N Fermion Ground State of Calogero-Sutherland type Models in Two and Higher Dimensions

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Abstract

I obtain the exact ground state of N-fermions in D-dimensions $(D \geq 2)$ in case the N particles are interacting via long-ranged two-body and three-body interactions and further they are also interacting via the harmonic oscillator potential. I also obtain the N-fermion ground state in case the oscillator potential is replaced by an N-body Coulomb-like interaction.

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Recently there is a revival of interest in the area of exactly solvable many body problems. A celebrated example of a solvable many-body system is the well known Calogero-Sutherland (CS) model in one dimension [1-3] which has found wide applications [4] in areas as diverse as quantum chaos and fractional statistics.In these models, one not only knows the exact N-boson as well as N-fermion ground states but also the complete excitation spectrum. What about two and higher dimensional many-body problems ? So far as I am aware off, the only N-body problem which is exactly solvable in D-dimensions ($D \geq 2$) is that of N-bosons or N-fermions experiencing pairwise (or one body) harmonic interaction. Apart from this example, there exist several N-body problems [5-10] for which the N-boson ground state as well as radial excitation spectrum over it is analytically known. In some of these cases, a class of N-fermion excited states are also known analytically. However, to the best of my knowledge, apart from the oscillator potential, there exist no other many-body problem in D-dimensions for which N-fermion ground state is exactly known.

In this note, I obtain the exact ground state of N-fermions in D-dimensions ($D \geq 2$) in case the N-particles are interacting via long ranged two-body and three-body interactions and further they are also interacting via an external harmonic oscillator potential. As a by-product, I also discuss its relevance in the context of the N-anyon ground state. Finally, I also obtain the N-fermion ground state in case the harmonic oscillator potential is replaced by an N-body Coulomb-like potential [8].

Soon after the seminal work of Calogero [1], it was shown by Calogero and Marchioro [5] that the N-boson ground state and radial excitations over it can be obtained in the case of a three-dimensional N-body problem with two-body inverse square interaction provided one also adds a long ranged three-body interaction which is not present in one dimension. Recently we [7] have generalized the Calogero-Marchioro result to arbitrary number of dimensions and also considered the Sutherland variant of the model. The model considered by us in D-dimensions is ($D \geq 2$) [7]

$$H = -\frac{1}{2}\sum_{i=1}^{N} \nabla_i^2 + g\sum_{i<j}^{N} \frac{1}{r_{ij}^2} + G \sum_{i<j,i\neq k,j\neq k}^{N} \frac{\mathbf{r}_{ki} \cdot \mathbf{r}_{kj}}{r_{ki}^2 r_{kj}^2} + \frac{1}{2}\sum_{i=1}^{N} r_i^2 \tag{1}$$

where we have set $\hbar = m = \omega = 1$ [11], $\mathbf{r}_i$ is the position of the i'th particle, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ denotes the relative separation of the i'th and j'th particles, while g and G are the strengths of the two and three-body interactions, respectively.

Unlike the N-boson ground state, it is in general far more difficult to obtain the N-fermion ground state. This is because, in view of the Pauli exclusion principle, the N-fermion ground state energy in D-dimensions, depends very sensitively on the value of N and D. As a result, even in the simplest case of N-fermions in an oscillator potential, no general expression can be written for the ground state energy even though, for any given N and D, one can immediately give its value.

To obtain the N-fermion ground state of the system as given by eq. (1), we start with the ansatz

$$\psi = \prod_{i<j}^{N} \mid \mathbf{r}_i - \mathbf{r}_j \mid^{\Lambda_D^f} \psi_1 \,. \tag{2}$$

On substituting this ansatz in the Schrödinger equation $H\psi = E\psi$ with H as given by eq. (1), it is easily shown that ψ_1 satisfies the equation

$$-\frac{1}{2}\sum_{i=1}^{N} \nabla_i^2 \psi_1 - \Lambda_D^f \sum_{i<j}^{N} \frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot(\nabla_i \psi_1 - \nabla_j \psi_1)}{r_{ij}^2} + [V - E + 2\Lambda_D^f \sum_{i<j}^{N} \frac{1}{r_{ij}^2}]\psi_1 = 0 \tag{3}$$

provided g and G are related to Λ_D^f by

$$\Lambda_D^f = \sqrt{G} = \frac{1}{2}[\sqrt{D^2 + 4g} - D] \,. \tag{4}$$

Note that in this particular case $V = \frac{1}{2}\sum_i r_i^2$. Now, since we want the N-fermion ground state, let us further substitute the ansatz

$$\psi_1 = \psi_S \phi(t \equiv \sum_{i=1}^{N} r_i^2) \tag{5}$$

where ψ_S is the N x N Slater determinant which is the N free-fermion ground state state wave function. For example, one of the free 3-fermion ground state wave function in $d \geq 3$ (while it is the unique one in D = 2) is given by

$$\psi_S^{N=3} = [(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3)] \,. \tag{6}$$

On substituting the ansatz (5) in eq. (3) we find that ϕ satisfies the equation

$$t\phi''(t) + \left[\frac{N(N-1)}{2}\Lambda_D^f + e_0^f\right]\phi'(t) + \frac{1}{2}(E - V)\phi(t) = 0 \tag{7}$$

where e_0^f is the ground state energy (including the center of mass) of N-fermions in an oscillator potential. For example, in the case of 3 particles (N=3), using eq. (6) it is easy to verify eq. (7) and show that

$$e_0^f(N=3, D \geq 2) = \frac{3D}{2} + 2\,. \tag{8}$$

In the case of the oscillator potential ($V = \frac{1}{2}t$), it is easily shown that the exact solution to eq. (7) is

$$\phi(t) = e^{-t/2}\; L_n^{\Gamma_D^f}(t) \tag{9}$$

while the corresponding energy is

$$E_n^f = \left[2n + \frac{N(N-1)}{2}\Lambda_D^f + e_0^f\right] \tag{10}$$

which gives us the exact N-fermion ground state (n=0) as well as the radial excitation spectrum over it. Here Γ_D^f is given by

$$\Gamma_D^f = \left[\frac{N(N-1)}{2}\Lambda_D^f + e_0^f - 1\right]. \tag{11}$$

Several comments are in order at this stage.

1. It is worth noting that whereas the exact N-fermion ground state is obtained when $g = G + D\sqrt{G}$ (see eq. (4)), the exact N-boson ground state is obtained only in case $g = G+(D-2)\sqrt{G}$ [7]. The corresponding ground state energies and eigenfunctions are

$$E_0^f = \frac{N(N-1)}{2}\Lambda_D^f + e_0^f \tag{12}$$

$$\psi_0^f = \prod_{i<j}^{N} \mid \mathbf{r}_i - \mathbf{r}_j \mid^{\Lambda_D^f} \psi_S exp(-\frac{1}{2}\sum_i r_i^2) \tag{13}$$

$$E_0^b = \frac{N(N-1)}{2}\Lambda_D^b + \frac{ND}{2} \tag{14}$$

$$\psi_0^b = \prod_{i<j}^{N} \mid \mathbf{r}_i - \mathbf{r}_j \mid^{\Lambda_D^b} exp(-\frac{1}{2}\sum_{i=1}^{N} r_i^2) \tag{15}$$

where

$$\Lambda_D^b = \sqrt{G} = \frac{1}{2}[\sqrt{(D-2)^2+4g} - (D-2)]\,. \tag{16}$$

Thus, as yet we do not have a model (apart from the pure harmonic oscillator) in which the exact N-boson as well as N-fermion ground state energies can be simultaneously obtained (for the same value of the parameters) in a given dimension D. Instead, what we find is that if for a given set of parameters, the ground state energy of N-bosons in potential (1) can be obtained in D dimensions, then for the *same* set of parameters one can obtain the ground state energy of N-fermions but in $D-2$ dimensions. Conversely, if one can obtain the ground state energy of N-fermions in D-dimensions experiencing potential (1), then for the same set of parameters, one can also obtain the ground state energy of N-bosons in $D+2$ dimensions. It is worth pointing out that long time ago, a similar connection was obtained by Parisi and Sourlas [12] between the critical exponents of the random field model (which is one of the simplest disordered system) in D-dimensions and the critical exponents of the same model in the absence of any disorder but in $D-2$ dimensions. Similarly, they also showed [13] that the branched polymers in D-dimensions belong to the same universality class as the Lee-Yang edge singularity in $D-2$ dimensions.

2. The fact that eq. (12) indeed represents the ground state energy of N-fermions can be shown as follows. Firstly, for $g = G = 0 = \Lambda_D^f$ (i.e. pure oscillator potential), obviously e_0^f is the ground state energy. Further, one can define the supersymmetric charges

$$Q_{x_i} = p_{x_i} - ix_i + i\Lambda_D^f \sum_{j(>i)} \frac{x_i - x_j}{r_{ij}^2} + i\frac{\partial}{\partial x_i}(log\psi_S) \tag{17}$$

$$Q_{y_i} = p_{y_i} - iy_i + i\Lambda_D^f \sum_{j(>i)} \frac{y_i - y_j}{r_{ij}^2} + i\frac{\partial}{\partial y_i}(log\psi_S) \tag{18}$$

etc. and their Hermitian conjugates and show that the H as given by eq. (1) can be written as

$$H = \frac{1}{2}\sum_{i=1}^{N}(Q_{x_i}^+ Q_{x_i} + Q_{y_i}^+ Q_{u_i} + ...) + E_0^F \tag{19}$$

where E_0^f is as given by eq. (12) while g and G are related to Λ_D^f by eq. (4). Further, one can show that the Q's annihilate the N-fermion ground state as given by eq. (13). Clearly, since the operator on the right hand side of eq. (19) is positive definite and it annihilates the ground state wave function, hence E_0^f must be the N-fermion ground state energy corresponding to the Hamiltonian (1).

3. For large N, how does the N-fermion ground state energy behaves as a function of N ? This is easily answered by noting that for large N, e_0^f in D-dimensions is given by

$$e_0^f = \frac{D}{D+1}(D!)^{1/D} N^{\frac{D+1}{D}} + 0(N^{\frac{D-1}{D}})\,. \tag{20}$$

Hence for any $D(\geq 2)$, and for large N, the ground state energy of N-fermions goes like

$$E_0^f \stackrel{N\to\infty}{\longrightarrow} \frac{N^2}{2}\Lambda_D^f + \frac{D}{D+1}(D!)^{1/D} N^{\frac{D+1}{D}} - \frac{N}{2}\Lambda_D^f + 0(N^{\frac{D-1}{D}}) \tag{21}$$

In particular, note that for large N, the ground state energy of N-fermions is maximum in $D = 2$ and monotonically decreases with the number of dimensions. On the other hand, the N-boson ground state energy as given by eq. (14) is least in two dimensions and monotonically increases with the number of dimensions.

4. The bosonic excited state spectrum is easily obtained by running through eqs. (2) to (9) putting $\psi_S = 1$ and replacing Λ_D^f by Λ_D^b as given by eq. (16). One finds that the bosonic radial eigenstates are given by

$$E_n = \left[2n + \frac{N(N-1)}{2}\Lambda_D^b + \frac{ND}{2}\right] \tag{22}$$

$$\psi_n = \prod_{i<j}^{N} \mid \mathbf{r}_i - \mathbf{r}_j \mid^{\Lambda_D^b} exp(-\frac{1}{2}\sum_{i=1}^{N} r_i^2) L_n^{\Gamma_D^b} \tag{23}$$

where

$$\Gamma_D^b = \frac{N(N-1)}{2}\Lambda_D^b + \frac{ND}{2} - 1 \tag{24}$$

I would now like to show that apart from the oscillator potential, the exact N-fermion ground state energy can also be obtained in case one replaces the oscillator potential in the Hamiltonian (1) by the N-body potential

$$V(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = -\frac{\alpha}{\sqrt{\sum_{i=1}^{N} r_i^2}} \tag{25}$$

To this end note that the steps as given by eqs. (2) to (7) are independent of the form of the potential. On substituting $t(=\sum_{i=1}^{N} r_i^2) = \rho^2$ in eq. (7), we find that in the case of the potential (23) it takes the form

$$\phi''(\rho) + \frac{1}{\rho}[N(N-1)\Lambda_D^f + 2e_D^f - 1]\phi'(\rho) + 2(\frac{\alpha}{\rho} - \mid E \mid) = 0 \tag{26}$$

where e_0^f as before is the ground state energy of N-fermions in D-dimensions in an oscillator potential.. It is easily shown shown that the solution to this equation is

$$\phi(\rho) = e^{-\sqrt{2|E|}\rho} L_n^{2\Gamma_D^f}(2\sqrt{2 \mid E \mid}\rho) \tag{27}$$

$$E_n = -\frac{\alpha^2}{2\left[n + \frac{N(N-1)}{2}\Lambda_D^f + e_0^f - \frac{1}{2}\right]^2}, \; n = 0, 1, 2, ... \tag{28}$$

where Γ_D^f is as given by eq. (11). For $g = G = 0$ (i.e. $\Lambda_D^f = 0$), this gives us the exact N-fermion ground state and radial excitations over it in the potential (25). This then provides another instance where exact fermionic ground state can be obtained in the case of both the oscillator and the N-body potential (25) thereby providing one more support to the conjecture that whenever an exact eigenstate can be obtained in the oscillator potential, similar exact eigenstate can also be obtained in the case of the N-body potential as given by (25) [8].

All these results are easily extended to the Calogero variant i.e. by replacing the one body potential $\sum_{i=1}^{N} r_i^2$ by $\frac{1}{2}\sum_{i<j}^{N} r_{ij}^2$ in Hamiltonian (1) as well as in eq. (25). Everything goes through except for the fact that now the energy is that of N-fermions minus the center of mass energy ($=D/2$). For example, it is easily shown that the N-fermion ground state and radial excitations over it are given by

$$\hat{\psi}_n = \prod_{i<j}^{N} \mid \mathbf{r}_i - \mathbf{r}_j \mid^{\Lambda_D^f} \psi_S exp(-\frac{1}{2}\frac{1}{\sqrt{2N}}\sum r_{ij}^2) L_n^{\Gamma_D^f - \frac{D}{2}}(\frac{1}{\sqrt{2N}}\sum r_{ij}^2) \tag{29}$$

$$\hat{E}_n^f == \sqrt{\frac{N}{2}}\Big[2n + \frac{N(N-1)}{2}\Lambda_D^f + e_0^f - \frac{D}{2}\Big]\,. \tag{30}$$

It is worth pointing out that in the special case of three dimensions ($D = 3$), Calogero and Marchioro [5] had obtained these results long time ago [5] in case N = 3,4. However, on comparing their Eqs. (12) to (18) with our results, we find that there are misprints/mistakes in their expressions. For example, according to them the 3-fermion energy is $\sqrt{\frac{3}{2}}(2n + 3\Lambda_{D=3}^f + 4)$ while the correct expression should be $\sqrt{\frac{3}{2}}(2n + 3\Lambda_{D=3}^f + 5)$ as seen from eq. (30). The fact that our expression is correct is easily seen by noting that in the limit of $g = G = 0$ (hence $\Lambda_D^f = 0$), the ground state energy of 3-fermions (in an oscillator potential) in 3-dimensions is known to be 5 (and not 4) after center of mass has been taken out. Similarly, according to them $\Gamma_{D=3}^f(N = 3) - \frac{3}{2} = 3\Lambda_{D=3}^f + 3$ while the correct expression is $3\Lambda_{D=3}^f + 4$ as is easily verified by using eqs. (11) and (29). Similarly, their 4-fermion energy and eigenfunctions are not correct. As is clear from eq. (30), the correct expression for energy is $\sqrt{2}(2n + 6\Lambda_{D=3}^F + 15/2)$ and not $\sqrt{2}(2n + 6\Lambda_{D=3}^F + 6)$ as claimed by them. Note that the ground state energy of 4-fermions in 3-dimensions in pure oscillator potential is 15/2 (and not 6) after the center of mass has been taken out. Similarly, it may be noted that $\Gamma_{D=3}^f(N = 4) - \frac{3}{2} = 6\Lambda_{D=3}^f + \frac{13}{2}$ (see eq. (29)) and not $6\Lambda_{D=3}^f + 5$ as claimed by them.

The results obtained in this paper have direct relevance to the problem of N-anyons in an oscillator potential [10,14]. For example, one can now understand as to why N-anyon linear state starting from the N-fermion ground has not been analytically obtained so far. To see this, let us recall that the N-anyon Hamiltonian in the oscillator potential is related to the Hamiltonian (1) at $g = G(= \alpha^2$ say) by

$$H_{anyon} = H + \alpha \sum_{j>i=1}^{N} \frac{l_{ij}}{r_{ij}^2} \tag{31}$$

where $l_{ij} = \mathbf{r}_{ij} \times \mathbf{p}_{ij}$, is the angular momentum operator. However, as is clear from eqs. (16) and (4), at $g = G$, only N-boson ground state can be obtained exactly but **not** the N-fermion ground state (it may be noted that anyons can only exist in two dimensions). In this context it may also be noted that N-bosons, in their ground state, always have zero angular momentum and hence the noninteracting N-anyon ground state starting from the N-boson ground state is identical with the interacting N-boson ground state in the Calogero-Marchioro model (with $g = G$).

What about the noninteracting N-anyon ground starting from the N-fermion ground state ? Using eq. (4) it is clear that it cannot be a linear state since such a linear state is possible if $g \neq G$ while in the anyon model g is always equal to G.

However, we can use the exact N-fermion ground state obtained in this paper to get some information about the nature of the N-anyon ground state in an oscillator potential. In particular, note that the N-anyon Hamiltonian is related to the Hamiltonian (1) at $G = \alpha^2, g = \alpha^2 + 2\alpha$ by

$$H_{anyon} = H - \alpha \sum_{j>i=1}^{N} \frac{l_{ij}}{r_{ij}^2} - 2\alpha \sum_{i<j}^{N} \frac{1}{r_{ij}^2} \, . \tag{32}$$

Now what one could do is to treat second and third terms on the right hand side of eq. (32) as perturbation. In that case, to the zeroth order, there is an exact linear 3-anyon state starting from the 3-fermion ground state. One now has to calculate the effect of the last two terms by using perturbation theory and study the nature of crossing with the exact N-anyon state starting from the N-boson ground state. We hope to study this question in the near future.

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