \tag{41}$$

Following the steps in Refs. [10, 14], it is straightforward to prove that p'_i and p_i are related by

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ -a & b & 0 \\ b & 0 & -a \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \tag{42}$$

where

$$a = \frac{\sin(\frac{\pi}{3} - \frac{\pi}{3d})}{\sin(\pi/3)}, \qquad b = \frac{\sin(\pi/3d)}{\sin(\pi/3)}. \tag{43}$$

It is interesting to note that the relation between p'_i and p_i is similar to that in the translation case [14], except that a and b are different in the two cases. As expected, for $d = 1$, one recovers the known relation between p'_i and p_i for the Calogero model [1].

A further generalization of Eqs. (16) and (17) is also possible, and in that case B^2 is given by ($0 \le \delta \le \pi/6d$)

$$B^2 = -\frac{\partial^2}{\partial\phi^2} + \frac{9d^2 g}{\sin^2(3d\phi + 3d\delta)}. \tag{44}$$

One can now show that for even d, $p'_i = -p_i$, while for odd d, p'_i and p_i are related by Eq. (42), but where a and b are now given by

$$a = \frac{\sin(\frac{\pi}{3} - \frac{\pi}{3d} + 2\delta)}{\sin(\pi/3)}, \qquad b = \frac{\sin(\frac{\pi}{3d} - 2\delta)}{\sin(\pi/3)}. \tag{45}$$

For $\delta = \pi/6d$, we obtain $a = 1$, $b = 0$, so that $p'_1 = -p_2$, $p'_2 = -p_1$, $p'_3 = -p_3$ as in the CMW case [10].

Finally, one can also consider a generalization of the CMW problem characterized by Eqs. (36) and (37), but where

$$B^2 = -\frac{\partial^2}{\partial\phi^2} + 9d^2\left[\frac{g}{\sin^2 3d\phi} + \frac{f}{\cos^2 3d\phi}\right], \tag{46}$$

with $d = 1, 2, 3, \ldots$. In this case one can show that for every integral d, p_i and p'_i satisfy $p'_i = -p_i$ ($i = 1, 2, 3$). A further generalization consists in replacing ϕ by $\phi + \delta$ ($0 \le \delta \le \pi/6d$), and one can show that irrespective of the values of δ and d, $p'_i = -p_i$.

Thus one has obtained a wide class of new exactly solvable three-body problems where there is a simple but interesting relationship between p'_i and p_i. In all these cases, one can also add either the oscillator or the Coulomb-like potential [11], and the full bound state problem is exactly solvable in both the cases.

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