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Analytically Solvable PT-Invariant Periodic Potentials

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Abstract: Associated Lamé potentials $V(x) = a(a+1)m\mathrm{sn}^2(x,m) + b(b+1)m\mathrm{cn}^2(x,m)/\mathrm{dn}^2(x,m)$ are used to construct complex, PT-invariant, periodic potentials using the anti-isospectral transformation $x \to ix + \beta$, where β is any nonzero real number. These PT-invariant potentials are defined by $V^{PT}(x) \equiv -V(ix+\beta)$, and have a different real period from $V(x)$. They are analytically solvable potentials with a finite number of band gaps, when a and b are integers. Explicit expressions for the band edges of some of these potentials are given. For the special case of the complex potential $V^{PT}(x) = -2m\mathrm{sn}^2(ix+\beta,m)$, we also analytically obtain the dispersion relation. Additional new, solvable, complex, PT-invariant, periodic potentials are obtained by applying the techniques of supersymmetric quantum mechanics.

In the past few years, Bender and others [1, 2] have looked at several complex potentials with PT-symmetry and have shown that the energy eigenvalues are real when PT-symmetry is unbroken, whereas they come in complex conjugate pairs when PT-symmetry is spontaneously broken. Recently, Mostafazadeh [3] has clarified this issue by showing that there is a more general setting of pseudo-hermiticity (of which PT-symmetry forms a special case) in which eigenvalues are either real or occur in complex conjugate pairs. Scattering problems with complex, PT-invariant potentials have also been investigated [4, 5]. However, there have been very few papers discussing periodic potentials with PT-symmetry. Only two types of PT-invariant, periodic potentials have been considered in detail in the literature, namely, $i\sin^{2N+1}(x)$ [6] and delta function potentials with complex couplings [7, 8]. These potentials have been shown to possess real band spectra with an infinite number of band gaps. It might be noted here that obtaining these results required extensive numerical analysis and no analytic results for band edge energies were possible [9]. The purpose of this letter is to construct and study several new classes of analytically solvable, complex, PT-invariant, periodic potentials with a finite number of band gaps. Our approach will consist of (i) making use of the anti-isospectral transformation $x \to ix + \beta$ [10], and (ii) constructing supersymmetric partner potentials from techniques developed in supersymmetric quantum mechanics [11].

Anti-Isospectral Transformations: We begin with the simple observation that if $\psi(x)$ is a solution of the Schrödinger equation for the real potential $V(x)$ with energy E, then $\psi(ix+\beta)$ is a solution of the Schrödinger equation for the complex potential $-V(ix+\beta)$ with energy $-E$, where β is an arbitrary constant. The new potential $-V(ix+\beta)$, generated by the anti-isospectral transformation $x \to ix+\beta$, is clearly PT-symmetric and will be denoted by $V^{PT}(x)$. Further, if $\psi(x)$ and $\psi(ix+\beta)$ satisfy appropriate boundary conditions, they are eigenfunctions of $V(x)$ and $V^{PT}(x)$ respectively. Since the ordering of energy levels for $V^{PT}(x)$ is the opposite of the ordering of energy levels for $V(x)$, this is presumably why the transformation $x \to ix+\beta$ is called "anti-isospectral".

In this letter, we focus on periodic potentials by choosing $V(x)$ to be the associated Lamé potential

$$V(x) = a(a+1)m\,\mathrm{sn}^2(x,m) + b(b+1)m\frac{\mathrm{cn}^2(x,m)}{\mathrm{dn}^2(x,m)}\,, \tag{1}$$

which has a real period $2K(m)$. Note that if either a or b is zero, this potential is called the Lamé potential. Recall that when a and b are non-negative integers, the associated Lamé potential has many analytically solvable eigenstates and only a finite number of band gaps [12, 13]. Here, $\mathrm{sn}\,(x,\mathrm{m}\,)$, $\mathrm{cn}\,(x,\mathrm{m}\,)$, $\mathrm{dn}\,(x,\mathrm{m}\,)$ are Jacobi elliptic functions with elliptic modulus parameter m $(0 \leq \mathrm{m}\ \leq 1)$. They are doubly periodic functions with periods $[4K(\mathrm{m}\,), i2K'(\mathrm{m}\,)]$, $[4K(\mathrm{m}\,), 2K(\mathrm{m}\,) + i2K'(\mathrm{m}\,)]$, $[2K(\mathrm{m}\,), i4K'(\mathrm{m}\,)]$ respectively [14, 15], where $K(\mathrm{m}\,) \equiv \int_0^{\pi/2} d\theta [1 - \mathrm{m}\,\sin^2\theta]^{-1/2}$ denotes the complete elliptic integral of the first kind, and $K'(\mathrm{m}\,) \equiv K(1-\mathrm{m}\,)$. The complex, PT-invariant potential obtained via an anti-isospectral transformation applied to eq. (1) is

$$V^{PT}(x) = -a(a+1)m\mathrm{sn}^2(ix+\beta,m) - b(b+1)m\frac{\mathrm{cn}^2(ix+\beta,m)}{\mathrm{dn}^2(ix+\beta,m)}\,. \tag{2}$$

It is also periodic with a different real period $2K'(m)$. Furthermore, it is analytically solvable with a finite number of band gaps. It is important to understand that the key point for obtaining the above results is that unlike trigonometric and other periodic functions, Jacobi elliptic functions are doubly periodic functions. This allows both $V(x)$ and $V^{PT}(x)$ to be simultaneously periodic, even though the periods are different. Note that the arbitrary nonzero constant β in the anti-isospectral transformation, $x \to ix + \beta$, is chosen so as to avoid the singularities of Jacobi elliptic functions [14].

Let us first apply our approach to the Lamé potentials $(b = 0)$

$$V(x) = a(a+1)m\mathrm{sn}^2(x,m)\,, \quad a = 1,2,3,...\,, \tag{3}$$

which are known to have $2a+1$ eigenstates (band edges) and a band gaps. Let $E_j(m)$ and $\psi_j(x,m)$ with $j = 0,1,\ldots,2a$ denote the band edge energies and wave functions. The anti-isospectral transformation $x \to ix + \beta$ [10] yields the PT-invariant potential

$$V^{PT}(x) = -a(a+1)m\mathrm{sn}^2(ix+\beta,m)\,, \quad a = 1,2,3,...\,, \tag{4}$$

with real period $2K'(m)$. The band-edge eigenvalues and eigenfunctions of $V^{PT}(x)$ are related to those of the Lamé potential (3) by

$$E_j^{PT}(m) = -E_{2a-j}(m)\,, \quad \psi_j^{PT}(x,m) \;\propto\; \psi_{2a-j}(ix+\beta,m)\;, \quad j = 0,1,...,2a\,. \tag{5}$$

Thus, the PT-invariant, periodic potential (4) also has precisely a band gaps and $2a+1$ band edges at energies given by eq. (5). Special mention should be made of the remarkable fact that for any integer a, all bands and band gaps exchange their role as one goes from a Lamé potential to its PT-invariant version $V^{PT}(x)$.

For any band structure problem, an important quantity is the discriminant Δ [16] which gives information about the number of band gaps as well as their widths. The question is whether one can relate the discriminant for the potential $V^{PT}(x)$ with the discriminant Δ for the corresponding Lamé potential. Unfortunately, this is not directly possible by using eq. (5), since it only relates energies of states with different numbers of nodes. However, we now derive a remarkable relation using which we can relate the two discriminants Δ and Δ^{PT}.

We start from the Schrödinger equation for the Lamé potential (3)

$$-\psi''(x) + a(a+1)m\,\mathrm{sn}^2(x,m)\psi(x) = E(m)\psi(x)\,, \tag{6}$$

where a prime denotes a derivative with respect to the argument. On using the relation [14, 15]

$$\sqrt{m}\,\mathrm{sn}(x,m) = -\mathrm{dn}[ix + K'(m) + iK(m), 1-m]\,, \tag{7}$$

and then defining a new variable $y = ix + K'(m) + iK(m)$, the Schrödinger eq. (6) takes the form

$$-\psi''(y) + a(a+1)(1-m)\mathrm{sn}^2(y,1-m)\psi(y) = [a(a+1) - E(m)]\psi(y)\,, \tag{8}$$

so that for the Lamé potentials (3) we obtain the remarkable relations

$$E_j(m) = a(a+1) - E_{2a-j}(1-m)\,,\ \psi_j(x,m) \propto \psi_{2a-j}(ix + K'(m) + iK(m), 1-m), \quad j = 0,1,...,2a. \tag{9}$$

In passing, note that for the special choice $m = 1/2$, one has several interesting relations:

$$E_j(m=1/2) + E_{2a-j}(m=1/2) = a(a+1)\,, \quad E_a(m=1/2) = a(a+1)/2\,. \tag{10}$$

On combining eqs. (5) and (9) we obtain

$$E_j^{PT}(m) = E_j(1-m) - a(a+1)\,, \quad j = 0,1,...,2a\,, \tag{11}$$

and hence the corresponding discriminants are related by

$$\Delta^{PT}(E,m) = \Delta[E + a(a+1), 1-m]\,. \tag{12}$$

As an illustration, in Figure 1 we plot the real and imaginary parts of the PT-invariant, complex potential $V^{PT}(x) = -12m\mathrm{sn}^2(ix+\beta, m)$. Using the well known results for the Lamé potential with $a = 3$ [12] and eq. (5), the ground state (lowest band edge) eigenvalue and eigenfunction is

$$\psi_g(x) = \mathrm{sn}(ix+\beta, m)[2+2m-\delta_3 - 5m\mathrm{sn}^2(ix+\beta, m)] \ , \ E_g = -5-5m-2\delta_3 \,, \tag{13}$$

where $\delta_3 \equiv \sqrt{4-7m+4m^2}$. In Table I we have given all the seven band edge eigenvalues and eigenfunctions. We have subtracted off the ground state energy from the potential so that the lowest band edge by construction is at zero energy. Observe from the table that the band edges are both periodic as well as anti-periodic with periods $2K'(m)$ and $4K'(m)$ respectively.

For the special case of the Lamé potential with $a = 1$, the dispersion relation is also analytically known [17]. We now obtain the dispersion relation for the corresponding PT-invariant potential $V^{PT}(x) = -2m\mathrm{sn}^2(ix+\beta, m)$. To that end, we start from the Schrödinger equation:

$$-\psi''(x) + [1+m-2m\mathrm{sn}^2(ix+\beta, m)]\psi(x) = E\psi(x) \,, \tag{14}$$

where we have subtracted the ground state energy $E_g = -1-m$ from the potential so that the new potential $[V^{PT}]_-(x) = V^{PT}(x) - E_g$ has zero ground state energy. On substituting $y = ix+\beta$, eq. (14) takes the form

$$-\psi''(y) + [-m+2m\mathrm{sn}^2(y, m)]\psi(y) = (1-E)\psi(y) \,. \tag{15}$$

Now it is well known that two independent solutions of this equation are given by [17]

$$\psi(x) = \frac{H(ix+\beta\pm\alpha_1)\exp[\mp(ix+\beta)Z(\alpha_1)]}{\theta(ix+\beta)} \,, \tag{16}$$

where H, θ, Z are the Jacobi eta, theta and zeta functions, while α_1 is related to the energy E of eq. (15) by

$$E = m\mathrm{sn}^2(\alpha_1, m) \,. \tag{17}$$

On using the Bloch condition and the fact that while $\theta(ix+\beta)$ is a periodic function with period $2K'(m)$, $H(ix+\beta)$ is only quasi-periodic [17], i.e.

$$H(i[x+2K'(m)]+\beta) = H(ix+\beta)\exp[-\pi K'(m)/K(m)] \,, \tag{18}$$

it is easily shown that the complex, PT-invariant potential $[V^{PT}]_-(x)$ has a dispersion relation given by

$$k = \mp \frac{\pi}{2K'(m)} \pm iZ(\alpha_1) + i\frac{\pi}{2K(m)}\,, \tag{19}$$

where α_1 is given by eq. (17).

We now turn to the associated Lamé potentials of eq. (1), where without loss of generality we consider $a > b$ with both being positive integers. We shall later comment about the case $a = b$. As has been shown by us [12, 13], these are also exactly solvable problems with precisely a band gaps and $2a+1$ band edges. However, in many of these cases, some of the bands are unusual in that both the band edges have the same period, since some band gaps vanish. In this sense, the associated Lamé potentials are much richer than the Lamé potentials. On using the anti-isospectral transformation, it is easy to see that the band edges of the potential (1) and its PT-invariant counterpart

$$V^{PT}(x) = -a(a+1)m\mathrm{sn}^2(ix+\beta,m) - b(b+1)m\frac{\mathrm{cn}^2(ix+\beta,m)}{\mathrm{dn}^2(ix+\beta,m)}\,, \tag{20}$$

are again connected by the relation (5). However, we are unable to relate the discriminants of the two potentials since we have not been able to derive an analogue of the relation (9). As an illustration, let us consider the (a=2, b=1) associated Lamé potential and its corresponding PT-invariant potential $V^{PT}(x) = -6m\mathrm{sn}^2(ix+\beta,m) - 2m\mathrm{cn}^2(ix+\beta,m)/\mathrm{dn}^2(ix+\beta,m)$. The ground state eigenvalue and eigenfunction is given by

$$\psi_g(x) = \frac{\mathrm{cn}(ix+\beta,m}{\mathrm{dn}(ix+\beta,m)}[3m\mathrm{sn}^2(ix+\beta,m) - 2 + \sqrt{4-3m}]\,, \quad E_g = -5 - m - 2\sqrt{4-3m}\,. \tag{21}$$

In Table 2 we have given all the band edge eigenvalues and eigenfunctions.

Finally, let us discuss the associated Lamé potentials (1) for the case $a = b =$ integer. In view of the well known Landen transformation formula [14, 18]

$$\mathrm{dn}(x,m) + \mathrm{dn}(x+K(m),m) = \frac{1}{\alpha}\mathrm{dn}\left[\frac{x}{\alpha},\tilde{m}\right]\,, \quad \alpha = \frac{1}{1+\sqrt{1-m}}\,, \quad \tilde{m} = \left[\frac{1-\sqrt{1-m}}{1+\sqrt{1-m}}\right]^2\,, \tag{22}$$

the associated Lamé potentials with $a = b$ can be rewritten, apart from an overall constant as Lamé potentials and so the results derived above for the Lamé potentials will go through with a modified modulus parameter $\tilde{m}$.

Supersymmetric Partner Potentials: Additional analytically solvable finite band gap potentials can be obtained from our previous results by using supersymmetry. The procedure is standard [11]. Consider a periodic potential $V_-(x)$ whose ground state energy is zero, eigenvalues are $E_n^{(-)}$ and eigenfunctions (band edges) are $\psi_n^{(-)}(x)$. Let the ground state wave function be denoted by $\psi_g(x) \equiv \psi_0^{(-)}(x)$. One constructs the superpotential $W(x) = -\psi_g'(x)/\psi_g(x)$. The original potential and its supersymmetric partner potential are then given by $V_\pm(x) = W^2(x) \pm W'(x)$. The eigenvalues are the same for both potentials and their un-normalized eigenfunctions are related by

$$\psi_0^{(+)}(x) \propto 1/\psi_0^{(-)}(x) \quad , \quad \psi_n^{(+)}(x) \propto \left[\frac{d}{dx} + W(x)\right] \psi_n^{(-)}(x) \quad , \quad n \geq 1 \ . \tag{23}$$

This technique is immediately applicable to PT-invariant potentials $V^{PT}(x)$ of the type given in eq. (20) from which the ground state energy has been subtracted. The only caution to keep in mind is that the ground state of $V^{PT}(x)$ corresponds to the highest eigenstate of $V(x)$ as indicated by eq. (5).

Let us apply the above formalism to the Lamé potentials. First, for the special case $a = 1$, one has

$$V_-(x) = 2m\mathrm{sn}^2(x,m) - m \quad , \quad [V^{PT}]_-(x) = -2m\mathrm{sn}^2(ix+\beta,m) + m + 1 \quad , \quad \psi_g(x) = \mathrm{sn}(ix+\beta,m) \ ,$$

$$W^{PT}(x) = -i\frac{\mathrm{cn}(ix+\beta,m)\ \mathrm{dn}(ix+\beta,m)}{\mathrm{sn}(ix+\beta,m)}, \ [V^{PT}]_+(x) = -2m\mathrm{sn}^2(ix+\beta+iK'(m),m) + m + 1. \tag{24}$$

Here, the result of invoking supersymmetry is basically a translation of the independent variable in $[V^{PT}]_-(x)$. Such potentials are usually called self-isospectral potentials [19].

For higher a values, the two supersymmetric partner potentials are quite different in shape from each other [12], even though they both have the same band edge eigenvalues. Let us now explicitly consider the case $a = 3$. Here, using the ground state eigenfunction of the PT-invariant potential $V^{PT}(x) = -12m\mathrm{sn}^2(ix+\beta,m) - E_g$, where E_g is given in eq. (13), we find that the corresponding superpotential is

$$W^{PT}(x) = -i\frac{\mathrm{cn}(ix+\beta,m)\mathrm{dn}(ix+\beta,m)}{\mathrm{sn}(ix+\beta,m)} + 10im\frac{\mathrm{cn}(ix+\beta,m)\mathrm{sn}(ix+\beta,m)\mathrm{dn}(ix+\beta,m)}{[2+2m-\delta_3-5m\mathrm{sn}^2(ix+\beta,m)]} \ , \tag{25}$$

so that the supersymmetric partner potential is $[V^{PT}]_+(x) = [W^{PT}(x)]^2 + [W^{PT}(x)]'$. In this way, one has discovered another PT-invariant complex potential with a finite number of band gaps. It is plotted in Figure 2.

We can obtain yet other analytically solvable, complex, PT-invariant potentials by exchanging the orders of applying anti-isospectral transformations and supersymmetry. For example, we could first determine the supersymmetric partner of a solvable associated Lamé potential and then compute the corresponding PT-invariant potential. The results for the $a = 1$ Lamé potential are:

$$V_+(x) = 2m\mathrm{sn}^2(x - K(m), m) - m \ , \ [V_+]^{PT}(x) = -2m\mathrm{sn}^2(ix + \beta - K(m), m) + m + 1 \ , \quad (26)$$

where the potentials have been adjusted so as to have zero ground state energy. Again, for this $a = 1$ example, one finds that $[V_+]^{PT}(x)$ is essentially the same as $[V^{PT}]_+(x)$ with a constant complex shift of the independent variable x.

The situation is much richer and more interesting for Lamé potentials with higher a values. For instance, it is shown in ref. [12] that the supersymmetric partner potential of the $a = 3$ Lamé potential is given by

$$V_+(x) = -12m\mathrm{sn}^2(x, m) + \frac{2m^2\,\mathrm{sn}^2(x, m)\mathrm{cn}^2(x, m)}{\mathrm{dn}^2(x, m)} \frac{[2m + \delta_1 + 11 - 15m\mathrm{sn}^2(x, m)]^2}{[2m + \delta_1 + 1 - 5m\mathrm{sn}^2(x, m)]^2} \ , \quad (27)$$

and the corresponding PT-invariant complex potential $[V_+]^{PT}$ is simply obtained from here by using the anti-isospectral transformation $x \to ix + \beta$ and subtracting off the ground state energy E_g given by eq. (25) from it. This potential $[V_+]^{PT}$ is plotted in Figure 3. Clearly its band edge energy eigenvalues are simply related to those of $V_+(x)$ and hence to the $a = 3$ Lamé potential by relation (5). Hence the band edge energy eigenvalues of $[V_+]^{PT}$ are identical to those of $[V^{PT}]_+$ and $[V^{PT}]_-$ even though the three potentials are distinct. For the example under consideration, this is just the statement that the three different complex PT-invariant potentials $[V^{PT}]_-(x), [V^{PT}]_+(x), [V_+]^{PT}(x)$ plotted in Figures 1,2,3 all have the same band structure.

In Table 2 we have given the expression for the band edge eigenvalues and eigenfunctions for the PT-invariant complex potential $V^{PT}(x) = -6m\mathrm{sn}^2(ix + \beta, m) - 2m\mathrm{cn}^2(ix + \beta, m)/\mathrm{dn}^2(ix + \beta, m)$ suitably adjusted by subtracting its ground state energy $E_g = -5 - m - 2\sqrt{4 - 3m}$ so that the lowest band edge is at zero energy. Using the ground state wave function of this potential, the corresponding superpotential turns out to be

$$\begin{aligned} W^{PT} = {} & i\frac{\mathrm{sn}(ix + \beta, m)\mathrm{dn}(ix + \beta, m)}{\mathrm{cn}(ix + \beta, m)} - im\frac{\mathrm{cn}(ix + \beta, m)\mathrm{sn}(ix + \beta, m)}{\mathrm{dn}(ix + \beta, m)} \\ & - 6im\frac{\mathrm{sn}(ix + \beta, m)\mathrm{dn}(ix + \beta, m)\mathrm{cn}(ix + \beta, m)}{3m\mathrm{sn}^2(ix + \beta, m) - 2 + \sqrt{4 - 3m}} \ . \end{aligned} \quad (28)$$

Hence the corresponding supersymmetric partner potential $[V^{PT}]_+(x) = [W^{PT}(x)]^2 + [W^{PT}(x)]'$ is easily calculated. On the other hand, yet another PT-invariant potential with the same band edges can be obtained by starting from the partner potential of the $(a = 2, b = 1)$ associated Lamé potential $V(x) = 6m\mathrm{sn}^2(x,m) + 2m\mathrm{cn}^2(x,m)/\mathrm{dn}^2(x,m)$, and applying the anti-isospectral transformation. We get

$$[V_+]^{PT}(x) = -2m\mathrm{sn}^2(ix+\beta,m) - 6m\frac{\mathrm{cn}^2(ix+\beta,m)}{\mathrm{dn}^2(ix+\beta,m)} + 5 + m + 2\sqrt{4-3m}\,. \tag{29}$$

It is worth noting that again the two potentials $[V_+]^{PT}$ and $[V^{PT}]_+$ are quite different even though they have the same band edge eigenvalues. Further, while the initial associated Lamé potential is self-isospectral, its PT-transform is not so. In fact, this seems to be true in general. In particular, whereas the associated Lamé potentials with $b = a - 1$ are isospectral, we find that the corresponding PT-invariant periodic potentials are not self isospectral except when $a = 1$.

Finally, let us comment that for the PT-invariant potential $i\sin^{2N+1}(x)$, Bender et al. [6] found that the band edge eigenfunctions are always 2π periodic and unlike other lattice problems, the anti-periodic band edge eigenfunctions of period 4π were absent. They speculated whether this could perhaps be a unique signal of PT symmetry. However, in this letter, we have seen many examples where this is not true. In particular, Table I shows an example with both periodic as well as anti-periodic band edges, showing that the absence of anti-periodic band edges is not a general property of PT-invariant periodic potentials.

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Figure Captions

Figure 1: A plot of the real and imaginary parts of the complex PT-invariant potential $[V^{PT}]_-(x) = -12m\text{sn}^2(ix+\beta,m)-E_g$, where $E_g = -5-5m-2\delta_3$, and $\delta_3 \equiv \sqrt{4-7m+4m^2}$ (see ref. [12]). The potential has been defined so as to have zero ground state energy. The plot is for the choice $m = 0.75,\ \beta = 0.5$. The potential has a period $2K'(0.75) = 3.3715$. The continuous curve denotes the real part and the dashed curve denotes the imaginary part.

Figure 2: A plot of the real (continuous curve) and imaginary (dashed curve) parts of the supersymmetric partner potential $[V^{PT}]_+(x)$ of the complex PT-invariant potential shown in Figure 1, for the choice $m = 0.75,\ \beta = 0.5$.

Figure 3: A plot of the real (continuous curve) and imaginary (dashed curve) parts of the PT-invariant potential $[V_+]^{PT}(x)$ obtained by first taking the supersymmetric partner of the $a = 3$ Lamé potential and then applying the anti-isospectral transformation $x \to x + i\beta$, for the choice $m = 0.75,\ \beta = 0.5$. The constant energy $-3-2\delta_3-2\delta_1$ (see Table 1) has been subtracted off, so that the ground state of $[V_+]^{PT}(x)$ has zero energy.

Table 1: The eigenvalues and eigenfunctions for the 7 band edges of the PT-invariant Lamé potential $[V^{PT}]_-(x) = -12m\text{sn}^2(ix+\beta,m) - E_g$, where $E_g = -5-5m-2\delta_3$ [eq. (13)], and $\delta_1 \equiv \sqrt{1-m+4m^2}$, $\delta_2 \equiv \sqrt{4-m+m^2}$, $\delta_3 \equiv \sqrt{4-7m+4m^2}$. The potential has a period $2K'(m)$. The real periods of various eigenfunctions are also tabulated.

E	$\psi^{(-)}$	Period
0	$\text{sn}(ix+\beta,m)[2+2m-\delta_3-5m\text{sn}^2(ix+\beta,m)]$	$2K'(m)$
$3m+2\delta_3-2\delta_2$	$\text{cn}(ix+\beta,m)[2+m-\delta_2-5m\text{sn}^2(ix+\beta,m)]$	$4K'(m)$
$3+2\delta_3-2\delta_1$	$\text{dn}(ix+\beta,m)[1+2m-\delta_1-5m\text{sn}^2(ix+\beta,m)]$	$4K'(m)$
$1+m+2\delta_3$	$\text{sn}(ix+\beta,m)\text{cn}(ix+\beta,m)\text{dn}(ix+\beta,m)$	$2K'(m)$
$4\delta_3$	$\text{sn}(ix+\beta,m)[2+2m+\delta_3-5m\text{sn}^2(ix+\beta)]$	$2K'(m)$
$3m+2\delta_3+2\delta_2$	$\text{cn}(ix+\beta,m)[2+m+\delta_2-5m\text{sn}^2(ix+\beta,m)]$	$4K'(m)$
$3+2\delta_3+2\delta_1$	$\text{dn}(ix+\beta,m)[1+2m+\delta_1-5m\text{sn}^2(ix+\beta,m)]$	$4K'(m)$

Table 2: The eigenvalues and eigenfunctions for the 5 band edges of the PT-invariant associated Lamé potential $[V^{PT}]_-(x) = -6m\text{sn}^2(ix+\beta,m) - 2m\text{cn}^2(ix+\beta,m)/\text{dn}^2(ix+\beta,m) - E_g$, where E_g is given by eq. (21). Here $\delta_4 \equiv \sqrt{4-5m+m^2}$. The potential has a period $2K'(m)$. The real periods of various eigenfunctions are also tabulated.

E	$\psi^{(-)}$	Period
0	$\frac{\text{cn}(ix+\beta,m)}{\text{dn}(ix+\beta,m)}[3m\text{sn}^2(ix+\beta,m)-2+\sqrt{4-3m}]$	$2K'(m)$
$2\sqrt{4-3m}-m-2\delta_4$	$\frac{\text{sn}(ix+\beta,m)}{\text{dn}(ix+\beta,m)}[3m\text{sn}^2(ix+\beta,m)-2-m+\delta_4]$	$4K'(m)$
$2\sqrt{4-3m}-m+2\delta_4$	$\frac{\text{sn}(ix+\beta,m)}{\text{dn}(ix+\beta,m)}[3m\text{sn}^2(ix+\beta,m)-2-m-\delta_4]$	$4K'(m)$
$4\sqrt{4-3m}$	$\frac{\text{cn}(ix+\beta,m)}{\text{dn}(ix+\beta,m)}[3m\text{sn}^2(ix+\beta,m)-2-\sqrt{4-3m}]$	$2K'(m)$
$5-3m+2\sqrt{4-3m}$	$\text{dn}^2(ix+\beta,m)$	$2K'(m)$

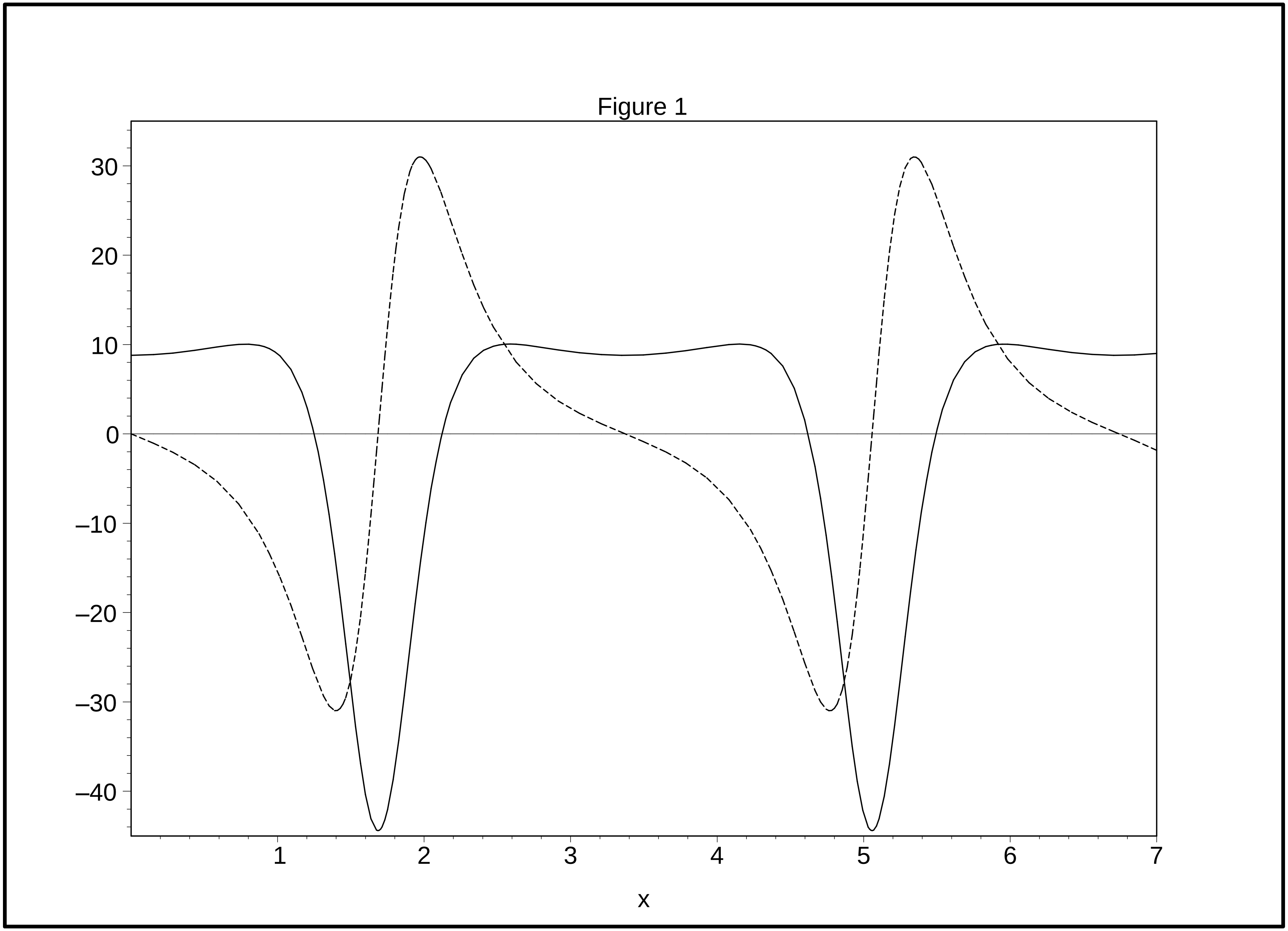

Figure 1
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1
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5
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7
x

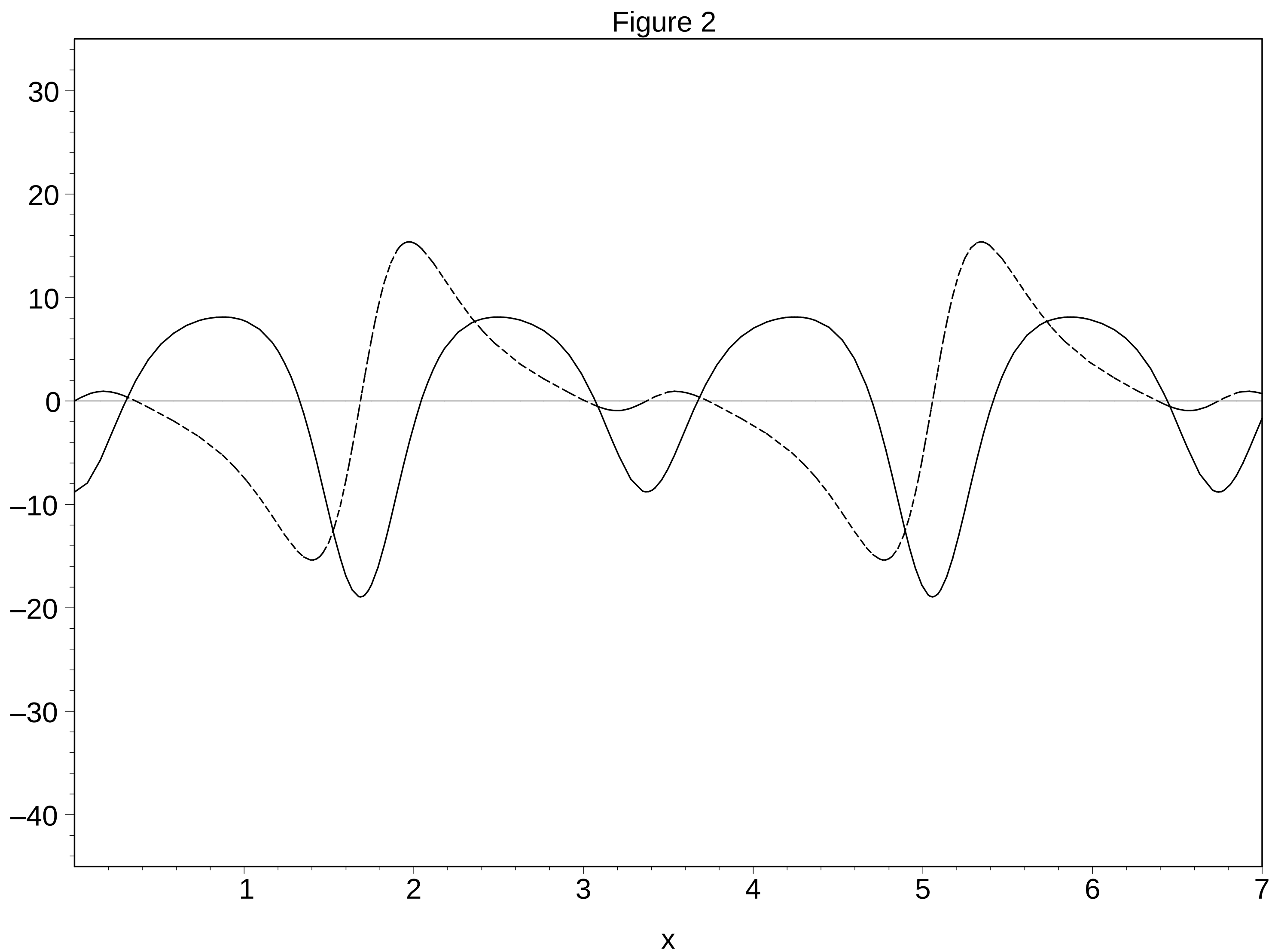

Figure 2
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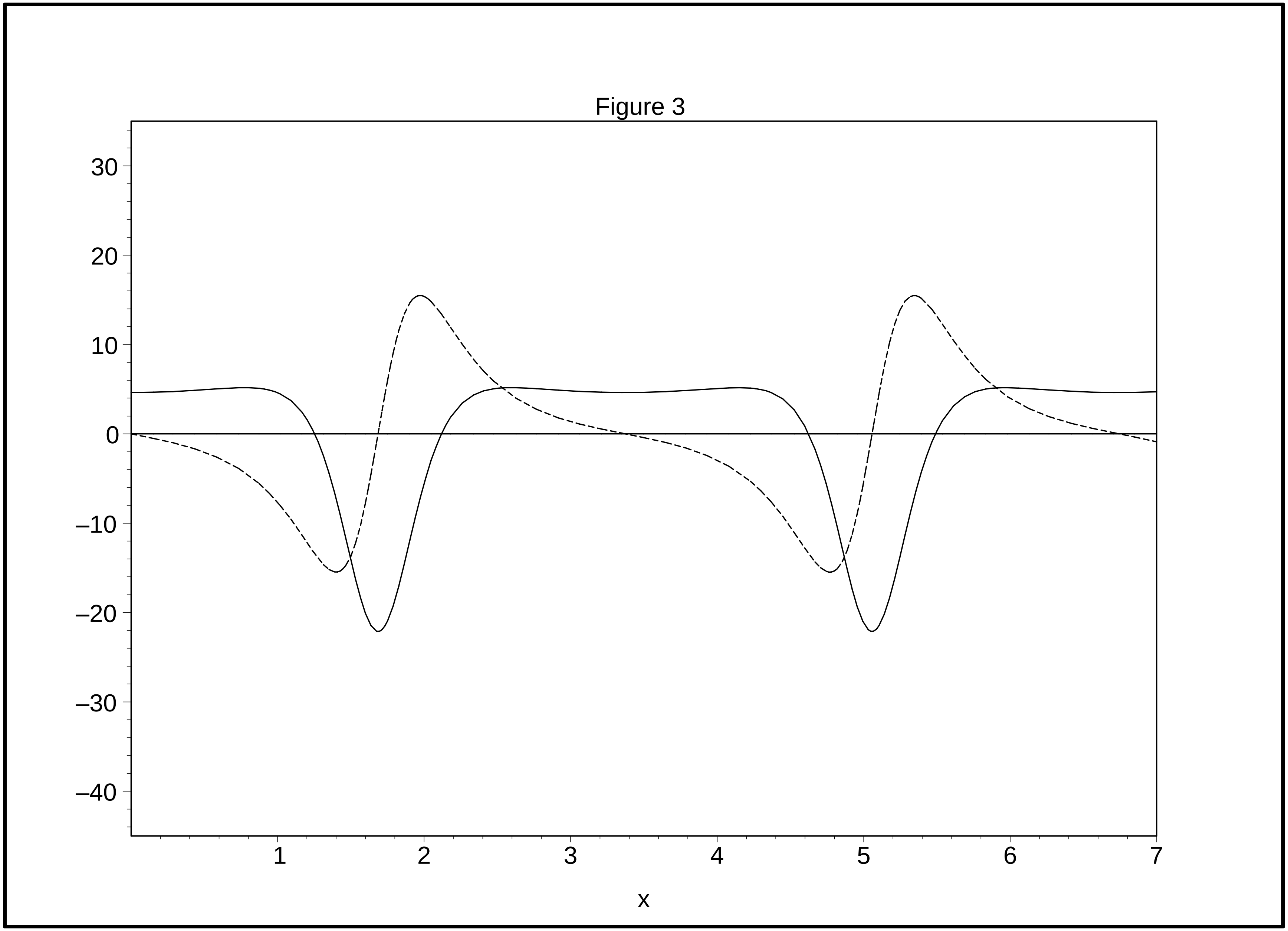

Figure 3
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