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COHERENT STATES FOR ISOSPECTRAL HAMILTONIANS

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Abstract

We show that for the strictly isospectral Hamiltonians, the corresponding coherent states are related by a unitary transformation. As an illustration, we discuss, the example of strictly isospectral one-dimensional harmonic oscillator Hamiltonians and the associated coherent states.

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In recent years extensive work has been done on the various aspects of coherent states.[1,2] The interest in coherent states is largely due to the fact that they provide an alternative set of basis vectors (non-orthogonal, overcomplete), they label the phase space of the system, and in some cases (the harmonic oscillator being the best known example) they have minimum fluctuations (allowed by the Heisenberg uncertainty principle) in the canonically conjugate variables, and hence are closest to the classical states. In recent years, there has been a lot of interest in constructing coherent states for potentials other than the harmonic oscillator, for example the Morse potential[3], the Coulomb potential[4], etc.

Very recently, coherent states have also been constructed[5] for the family of Hamiltonians which are strictly isospectral to the harmonic oscillator Hamiltonian[6,7]. Various approaches exit in the literature for the construction of such isospectral families, for example the factorization method[8], the Gelfand-Levitan method[9] and the approach of supersymmetry (SUSY) quantum mechanics[10], and all of them are essentially equivalent[11−13].

In this letter we address the following question: given a Hamiltonian whose coherent states are known, how does one construct coherent states for the corresponding strictly isospectral Hamiltonians. In particular we show that any two strictly isospectral Hamiltonians are related by a unitary transformation and as a consequence the corresponding coherent states are also related by the same unitary transformation. An explicit construction of the unitary operators is given. As an illustration, we discuss the specific example of strictly isospectral oscillator family, and show that a recent construction[5] of the coherent states for this family of potentials is not satisfactory. This is because these coherent states do not go over to the harmonic oscillator coherent states in the appropriate limit. In contrast, our coherent states do.

Consider the operators ($\hbar = m = 1$)

$$a \;=\; \frac{1}{\sqrt{2}}\left(\frac{d}{dx} \;+\; W(x)\right)\;,\; a^\dagger \;=\; \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} \;+\; W(x)\right)\;, \tag{1}$$

where $W(x)$ is an arbitrary function of x. It is well-known that the Hamiltonians $a^\dagger a$ and $aa^\dagger$ are SUSY partners. As a consequence, eigenvalues, eigenfunctions and S-matrices of the two Hamiltonians are related, and if the eigenfunctions of $a^\dagger a$ can be solved for, then, the eigenfunctions of $aa^\dagger$ can be obtained in terms of those of $a^\dagger a$.

Further it is also well-known that one can construct a family of strictly isospectral Hamiltonians $b^\dagger b$. Consider the operators

$$b \;=\; \frac{1}{\sqrt{2}}\left(\frac{d}{dx} \;+\; \hat{W}(x)\right) \;,\; b^\dagger \;=\; \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} \;+\; \hat{W}(x)\right) \;, \tag{2}$$

such that

$$bb^\dagger \;=\; aa^\dagger \;. \tag{3}$$

The condition (3) leads to a Riccati equation which can be solved to give

$$\hat{W}(x) \;=\; W(x) \;+\; \phi_\lambda(x) \;, \tag{4}$$

where

$$\phi_\lambda(x) \;=\; \psi_0^2(x) \;\left[\lambda \;+\; \int_{-\infty}^{x} dy\; \psi_0^2(y)\right]^{-1} \;, \tag{5}$$

where λ is a real number not lying in the closed interval $[-1,0]$ and $\psi_0(x)$ is the normalized ground state eigenfunction of the Hamiltonian $a^\dagger a$.

The eigenstates $|\theta_n>$ of the strictly isospectral family of Hamiltonians $H_\lambda = b^\dagger b + E_0$, ($E_0$ being the lowest energy eigenvalue) can be obtained in terms of the eigenstates $|\psi_n>$ of $H = a^\dagger a + E_0$, by noting the fact that

$$(b^\dagger b) b^\dagger a \;=\; b^\dagger a (a^\dagger a) \;. \tag{6}$$

Hence the normalized eigenstates of H_λ are

$$|\theta_n> = \;\frac{1}{E_n - E_0}\; b^\dagger a|\psi_n> \;,\; n = 1, 2, \ldots \tag{7}$$

The ground state of H_λ is determined from the condition $b|\theta_0>= 0$. Hence the normalized eigenfunctions of H_λ in the coordinate representation are given by

$$\begin{aligned}\theta_0(x) \;&=\; \sqrt{\lambda(\lambda+1)}\left[\lambda \;+\; \int_{-\infty}^{x} dy\; \psi_0^2(y)\right]^{-1} \psi_0(x) \;, \\ \theta_n(x) \;&=\; \psi_n(x) \;+\; \frac{1}{2(E_n - E_0)}\phi_\lambda(x)\left(\frac{d}{dx} + W(x)\right)\psi_n(x) \;,\; n = 1, 2, \ldots\end{aligned} \tag{8}$$

Let us now show that the Hamiltonians $H = a^\dagger a + E_0$ and $H_\lambda = b^\dagger b + E_0$ are related by a unitary transformation. The condition (3) implies that if one writes

$$b \;=\; aU^\dagger \;,\; b^\dagger \;=\; Ua^\dagger \;, \tag{9}$$

then

$$U^\dagger U \;=\; 1 \;. \tag{10}$$

We now define the operators

$$A \;=\; UaU^\dagger \;,\; A^\dagger \;=\; Ua^\dagger U^\dagger \;, \tag{11}$$

so that $b^\dagger b = A^\dagger A = Ua^\dagger aU^\dagger$ and hence we have the relation

$$H_\lambda \;=\; UHU^\dagger \;. \tag{12}$$

Further, the fact that the Hamiltonians H and H_λ are isospectral and diagonal in the orthonormal bases $|\psi_n>$ and $|\theta_n>$ respectively implies that

$$E_n \;=< \theta_n|H_\lambda|\theta_n> = <\psi_n|H|\psi_n> \;,\; n = 0, 1, \ldots \tag{13}$$

Hence on using the relation (12) one has the equality (upto a phase factor that can be taken to be unity), i.e.,

$$U^\dagger|\theta_n> = \;|\psi_n> \;\;. \tag{14}$$

The fact that the sets $|\psi_n>$ and $|\theta_n>$ are orthonormal implies that

$$<\psi_n|\psi_m> = <\theta_n|UU^\dagger|\theta_m> = \;\delta_{nm} \;, \tag{15}$$

and hence

$$UU^\dagger \;=\; 1 \;. \tag{16}$$

Thus we have shown that the strictly isospectral Hamiltonians and their respective eigenfunctions are related by a unitary transformation.

One way of defining coherent states for an arbitrary Hamiltonian is based on the dynamical symmetry group of the Hamiltonian[2]. Let G be the symmetry group of the Hamiltonian

H. Hence one can express H as a linear combination of the generators J_i of the Lie algebra of G. Thus

$$H \ = \ \sum_i \ d_i \ J_i \ , \tag{17}$$

where d_i are complex numbers and J_i are closed under the commutation relations

$$[J_i \ , \ J_j] \ = \ c_{ij}^k \ J_k \ . \tag{18}$$

Now, since H_λ is unitarily related to H, one has

$$H_\lambda \ = \ \sum_i \ d_i \ \tilde{J}_i \ , \ \tilde{J}_i \ = \ U J_i U^\dagger \ , \tag{19}$$

where $\tilde{J}_i$ obey the same Lie algebra (18) as do J_i, since the structure constants c_{ij}^k do not change under a unitary transformation. Thus H_λ has the same symmetry group as that of H. Let $D(z)$ be the element belonging to the coset space of G with respect to its maximally stability subgroup. Note that $D(z)$ is an operator function of the generators J_i. If the Perelomov coherent states associated with H and H_λ are defined respectively as

$$|z> = \ D(z) \ |\psi_0> \ ,$$

$$|z;\lambda> = \ D_\lambda(z) \ |\theta_0> \ , \tag{20}$$

then, as consequence of Eq.(19) one has

$$D_\lambda(z) = U D(z) U^\dagger \ , \tag{21}$$

and hence it follows from Eq.(14) that

$$|z;\lambda> = \ U \ |z> \ \ . \tag{22}$$

Thus we have shown that the Perelomov coherent states associated with the isospectral Hamiltonians H and H_λ are unitarily related.

The explicit structure of the unitary operator U can be easily obtained. Expanding U in the complete set of eigenstates of H and using Eq.(14) we have

$$U \ = \ \sum_{n,m=0}^{\infty} \ U_{nm} \ |\psi_n><\psi_m| \ , \ U_{nm} \ = <\psi_n|\theta_m> \ \ . \tag{23}$$

Using Eq.(8) one can write down explicit expressions for the matrix elements U_{nm}. Thus

$$
\begin{aligned}
U_{n0} &= \int dx\, \sqrt{\lambda(\lambda+1)} \left[\lambda \,+\, \int_{-\infty}^{x} dy\, \psi_0^2(y)\right]^{-1} \psi_n^*(x)\psi_0(x) \;, \\
U_{n,m+1} &= \delta_{n,m+1} + \frac{1}{2(E_{m+1}-E_0)} \int dx \psi_n^*(x)\phi_\lambda(x) \left(\frac{d}{dx} + W(x)\right) \psi_{m+1}(x) \;,\; n,m = 0,1,\ldots
\end{aligned} \tag{24}
$$

It is simple to see from Eqs.(5) and (24) that in the limit $|\lambda| \to \infty$ one has $\phi_\lambda(x) \to 0$ and hence $U^\dagger$, $U \to 1$. As a result $H_\lambda \to H$, and hence the eigenstates as well as the coherent states associated with H_λ reduce to those of H in this limit.

We next consider the specific example of the strictly isospectral oscillator Hamiltonians in order to illustrate the general arguments made in the foregoing. The harmonic oscillator is described by the Hamiltonian $H = a^\dagger a + \frac{1}{2}$ $(\hbar = m = \omega = 1)$ where

$$
a \;=\; \frac{1}{\sqrt{2}}\left(\frac{d}{dx} \,+\, x\right) \;,\; a^\dagger \;=\; \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} \,+\, x\right) \;, \tag{25}
$$

where the annihilation and creation operators a and $a^\dagger$ obey the usual commutation relations $[a, a^\dagger] = 1$. The isospectral oscillator family is then described by the Hamiltonian H_λ (see Eq.(12))

$$
H_\lambda \;=\; A^\dagger A + \frac{1}{2} \;,\; A \;=\; UaU^\dagger \;,\; A^\dagger \;=\; Ua^\dagger U^\dagger \;. \tag{26}
$$

The structure of the unitary operator U [see Eq.(24)] in this case will depend on the eigenvalues and eigenfunctions of the harmonic oscillator H. Note that A and $A^\dagger$ obey the same commutation relations as do a and $a^\dagger$, i.e., $[A, A^\dagger] = 1$. Thus A and $A^\dagger$ are indeed the annihilation and creation operators associated with the isospectral Hamiltonian H_λ. Let us define new canonically conjugate operators, namely

$$
\begin{gathered}
\hat{X} \;=\; \frac{1}{\sqrt{2}}(A^\dagger + A) \;,\; \hat{P} \;=\; \frac{i}{\sqrt{2}}(A^\dagger - A) \;, \\
[\hat{X}, \hat{P}] \;=\; i \;,
\end{gathered} \tag{27}
$$

so that the Hamiltonian H_λ can be written in the form

$$
H_\lambda \;=\; \frac{1}{2}(\hat{P}^2 \,+\, \hat{X}^2) \;. \tag{28}
$$

Note that $\hat{X}$ and $\hat{P}$ are related to the position and momentum operators $\hat{x}$ and $\hat{p}$ of the harmonic oscillator by the unitary transformation as in Eq.(26). Thus the family of isospectral oscillators can be viewed as harmonic oscillators but expressed in terms of appropriately transformed position and momentum operators. As a consequence, the coherent states associated with the isospectral oscillator family $|z;\lambda>$ may be defined as eigenstates of the annihilation operator A, namely

$$A\ |z;\lambda> = \ z\ |z;\lambda> \ , \tag{29}$$

where the eigenvalue z is λ-independent, or equivalently, in the Perelomov sense as the displaced ground state, namely

$$|z;\lambda> = \ D_\lambda(z)|\theta_0> \ , \ D_\lambda(z) \ = \ exp\{zA^\dagger - z^* A\} \ , \tag{30}$$

or, equivalently, as the state which has the minimum uncertainty product

$$\Delta X \Delta P \ = \ \frac{1}{2} \ , \tag{31}$$

(note that we have chosen $\hbar = m = \omega = 1$) with the uncertainties in X and P being equal. It must be noted that these coherent states are not minimum uncertainty states with respect to the position and momentum of the particle, viz., x and p. As is well known, only the Gaussian states minimize the product $\Delta x \Delta p$. As argued more generally in the foregoing the coherent states of the isospectral oscillator $|z;\lambda>$ are related to the harmonic oscillator coherent states by a unitary transformation as in Eq.(26).

One may also define a more general state which minimizes the uncertainty product $\Delta X \Delta P$, in analogy with the squeezed coherent state[14] of the usual harmonic oscillator, as follows:

$$|\xi, z;\lambda> = \ S_\lambda(\xi) D_\lambda(z)|\theta_0> \ , \ S_\lambda(\xi) \ = \ exp\{\frac{1}{2}\xi(A^\dagger)^2 - \frac{1}{2}\xi^* A^2\} \ , \tag{32}$$

where the displacement operator $D_\lambda(z)$ is as defined in Eq.(30). Note that in the state $|\xi, z;\lambda>$ the uncertainties ΔX and ΔP are unequal while the product is one-half. As a consequence of Eq.(26) it follows that the state $|\xi, z; \lambda>$ is related to the squeezed coherent state of the harmonic oscillator by a unitary transformation.

There has been some discussion in the literature[5,6] about what is the correct set of annihilation and creation operators, and the coherent states associated with the isospectral oscillator family. For example in Ref.6, annihilation and creation operators $(A_M, A_M^\dagger)$ are constructed. However they do not connect the ground state $|\theta_0>$ to $|\theta_n>$ $(n \geq 1)$. Further they do not reduce to the oscillator operators $(a, a^\dagger)$ in the limit $|\lambda| \to \infty$ but instead reduce to $(a^\dagger a^2,\ (a^\dagger)^2 a)$. In Ref.5, coherent states are constructed as the eigenstates of the annihilation operator A_M of Ref.6 and these consequently do not reduce to the harmonic oscillator coherent states in the limit $|\lambda| \to \infty$. We would like to emphasize that unlike Ref.6 our $(A, A^\dagger)$ defined by Eq.(27) are the correct set of annihilation and creation operators for the isospectral oscillator family, they act on *all* the eigenstates of the Hamiltonian H_λ, and they reduce to $(a, a^\dagger)$ in the limit $|\lambda| \to \infty$. Further the coherent states associated with H_λ also reduce, unlike in Ref.5, to the harmonic oscillator coherent states in this limit. The new coherent states $|z; \lambda>$ possess all the properties of the usual coherent states $|z>$ such as non-orthogonality, overcompleteness, etc., as these properties are invariant under a unitary transformation.

In conclusion, we have demonstrated that the strictly isospectral family of Hamiltonians are related bya unitary transformation, and argued that, as a consequence, the coherent states associated with these isospectral Hamiltonians are also related by the same transformation. We have given an explicit construction of the unitary transformation. We would like to remark that the conclusions of this letter are valid even in the case of n-parameter isospectral families of Hamiltonians[15].

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